corresponding to a temperature difference over 5 cm of 1.7°C. While the field of motion at the top was somewhat irregular and not strictly cellular, some cells of about 2-cm diameter were always visible in the overstable range and at least two were observed to go through two full cycles of reversal of the axial rotation without moving away or deforming appreciably. The period of oscillation of the cells ranged from 15.1 to 16.2 sec in a set of several determinations. The theoretical data available at present are not strictly applicable because the boundary conditions are slightly different but the above values may roughly be compared with theoretical values for the given P and T of $R=2.1\times 10^5$ and oscillation period $=13.9$ sec. Systematic experiments along these lines are planned at this laboratory and should be the subject of later reports.

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Theory of Elementary Particles in General Relativity

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The matrices of elementary particles are generalized to general coordinates, and a new covariant displacement operator is defined, in order to generalize the theory of elementary particles to general relativity, and obtain general commutation rules. The elementary particle is considered as a singularity, with spin-structure of a gravitational field which gives Riemannian structure to the space-time. It is shown that the transformation properties of the space-time are determined not only by the affine structure due to the gravitational field but also by the spin-structure of the particle singularity.

INTRODUCTION

DIRAC'S equation has been generalized by Bhabha[1] to describe elementary particles of any integral or half-integral spin. However, in his theory of elementary particles[2] Bhabha assumes that his theory is valid in the framework of special relativity only.

Since a particle and its gravitational field are inseparable, it is desirable to extend the theory of elementary particles to general relativity by making it covariant for general continuous groups of transformation of Riemannian space-time. Even though, for particles of integral spin, tensor formalism is easier to handle, the spinor form will be retained here in order to keep the underlying unity of the theory for particles of integral and half-integral spins always evident. Pauli[3] has generalized Dirac's equation (for spin $\frac{1}{2}$) to general coordinates; we shall show that Bhabha's equation can be generalized in a similar manner. To do this we have to modify Einstein's idea of considering a particle as a singularity in its gravitational field which gives a Riemannian structure to space-time. The spin structure will be described by the operation of a set of spin matrices β^μ on the spin variables which are components of a wave function ψ. It will be shown that the β^μ satisfy a certain commutation relation (dependent upon the spin of the particle) which we shall for the moment write in the symbolic functional form:[4]

$$\mathbf{G}(\beta^\mu,\beta^\nu,\beta^\sigma,\cdots;g^{\mu\nu},g^{\sigma\tau},\cdots)=0, \tag{1}$$

where $g^{\mu\nu}g_{\nu\sigma}=\delta^\mu{}_\sigma$, $g_{\mu\nu}$ being the metric tensor of Riemannian space-time.

SPIN AFFINE TRANSFORMATIONS

In Eq. (1) we assume that the matrices β^μ are contravariant components of a vector point function whose operation on ψ is determined by $g_{\mu\nu}$ at each point of space-time. The effect of the gravitational field is to make this operation nonintegrable, i.e., dependent upon the path chosen. Hence, under a parallel displacement along a path, the increment in β^μ is

$$\delta\beta^\mu=-\Gamma_{\alpha\nu}{}^\mu\beta^\alpha dx^\nu,$$

where $\Gamma_{\alpha\nu}{}^\mu$ is the affine connection constructed from the gravitational potentials $g_{\mu\nu}$.

Under an infinitesimal affine transformation, the β^μ transform as

$$\beta'^\mu=\beta^\mu+\epsilon^\tau[\beta^\mu{}_{,\tau}+\Gamma_{\alpha\tau}{}^\mu\beta^\alpha]=\beta^\mu+\epsilon^\tau\beta^\mu{}_{;\tau}, \tag{2}$$

where $\beta^\mu{}_{,\tau}=\partial\beta^\mu/\partial x^\tau$, ϵ^τ is an infinitesimal parameter, and $\beta^\mu{}_{;\tau}$ defines the covariant derivative of β^μ.

