An Instance in Thermal Convection of Eddington's "Overstability"*

D. FULTZ, Y. NAKAGAWA, AND P. FRENZEN
Hydrodynamics Laboratory, Department of Meteorology, University of Chicago, Chicago, Illinois
(Received December 28, 1954)

The existence of modes of cellular convection driven by thermal instability which are oscillatory in type, rather than steady, has been predicted theoretically for systems that are rotating as a whole. Such modes are of a type called "overstable" by Eddington in a different connection. The result of an initial successful experiment with mercury is given.

THE theoretical study of the convective fluid motions due to thermal instability stems from Rayleigh's 1916 attempt[1] to explain the experimental observations of Benard on cellular convection. The standpoint in Rayleigh's and the subsequent studies has uniformly been to investigate the conditions just sufficient for the initiation of motions of a cellular type by means of perturbation techniques relative to a state of rest. All dependent variables in the perturbations are assumed proportional to a factor $e^{\sigma t}$. When σ is real, the conditions are investigated for σ first to become positive. The critical dividing point is taken at marginal stability $\sigma=0$, thus corresponding to a neutral or steady perturbation. The steady cells observed experimentally when the work is done carefully are then expected first at the physical conditions corresponding to the marginal stability point.

While the possibility that σ may be complex had at least been noted by Rayleigh himself[1] and by Jeffreys,[2] none of the published work in this area until recently dealt with problems in which σ could actually be complex in such a way as to give amplified motions. Pellew and Southwell[3] for example, show that in ordinary Benard convection problems, motions corresponding to such complex σ's are always damped. It is apparent that in more general problems of cellular motion the possibility may occur of first attaining a positive real part for σ with a nonzero imaginary part. The corresponding perturbation will be an exponentially amplifying oscillation or, at the point where the real part$=0$, a neutral oscillation. The former type of behavior has been named "overstability" by Eddington[4] who made some controversial applications of the idea to astrophysics. His reasons for the name are best described in his own words: "In the usual kind of *instability* a slight displacement provokes forces tending away from equilibrium; in *overstability* it provokes restoring forces so strong as to overshoot the corresponding position on the other side of equilibrium and set up an increasing oscillation." In the last two years Chandrasekhar[5] and, independently, Frenzen and Nakagawa[6] have shown theoretically that overstability can occur for cellular convection in a *rotating* system and Chandrasekhar has shown that the same is true of certain cases in hydromagnetics. The perturbation system governing rotating cellular convection can be arranged so as to depend on three nondimensional parameters: the Rayleigh number $R=g\alpha\beta d^4/\kappa\nu$, the Taylor number $T=4\Omega^2 d^4/\nu^2$, and the Prandtl number $P=\nu/\kappa$, where g is the acceleration of gravity, α the volume coefficient of expansion of the fluid, β the vertical adverse temperature gradient, d the depth of the fluid, ν the kinematic viscosity of the fluid, κ the thermometric conductivity, and Ω the basic rotation rate of the system. The above authors show in one particular case that, provided $P<0.6766$, for all sufficiently large T the criterion for overstability (oscillating cells) is reached before the marginal stability criterion for ordinary cells.

Mercury is a liquid whose Prandtl number is about 0.025 at ordinary temperatures and consequently it is suitable for a search for overstable motions. In addition, convection cells in a rotating system have at the top surface individual rotations about their axes which depend in sign on whether the motion at the center of the cell is up or down. This gives a very simple means of detecting whether oscillating cells are occurring simply by noting whether particles floating on the top free surface show periodic reversals of the axial rotation.

A trial carried out with preliminary equipment on November 20, 1953 showed rather conclusively that overstable (oscillating) cellular motions are obtained at approximately the expected conditions. A glass cylinder of $14\frac{1}{2}$-cm diameter containing 5.0 cm of mercury was rotated at about 10 rpm. Heating was applied uniformly from below by means of a Corning electrically-conducting glass plate. The estimated data for the point where the cells first became visible in the particle motions at the free surface and in the temperature gradient measurements were:

$$P=0.024, \quad T=2.2\times10^9; \quad R=7.6\times10^5,$$

* The research reported herewith has been made possible through support and sponsorship extended by Air Force Cambridge Research Center under a contract.

[1] Lord Rayleigh, Phil. Mag. (6) **32**, 529 (1916).

[2] H. Jeffreys, Phil. Mag. (7) **2**, 833 (1926), see p. 834.

[3] A. Pellew and R. V. Southwell, Proc. Roy. Soc. (London) **A176**, 312 (1940), see p. 322, ff.

[4] A. S. Eddington, *Internal Constitution of the Stars* (Cambridge University Press, London, 1926), p. 201.

[5] S. Chandrasekhar, Proc. Roy. Soc. (London) **A217**, 306 (1952).

