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General Relativistic Variational Principle for Perfect Fluids

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The field equations for the gravitational field of a perfect compressible fluid and the equations of motion of the fluid in its own gravitational field are derived from a single variational principle in which the variations of the various field quantities are restricted so that mass is conserved.

1. INTRODUCTION

In general relativity, the gravitational field due to matter is determined by the field equations.

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-\kappa T_{\mu\nu}, \tag{1.1}$$

where the tensor $T_{\mu\nu}$ describes the stress energy tensor of the matter. If the matter is a perfect fluid (i.e., one with no viscosity or heat conductivity), the stress energy tensor is

$$T^{\mu\nu}=\rho^0\lambda u^\mu u^\nu-pg^{\mu\nu}, \tag{1.2}$$

where u^μ are the components of the four-dimensional velocity vector,

$$\lambda=c^2+\epsilon+p/\rho^0, \tag{1.3}$$

$$g_{\mu\nu}u^\mu u^\nu=1, \tag{1.4}$$

ρ^0 is the rest density of the fluid (i.e., the number of particles per unit volume as measured by an observer moving with the mean velocity of these particles) p the pressure, and ϵ is the specific internal energy of the fluid as measured by an observer at rest with respect to the fluid. The equations of motion of the fluid are

$$T^{\mu\nu}{}_{;\nu}=0, \tag{1.5}$$

where the semicolon denotes covariant differentiation.

Equations (1.5) are a consequence of (1.1). However, if the $g_{\mu\nu}$ are known, Eqs. (1.5) together with the equation of conservation of mass, namely,

$$(\rho^0u^\nu)_{;\nu}=0, \tag{1.6}$$

serve to determine the kinematical variables u^σ and two thermodynamic variables ρ^0 and p (or any function of these two) which describe the motion of the fluid in the gravitational field given by the $g_{\mu\nu}$.

It is the purpose of this paper to give a variational principle which uses a Lagrangean which is a function of the hydrodynamic variables u^σ, ρ^0, and T^0, the rest temperature of the fluid, and the gravitational field variables $g_{\mu\nu}$. We shall consider variations of all of these field quantities such that (1.4) and (1.6) are satisfied, and shall show that Eqs. (1.1) and (1.2) are the Euler equations due to the variation of $g_{\mu\nu}$, and that Eqs. (1.5) with $T^{\mu\nu}$ given by (1.2) are the Euler equations due to the variation of the hydrodynamical field variables, namely ρ^0, T^0, and the world lines of elements of the fluid. Thus, from a single variational principle, we shall obtain both the field equations for the gravitational field created by the fluid and the equations of motion of the fluid in this gravitational field.

2. THE LAGRANGEAN

Consider the integral

$$I=\int[R-2\kappa\rho^0(c^2+H^0+\tfrac{1}{2}\mu g_{\mu\nu}u^\mu u^\nu)]\surd(-g)d^4x, \tag{2.1}$$

where R is the scalar curvature formed from the metric tensor $g_{\mu\nu}$, κ is the Einstein gravitational constant, μ is a Lagrange multiplier which must be chosen so that Eq. (1.4) is satisfied, and H^0 is the Helmholtz free energy defined as

$$H^0=\epsilon-T^0S^0, \tag{2.2}$$

where S^0 is the entropy as measured by an observer at rest with respect to the fluid. The integration in Eq. (2.1) is over a volume in space-time swept out by the world lines of an arbitrary member of "particles" of the fluid.

When we consider ρ^0 and T^0 as the independent thermodynamic variables, we have

$$\delta H^0=\left(\frac{\partial\epsilon^0}{\partial\rho^0}-T^0\frac{\partial S^0}{\partial\rho^0}\right)\delta\rho^0+\left(\frac{\partial\epsilon}{\partial T^0}-S^0-T^0\frac{\partial S^0}{\partial T^0}\right)\delta T^0.$$