Under a local infinitesimal spin-space rotation

$$S=1+\epsilon^\tau\Omega_\tau,$$

β^μ transforms as

$$\beta'^\mu=S^{-1}\beta^\mu S=\beta^\mu+\epsilon^\tau[\beta^\mu,\Omega_\tau], \tag{3}$$

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[1] H. J. Bhabha, Revs. Modern Phys. 17, 200 (1945).

[2] H. J. Bhabha, Revs. Modern Phys. 21, 451 (1949).

[3] W. Pauli, Ann. Physik 18, 337 (1933).

[4] $\mathbf{G}$ is a rational integral function of β^μ and $g^{\mu\nu}$.

where the usual commutator notation is used, as in

$$[\beta^\mu,\Omega_\tau]=\beta^\mu\Omega_\tau-\Omega_\tau\beta^\mu.$$

On varying $g_{\mu\nu}$, we assume β^μ to vary as

$$\delta\beta^\mu=\tfrac{1}{2}\beta_\nu\delta g^{\mu\nu};\quad \delta\beta_\mu=\tfrac{1}{2}\beta^\nu\delta g_{\mu\nu}. \tag{4}$$

This variation is also nonintegrable in general.

Definition A

Commutation rules $\mathbf{G}(\beta^\mu,\beta^\nu,\beta^\sigma,\cdots,g^{\mu\nu},g^{\sigma\tau},\cdots)=0$ will be called consistent under the operation of variation (4), if

$$\delta\mathbf{G}(\beta^\mu,\beta^\nu,\beta^\sigma,\cdots,g^{\mu\nu},g^{\sigma\tau},\cdots)=0,$$

that is if

$$\frac{1}{2}\left[\frac{\partial\mathbf{G}}{\partial\beta^\mu},\ \beta_\alpha\right]\delta g^{\alpha\mu}+\frac{1}{2}\left[\frac{\partial\mathbf{G}}{\partial\beta^\nu},\ \beta_\sigma\right]\delta g^{\sigma\nu}+\cdots=-\frac{\partial\mathbf{G}}{\partial g^{\mu\nu}}\delta g^{\mu\nu}-\cdots.$$

(Here square brackets mean that $\delta\beta^\mu$ must replace the differentiated β^μ without change of order.)

It can be shown easily that the following theorem always holds:

Theorem I: The commutation relation,

$$\mathbf{G}(\beta^\mu,\beta^\nu,\cdots g^{\mu\nu})=0,$$

remains invariant (i) under any local infinitesimal spin space rotation, (ii) under any infinitesimal affine transformation, provided it is consistent in the sense of definition A.

This theorem at once tells us that there always exists a spin-space rotation S which annuls the change in β^μ due to the affine transformation (2). Hence, equating the coefficients of ϵ^τ in (2) and (3), we get

$$\beta^\mu{}_{|\tau}=\beta^\mu{}_{,\tau}+\Gamma^\mu{}_{\alpha\tau}\beta^\alpha+[\Omega_\tau,\beta^\mu]=0. \tag{5}$$

Thus we have defined a new derivative which we may call the coefficient of covariant displacement (abbreviated as c.c.d.). Hence the c.c.d. of the fundamental matrix $\beta^\mu{}_{|\tau}=0$ determines the spin-space structure Ω_τ, just as the covariant derivative of the metric tensor $g^{\mu\nu}{}_{;\tau}=0$ determines the affine structure of Riemannian space. Therefore, $g_{\mu\nu}$ and β^ν both have fundamental significance in our combined spinor-tensor formalism.

The c.c.d. of the covariant β_μ is

$$\beta_{\mu|\tau}=\beta_{\mu,\tau}-\left\{\begin{matrix}\alpha\\ \mu,\tau\end{matrix}\right\}\beta_\alpha+[\Omega_\tau,\beta_\mu]. \tag{6}$$