[6] P. Frenzen and Y. Nakagawa, Tellus (to be published).

corresponding to a temperature difference over 5 cm of 1.7°C. While the field of motion at the top was somewhat irregular and not strictly cellular, some cells of about 2-cm diameter were always visible in the overstable range and at least two were observed to go through two full cycles of reversal of the axial rotation without moving away or deforming appreciably. The period of oscillation of the cells ranged from 15.1 to 16.2 sec in a set of several determinations. The theoretical data available at present are not strictly applicable because the boundary conditions are slightly different but the above values may roughly be compared with theoretical values for the given P and T of $R=2.1\times10^5$ and oscillation period=13.9 sec. Systematic experiments along these lines are planned at this laboratory and should be the subject of later reports.

PHYSICAL REVIEW VOLUME 94, NUMBER 6 JUNE 15, 1954

Theory of Elementary Particles in General Relativity

M. M. HATALKAR*
The Institute of Science, Bombay, India

(Received April 9, 1952; revised manuscript received March 10, 1954)

The matrices of elementary particles are generalized to general coordinates, and a new covariant displacement operator is defined, in order to generalize the theory of elementary particles to general relativity, and obtain general commutation rules. The elementary particle is considered as a singularity, with spin-structure of a gravitational field which gives Riemannian structure to the space-time. It is shown that the transformation properties of the space-time are determined not only by the affine structure due to the gravitational field but also by the spin-structure of the particle singularity.

INTRODUCTION

DIRAC'S equation has been generalized by Bhabha[1] to describe elementary particles of any integral or half-integral spin. However, in his theory of elementary particles[2] Bhabha assumes that his theory is valid in the framework of special relativity only.

Since a particle and its gravitational field are inseparable, it is desirable to extend the theory of elementary particles to general relativity by making it covariant for general continuous groups of transformation of Riemannian space-time. Even though, for particles of integral spin, tensor formalism is easier to handle, the spinor form will be retained here in order to keep the underlying unity of the theory for particles of integral and half-integral spins always evident. Pauli[3] has generalized Dirac's equation (for spin $\frac{1}{2}$) to general coordinates; we shall show that Bhabha's equation can be generalized in a similar manner. To do this we have to modify Einstein's idea of considering a particle as a singularity in its gravitational field which gives a Riemannian structure to space-time. The spin structure will be described by the operation of a set of spin matrices β^μ on the spin variables which are components of a wave function ψ. It will be shown that the β^μ satisfy a certain commutation relation (dependent upon the spin of the particle) which we shall for the moment write in the symbolic functional form:[4]

$$\mathbf{G}(\beta^\mu,\beta^\nu,\beta^\sigma,\cdots;g^{\mu\nu},g^{\sigma\tau},\cdots)=0, \tag{1}$$

where $g^{\mu\nu}g_{\nu\sigma}=\delta^\mu{}_\sigma$, $g_{\mu\nu}$ being the metric tensor of Riemannian space-time.

SPIN AFFINE TRANSFORMATIONS

In Eq. (1) we assume that the matrices β^μ are contravariant components of a vector point function whose operation on ψ is determined by $g_{\mu\nu}$ at each point of space-time. The effect of the gravitational field is to make this operation nonintegrable, i.e., dependent upon the path chosen. Hence, under a parallel displacement along a path, the increment in β^μ is

$$\delta\beta^\mu=-\Gamma_{\alpha\nu}{}^\mu\beta^\alpha dx^\nu,$$

where $\Gamma_{\alpha\nu}{}^\mu$ is the affine connection constructed from the gravitational potentials $g_{\mu\nu}$.

Under an infinitesimal affine transformation, the β^μ transform as

$$\beta'^\mu=\beta^\mu+\epsilon^\tau[\beta^\mu{}_{,\tau}+\Gamma_{\alpha\tau}{}^\mu\beta^\alpha]=\beta^\mu+\epsilon^\tau\beta^\mu{}_{;\tau}, \tag{2}$$

where $\beta^\mu{}_{,\tau}=\partial\beta^\mu/\partial x^\tau$, ϵ^τ is an infinitesimal parameter, and $\beta^\mu{}_{;\tau}$ defines the covariant derivative of β^μ.

Under a local infinitesimal spin-space rotation

$$S=1+\epsilon^\tau\Omega_\tau,$$

β^μ transforms as

$$\beta'^\mu=S^{-1}\beta^\mu S=\beta^\mu+\epsilon^\tau[\beta^\mu,\Omega_\tau], \tag{3}$$

* Fellow, The Institute of Science, Bombay, India. Deceased August 29, 1953. This research was completed while the author was doing graduate work at Harvard University.

[1] H. J. Bhabha, Revs. Modern Phys. 17, 200 (1945).

[2] H. J. Bhabha, Revs. Modern Phys. 21, 451 (1949).

[3] W. Pauli, Ann. Physik 18, 337 (1933).

[4] **G** is a rational integral function of β^μ and $g^{\mu\nu}$.