However, from the equation defining entropy, namely

$$T^0dS^0=d\epsilon^0+pd(1/\rho^0), \tag{2.3}$$

it follows that

$$\delta H^0=(p/\rho^{02})\delta\rho^0-S^0\delta T^0. \tag{2.4}$$

The world lines which are to be varied may be written as

$$x^\mu=x^\mu(u,v,w,s), \tag{2.5}$$

where u, v, w are variables labelling a particular world line and s is the proper time along this world line. It follows from (2.5) that

$$u^\mu=\partial x^\mu/\partial s,$$

and hence the variation in the velocity vector δu^μ produced by a variation δx^μ in the world lines is

$$\delta u^\mu=\partial\delta x^\mu/\partial s. \tag{2.6}$$

When we consider Eqs. (2.5) as a transformation in space time from the coordinates x^μ to the coordinates

u, v, w, s, we may evaluate all the quantities in the integrand of (2.1) as functions of the latter variables and write

$$I=\int[R^*-2\kappa\rho^{0*}(c^2+H^{0*}+\tfrac{1}{2}\mu g_{\mu\nu}{}^*u^{\mu*}u^{\nu*})]\times\sqrt{(-g^*)}dudvdwds, \quad (2.7)$$

where the starred quantities in the brackets are obtained from the unstarred ones by use of the scalar transformation law:

$$f^*(u)=f(x(u)),$$

and the tensor transformation laws. Further, we know that R^* is the same function of the $g_{\mu\nu}{}^*$ as R is of the $g_{\mu\nu}$.

In addition, it is a consequence of the transformation laws that

$$[-g^*(u)]^{\frac{1}{2}}=[-g(x)]^{\frac{1}{2}}\epsilon_{\mu\nu\sigma\tau}\frac{\partial x^\mu}{\partial u}\frac{\partial x^\nu}{\partial v}\frac{\partial x^\sigma}{\partial w}\frac{\partial x^\tau}{\partial s}, \quad (2.8)$$

where $\epsilon_{\mu\nu\sigma\tau}$ is the numerical tensor density which vanishes unless all its indices are different. If this is so it has the value plus one if the indices are an even permutation of 1, 2, 3, 4 and minus one, if the indices are an odd permutation of these numbers.

3. THE CONSERVATION OF MASS

Equation (1.6), the equation which states that mass is conserved may be written as

$$\frac{1}{\sqrt{(-g^*)}}\frac{\partial}{\partial x^{\sigma*}}(\sqrt{(-g^*)}\rho^{0*}u^{\sigma*})=0,$$

where $x^{1*}=u$, $x^{2*}=v$, $x^{3*}=w$, $x^{4*}=s$. Multiplying this equation by $\sqrt{(-g^*)}dudvdwds$ and integrating over a region of space-time swept out by world lines of the particles, we obtain

$$\int\frac{\partial}{\partial x^{\sigma*}}(\sqrt{(-g^*)}\rho^{0*}u^{\sigma*})dudvdwds=0.$$

This equation is equivalent to the statement that the three-dimensional integral,

$$\int\rho^0u^\tau[g(x(u))]^{\frac{1}{2}}\epsilon_{\lambda\mu\nu\sigma\tau}\frac{\partial x^\lambda}{\partial u}\frac{\partial x^\mu}{\partial v}\frac{\partial x^\nu}{\partial w}dudvdw,$$

is independent of s. That is, the requirement of conservation of mass for arbitrary amounts of fluid may be written as the condition

$$\rho^{0*}[-g^*(u)]^{\frac{1}{2}}=\rho^0[-g(x)]^{\frac{1}{2}}\epsilon_{\lambda\mu\nu\tau}\frac{\partial x^\lambda}{\partial u}\frac{\partial x^\mu}{\partial v}\frac{\partial x^\nu}{\partial w}\frac{\partial x^\tau}{\partial s}=M(u,v,w), \quad (3.1)$$

where M is not a function of s.

The variation in proper density $\delta\rho^0$ may be determined in terms of the variation in the metric field and the variation in the particle paths by the requirement that mass be conserved. Thus, from the requirement that $\delta M=0$, it follows from (3.1) that

$$\frac{\delta\rho^0}{\rho^0}+\tfrac{1}{2}g^{\mu\nu}\delta g_{\mu\nu}+\frac{1}{\sqrt{(-g)}}\frac{\partial}{\partial x^\sigma}(\sqrt{(-g)}\delta x^\sigma)=0. \quad (3.2)$$