Theorem I also tells us that there exists an infinitesimal spin-space transformation, $\mathbf{S}=1+\mathfrak{F}^{\sigma\alpha}\delta g_{\sigma\alpha}$, such that it makes the variation $\delta\beta_\mu$ integrable, i.e.,

$$d\beta^\mu=\delta\beta^\mu-[\mathfrak{F}^{\alpha\sigma},\beta^\mu]\delta g_{\sigma\alpha},$$

where $d\beta^\mu$ is a total differential. Hence the integrability condition,

$$\frac{\partial^2\beta^\mu}{\partial g_{\alpha\sigma}\partial g_{\nu\delta}}=\frac{\partial^2\beta^\mu}{\partial g_{\nu\delta}\partial g_{\alpha\sigma}},$$

gives

$$\frac{\partial\mathfrak{F}^{\alpha\sigma}}{\partial g_{\nu\delta}}-\frac{\partial\mathfrak{F}^{\nu\delta}}{\partial g_{\alpha\sigma}}-[\mathfrak{F}^{\alpha\sigma},\mathfrak{F}^{\nu\delta}]\equiv\tfrac{1}{4}[g^{\alpha\nu}\mathfrak{g}^{\sigma\delta}+g^{\sigma\delta}\mathfrak{g}^{\alpha\nu}+g^{\alpha\delta}\mathfrak{g}^{\sigma\nu}+g^{\sigma\nu}\mathfrak{g}^{\alpha\delta}],$$

where the matrices $\mathfrak{g}^{\mu\nu}$ represent a transformation

$$T=1+\epsilon_{\mu\nu}\mathfrak{g}^{\mu\nu}, \tag{7}$$

($\epsilon_{\mu\nu}=-\epsilon_{\nu\mu}$ are infinitesimal parameters) and form an infinitesimal algebra, defined over underlying Riemannian space-time, as follows:

$$\begin{aligned}
&[\mathfrak{g}^{\mu\nu},\mathfrak{g}^{\sigma\tau}]=-g^{\mu\sigma}\mathfrak{g}^{\nu\tau}-g^{\nu\tau}\mathfrak{g}^{\mu\sigma}+g^{\mu\tau}\mathfrak{g}^{\nu\sigma}+g^{\nu\sigma}\mathfrak{g}^{\mu\tau},\\
&[\beta^\mu,\mathfrak{g}^{\sigma\nu}]=g^{\mu\sigma}\beta^\nu-g^{\mu\nu}\beta^\sigma,\\
&\mathfrak{g}^{\mu\nu}=-\mathfrak{g}^{\nu\mu}.
\end{aligned} \tag{8}$$

Taking variations of (5) and (6) and using (8), one obtains

$$\delta\Omega_\tau=\tfrac{1}{2}g_{\alpha\sigma}\mathfrak{g}^{\sigma\nu}\delta\Gamma^\alpha{}_{\nu\tau}. \tag{9}$$

Hence the variation of spin-space structure is linearly connected with that of affine structure.

That the matrices $\mathfrak{g}^{\mu\sigma}$ are fundamental, i.e., $\mathfrak{g}^{\mu\nu}{}_{|\tau}=0$ can be seen by taking the c.c.d. of (8), which gives

$$\mathfrak{g}^{\mu\nu}{}_{|\tau}=\mathfrak{g}^{\mu\nu}{}_{,\tau}+\Gamma_{\alpha\tau}{}^\mu\mathfrak{g}^{\alpha\nu}+\Gamma_{\alpha\tau}{}^\nu\mathfrak{g}^{\mu\alpha}+[\Omega_\tau,\mathfrak{g}^{\mu\nu}]=0.$$

GENERALIZED WAVE FUNCTION ψ

Since the components of ψ are spinors, ψ transforms under a spin-space rotation as $\psi'=S^{-1}\psi$. We assume that there exists a Hermitian matrix D defined by $\bar\psi=\psi^\dagger D$, where $\psi^\dagger=$Hermitian conjugate of ψ and $\bar\psi=$Pauli conjugate of ψ. It can be verified that

$$\begin{aligned}
&\beta^{\mu\dagger}=D\beta^\mu D^{-1},\quad \beta^{\dagger\prime\mu}=D'\beta^\mu D'^{-1},\\
&D'=S^{-1}DS,\quad \bar\psi'=\bar\psi S
\end{aligned} \tag{10}$$

imply the invariance of $(\bar\psi\psi)$ and $(\bar\psi\beta^\mu\psi)$ under spin-space rotation. But under an affine transformation $\bar\psi\beta^\mu\psi$ transforms as a contravariant vector, hence its c.c.d. gives

$$\begin{aligned}
\psi_{|\tau}&=\psi_{,\tau}+\Omega_\tau\psi,\\
\bar\psi_{|\tau}&=\bar\psi_{,\tau}-\bar\psi\Omega_\tau,\\
\psi^\dagger{}_{|\tau}&=\psi^\dagger{}_{,\tau}+\psi^\dagger\Omega^\dagger{}_\tau.
\end{aligned} \tag{11}$$

The c.c.d. of β^μ gives

$$\Omega_\tau'=S^{-1}\Omega_\tau S+S^{-1}S_{,\tau}.$$

Hence for the invariance of Ω_τ we have $S_{,\tau}=-[\Omega_\tau,S]$. The $\psi_{|\tau}$ transforms as $\psi'{}_{|\tau}=S^{-1}\psi_{|\tau}$, and

$$\psi_{|\tau|\sigma}=\psi_{|\tau,\sigma}-\Gamma_{\tau\sigma}{}^\alpha\psi_{|\alpha}+\Omega_\sigma\psi_{|\tau}. \tag{12}$$

Hence, $\psi_{|\tau}$ is a covariant spinor-tensor of rank one. Equations (5) and (10) show that

$$D_{|\tau}=D_{,\tau}-D\Omega_\tau-\Omega^\dagger D=0, \tag{13}$$

i.e., the matrix D is fundamental. If we define a matrix Λ_κ so that $\bar{\psi}\cdot\Lambda_\kappa$ transforms like a vector,[5] then

$$\Lambda_{\kappa|\tau}=\Lambda_{\kappa,\tau}-\Gamma_{\kappa\tau}{}^{\alpha}\Lambda_\alpha+\Omega_\tau\Lambda_\kappa. \qquad (14)$$

From (12) and (15) one obtains

$$\psi_{|\mu\nu}-\psi_{|\nu\mu}=(\Omega_{\mu,\nu}-\Omega_{\nu,\mu}-[\Omega_\mu,\Omega_\nu])\psi, \qquad (15)$$

$$\begin{aligned} O&=\beta^{\tau}{}_{|\mu\nu}-\beta^{\tau}{}_{|\nu\mu}\\ &=\{\Gamma^{\tau}{}_{\alpha\mu,\nu}-\Gamma^{\tau}{}_{\alpha\nu,\mu}+\Gamma^{\tau}{}_{\kappa\nu}\Gamma^{\kappa}{}_{\alpha\nu}-\Gamma^{\tau}{}_{\kappa\mu}\Gamma^{\kappa}{}_{\alpha\nu}\}\beta^{\alpha}\\ &\qquad+[(\Omega_{\mu,\nu}-\Omega_{\nu,\mu}-[\Omega_\mu,\Omega_\nu]),\beta^{\tau}]. \qquad (16)\end{aligned}$$

The quantity in curly brackets is just the curvature tensor $R^{\tau}{}_{\mu\nu\alpha}$. Hence,

$$\psi_{|\mu\nu}-\psi_{|\nu\mu}=-\tfrac{1}{2}R_{\mu\nu\sigma\tau}I^{\sigma\tau}\psi, \qquad (17)$$

i.e.,

$$\Delta\psi=(\partial_1\partial_2-\partial_2\partial_1)\psi=-\tfrac{1}{2}R_{\mu\nu\sigma\tau}\mathcal{I}^{\sigma\tau}\psi.$$

Hence the total increment $\Delta\psi$ of a wave function ψ in an elementary circuital displacement depends linearly on the curvature and the matrices $\mathcal{I}^{\mu\nu}$ of the infinitesimal algebra generated locally in that region.

We shall therefore interpret $i\hbar$ times the c.c.d. as a displacement operator, covariant under any spin-affine transformations of Riemannian space-time in the presence of the gravitational field, and shall define $\mathfrak{Z}_\mu$ by

$$\mathfrak{Z}_\mu\psi=i\hbar\psi_{|\mu}. \qquad (18)$$

$\mathfrak{Z}$ will be seen to satisfy the following covariant commutation rules in general relativistic quantum mechanics:

$$\left.\begin{aligned}&[x_\nu,\mathfrak{Z}_\mu]=i\hbar\delta_{\mu\nu},\\ &[x_\mu,x_\nu]=0,\\ &[\mathfrak{Z}_\mu,\mathfrak{Z}_\nu]=-\tfrac{1}{2}\hbar^2R_{\mu\nu\sigma\tau}\mathcal{I}^{\sigma\tau},\\ &[\mathfrak{F},\mathfrak{Z}_\tau]=i\hbar\mathfrak{F}_{|\tau},\end{aligned}\right\} \qquad (19)$$

which may be compared with the noncommutation of the electromagnetic field displacement operator Π_μ,[6]

$$[\Pi_\mu,\Pi_\nu]=f_{\mu\nu}=\frac{i\hbar}{c}e\left(\frac{\partial\varphi_\mu}{\partial x^\nu}-\frac{\partial\varphi_\nu}{\partial x^\mu}\right).$$

This can be interpreted as follows: In general the matter and its field produce a topological deformation of the space-time by modifying its local properties so as to introduce nonintegrability. General relativity tells us that in the case of a gravitational field this topology can be metrized so that nonintegrability becomes related with the curvature.

[5] In case of spin 0 and 1, Λ_κ is just the generalization of Harish-Chandra's Υ_κ matrix which is useful in going from the particle to the wave aspect of mesons and photons [Harish-Chandra, Proc. Roy. Soc. (London) **186**, 503 (1946)].

[6] W. Pauli and M. Fierz, Proc. Roy. Soc. (London) **173**, 230 (1939).

LAGRANGIAN FORMULATION

Postulate

Laws of physics are covariant under any general spin-affine transformations of the space-time.

Taking this as our fundamental postulate, and assuming that a wave function ψ contains the maximum knowledge of the system, we observe that the wave equation must be linear in the derivatives of ψ and that an irreducible wave equation represents an elementary particle. We set up an invariant Lagrangian density as follows:

$$\begin{aligned}\mathfrak{L}=\frac{1}{16\pi}G^{-1}c^4(-g)^{\frac{1}{2}}g^{\mu\nu}[\Gamma^{\sigma}{}_{\rho\sigma}\Gamma^{\rho}{}_{\mu\nu}-\Gamma^{\sigma}{}_{\mu\rho}\Gamma^{\rho}{}_{\nu\sigma}]\\ +ic(-g)^{\frac{1}{2}}\bar{\psi}(\beta^{\mu}\mathfrak{Z}_\mu{}^{(s)}+\chi)\psi, \qquad (20)\end{aligned}$$

where $\mathfrak{Z}_\mu{}^{(s)}$ is the symmetrized covariant displacement operator, i.e.,

$$\mathfrak{Z}_\mu{}^{(s)}=\tfrac{1}{2}(\mathfrak{Z}_\mu+\mathfrak{Z}_\mu{}^{(\tau)}),$$

also where

$$\bar{\psi}\mathfrak{Z}_\mu{}^{(\tau)}=-i\hbar\bar{\psi}_{|\mu}=-\mathfrak{Z}_\mu\bar{\psi},$$

and G is the universal gravitational constant.

Variation of $\mathfrak{L}$ with respect to ψ, $\bar{\psi}$, and $g_{\mu\nu}$ gives the Euler-Lagrange equation

$$\beta^{\mu}\mathfrak{Z}_\mu\psi+\chi\psi=0,\quad \mathfrak{Z}_\mu\bar{\psi}\beta^{\mu}-\chi\bar{\psi}=0, \qquad (21)$$

and[7]

$$\begin{aligned}(-g)^{\frac{1}{2}}(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R)\\ =\frac{4\pi G}{c^3}(-g)^{\frac{1}{2}}\bar{\psi}(\mathfrak{Z}_\mu{}^{(s)}\beta_\nu+\beta_\mu\mathfrak{Z}_\nu{}^{(s)})\psi, \qquad (22)\end{aligned}$$

from which by putting

$$\mathcal{S}^{\mu}=(-g)^{\frac{1}{2}}c(\bar{\psi}\beta^{\mu}\psi),$$

and

$$\Theta_{\mu\nu}=\tfrac{1}{2}(-g)^{\frac{1}{2}}c\bar{\psi}(\mathfrak{Z}_\mu{}^{(s)}\beta_\nu+\beta_\mu\mathfrak{Z}_\nu{}^{(s)})\psi,$$

one obtains

$$\mathfrak{Z}_\mu\mathcal{S}^{\mu}=0,\quad\text{i.e.,}\quad \mathcal{S}^{\mu}{}_{|\mu}=0,$$

and

$$\mathfrak{Z}_\mu\Theta^{\mu\nu}=0,\quad\text{i.e.,}\quad \Theta^{\mu\nu}{}_{|\mu}=0. \qquad (23)$$

These are the conservation equations of the charge current vector $\mathcal{S}^{\mu}$ and the conservation equations of the symmetric stress energy tensor $\Theta_{\mu\nu}$ of an elementary particle in its gravitational field.

Introducing the operators $\mathbf{\mathfrak{Z}}=\beta^{\mu}\mathfrak{Z}_\mu$ and $\mathfrak{Z}^2=\mathfrak{Z}_\mu\mathfrak{Z}^{\mu}$, one sees that they satisfy the characteristic equation of even degree:

$$(\mathbf{\mathfrak{Z}}^2-\alpha_1{}^2\mathfrak{Z}^2)(\mathbf{\mathfrak{Z}}^2-\alpha_2{}^2\mathfrak{Z}^2)\cdots=0, \qquad (24)$$

or, of odd degree:

$$\mathbf{\mathfrak{Z}}(\mathbf{\mathfrak{Z}}^2-\alpha_1{}^2\mathfrak{Z}^2)(\mathbf{\mathfrak{Z}}^2-\alpha_2{}^2\mathfrak{Z}^2)\cdots=0; \qquad (25)$$

i.e.,

$$\mathfrak{Z}_\mu\mathfrak{Z}_\nu\mathfrak{Z}_\sigma\cdots(\beta^{\mu}\beta^{\nu}-\alpha_1{}^2g^{\mu\nu})(\beta^{\sigma}\beta^{\tau}-\alpha_2{}^2g^{\sigma\tau})\cdots=0,$$

[7] This equation can also be written as

$$(-g)^{\frac{1}{2}}(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R)=(8\pi G/c^4)\Theta_{\mu\nu}+\lambda g_{\mu\nu},$$

where λ = Einstein's cosmological constant. The conservation equation of $\Theta_{\mu\nu}$ remains unaffected.

which gives for even and odd degrees, respectively:

$$\Sigma(\beta^\mu\beta^\nu-\alpha_1{}^2g^{\mu\nu})(\beta^\sigma\beta^\tau-\alpha_2{}^2g^{\sigma\tau})\cdots=0,\qquad\qquad (26)$$
$$\Sigma\beta^\mu(\beta^\nu\beta^\sigma-\alpha_1{}^2g^{\nu\sigma})\cdots=0.$$

These are the general relativistic commutation relations previously written symbolically as **G** in (1) and in their irreducible form their degree represents the spin of the elementary particle as given by (8).[8] It may be noted that they are consistent according to definition A and hence satisfy theorem I. From (21) and (25) one gets,

$$(\chi^2-\alpha_1{}^2\mathfrak{Z}^2)(\chi^2-\alpha_2{}^2\mathfrak{Z}^2)\cdots=0.$$

Hence χ^2/α^2, the eigenvalues of $\mathfrak{Z}^2$, are squares of masses of elementary particles.

The matrices $\mathcal{S}^{\mu\nu}$ defined by (8) reduce to the nucleus of the representation of Lorentz transformation in a locally Cartesian frame; hence we can define an infinitesimal transformation $\mathcal{T}$ whose representation transforms as $\psi'=\mathrm{T}\psi=[1+\frac{1}{2}\epsilon_{\mu\nu}\mathcal{S}^{\mu\nu}]\psi$, where T is a representation matrix of $\mathcal{T}$ which may be called a local Lorentz transformation, and $\mathcal{S}_{\mu\nu}$ the nucleus of representation of $\mathcal{T}$. T forms a local group embedded in the general transformation group S and (9) shows us that S can be built up by successive variation of $\Gamma_{\alpha\tau}{}^\mu$ if we know the infinitesimal algebra of $\mathcal{S}^{\mu\nu}$ and β^μ defined by (8) and (26). In the particular case of spin $\frac{1}{2}$, $\mathcal{S}^{\mu\nu}=\frac{1}{4}[\beta^\mu,\beta^\nu]$, and for spin 0 and 1, $\mathcal{S}^{\mu\nu}=[\beta^\mu,\beta^\nu]$.

In the case of spin 2, ψ in (20), (21), (22) represents the wave function of gravitons (gravitational quanta). Now the metrical structure of physical space has been considered to be due to gravitation. One may ask how to reconcile this representation of gravitation with the one in terms of gravitational quanta. The apparent contradiction is resolved by considering matter with its field as causing a (topological) deformation of physical space-time and having the duality of metricity and discreteness, as observed by experiments which also deform space-time. We can observe metrical properties only on the macroscopic level, where approximate rigid bodies and local frames of reference exist, but in the case of the microscopic world where there exist no rigid rods to define distance (since the uncertainty principle applies) the other aspect of duality (discreteness) becomes apparent.

[8] For degrees two and three, these commutation rules reduce to the generalizations of Dirac's and Kemmer's commutation relations:

$$\beta^\mu\beta^\nu+\beta^\nu\beta^\mu=2g^{\mu\nu}$$

for spin $\frac{1}{2}$, and

$$\beta^\mu\beta^\sigma\beta^\nu+\beta^\nu\beta^\sigma\beta^\mu=g^{\mu\sigma}\beta^\nu+g^{\sigma\nu}\beta^\mu$$

for spins 0 and 1.

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Electron Energy Distributions in Stationary Discharges

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Calculations of the electron distribution function are presented for some simple examples of a stationary discharge in a dc space charge field. The treatment is valid when the predominant mechanism of energy exchange arises from motion in the dc space charge field. The computations indicate that the effect of dc space charge is, for a given external field strength, to increase the proportion of high energy electrons over that computed neglecting space charge. This results in a larger specific ionization rate, but the effect is not so great as to account for the low maintenance potentials observed in positive columns and in microwave discharges in inert gases.

I. INTRODUCTION

PAST theoretical analyses[1] of the energy distribution of electrons in gases have generally ignored the presence of space charge fields. In the microwave discharge, for example, the electric field is usually assumed to be of external origin; in positive columns of dc discharges, the relevant field is taken to be the longitudinal gradient. Now, in both these examples, the removal of charged particles takes place via the mechanism of ambipolar diffusion. This process requires the presence of a space charge field sufficiently strong to retard the motion of electrons, and to accelerate that of the positive ions to the boundary. Such fields are often comparable to or even larger than the external fields.

In order to obtain some idea of the effect of a space charge field (of the type prevalent in ambipolar diffusion) on electron energy distributions and associated quantities, such an average ionization rates, it has been deemed of interest to investigate the situation in which the space charge field is much larger than the applied field. This case represents the opposite extreme to that already treated, namely, space charge field very much less than external field. By this procedure one may hope to achieve an understanding of the generally encountered intermediate case by interpolation between the two extremes.

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[1] Morse, Allis, and Lamar, Phys. Rev. **48**, 412 (1935); J. A. Smit, Physica **3**, 543 (1936); T. Holstein, Phys. Rev. **70**, 367 (1946), to be referred to hereafter as I.