The derivation of (3.2) follows from (3.1), and the fact that the quantity,

$$\epsilon_{\lambda\mu\nu\tau}\frac{\partial x^\lambda}{\partial u}\frac{\partial x^\mu}{\partial v}\frac{\partial x^\nu}{\partial w}\frac{\partial x^\tau}{\partial s},$$

is the Jacobian of the transformation (2.5). Equation (3.2) may also be written in the u, v, w, s coordinate system as

$$\delta\rho^{0*}/\rho^{0*}+\tfrac{1}{2}g^{\mu\nu*}\delta g_{\mu\nu}{}^*+(\delta x^{\sigma*})_{;\sigma}=0, \quad (3.3)$$

where

$$\delta x^{\sigma*}=\delta x^\tau(\partial x^{*\sigma}/\partial x^\tau), \quad (3.4)$$

and

$$\delta g_{\mu\nu}{}^*=\delta g_{\sigma\tau}\frac{\partial x^\sigma}{\partial x^{\mu*}}\frac{\partial x^\tau}{\partial x^{\nu*}}. \quad (3.5)$$

It should be observed that $\delta g_{\mu\nu}{}^*$ is the variation in the metric tensor in the starred coordinate system. There is a variation in $g_{\mu\nu}{}^*$ due to a variation of the particle paths which may be computed from the definition of $g_{\mu\nu}{}^*$, namely

$$g_{\mu\nu}{}^*=g_{\sigma\tau}\frac{\partial x^\sigma}{\partial x^{\mu*}}\frac{\partial x^\tau}{\partial x^{\nu*}},$$

to be

$$\delta_x g_{\mu\nu}{}^*=(\delta x_\mu{}^*)_{;\nu}+(\delta x_\nu{}^*)_{;\mu}, \quad (3.6)$$

where the covariant derivatives are computed in the starred coordinate system and $\delta x_\mu{}^*$ are the covariant components of $\delta x^{\mu*}$.

For the purposes of the next section we shall need to evaluate

$$\begin{aligned}\delta_x(g_{\mu\nu}{}^*u^{\mu*}u^{\nu*})&=\delta_x(g_{\mu\nu}u^\mu u^\nu)\\&=(\partial g_{\mu\nu}/\partial x^\rho)\delta x^\rho u^\mu u^\nu+2g_{\mu\nu}u^\mu(\partial\delta x^\nu/\partial s)\\&=(\partial g_{\mu\nu}/\partial x^\rho)\delta x^\rho u^\mu u^\nu+2g_{\mu\nu}u^\mu u^\rho(\partial\delta x^\nu/\partial x^\rho)\\&=2g_{\mu\nu}u^\mu(\delta x^\nu)_{;\rho}u^\rho\\&=2g_{\mu\nu}{}^*u^{\mu*}(\delta x^{\nu*})_{;\rho}u^{\rho*}.\end{aligned}$$

4. THE EULER EQUATIONS

In order to compute the variation of I subject to the restrictions (1.4) and (1.6), we use the form of I given by Eq. (2.7). From this equation and the fact that

$\delta M=0$, it follows that

$$\delta I=\int\Big\{(-R^{*\mu\nu}+\tfrac{1}{2}g^{\mu\nu*}R)[\delta g_{\mu\nu}{}^*+(\delta x_\mu{}^*)_{;\nu}+(\delta x_\nu{}^*)_{;\mu}]$$
$$-2\kappa\rho^{0*}\bigg[\bigg(\frac{p}{\rho^{0*2}}\delta\rho^{0*}-S^{0*}\delta T^{0*}\bigg)+\tfrac{1}{2}\mu u^{\mu*}u^{\nu*}\delta g_{\mu\nu}{}^*$$
$$+\mu g_{\mu\nu}{}^*u^{\mu*}u^{\rho*}(\delta x^{\nu*})_{;\rho}\bigg]\Big\}\surd(-g^*)dudvdwds.$$

On using Eq. (3.3) to eliminate $\delta\rho^{0*}$ from this expression, we have

$$-\delta I=\int\{(R^{\mu\nu*}-\tfrac{1}{2}g^{\mu\nu*}R^*+\kappa\theta^{\mu\nu*})[\delta g_{\mu\nu}{}^*+2(\delta x_\mu{}^*)_{;\nu}]$$
$$-2\kappa\rho^{0*}S^{0*}\delta T^{0*}\}\surd(-g^*)dudvdwds, \quad (4.1)$$

where

$$\theta^{\mu\nu*}=\rho^{0*}\mu u^{\mu*}u^{\nu*}-pg^{\mu\nu*}. \quad (4.2)$$

If we now introduce a new variable by the equation

$$\delta T^0=(\delta\alpha^*)_{,\rho}u^{\rho*}=\partial\delta\alpha/\partial s,$$

where the comma denotes the ordinary derivative, Eq. (4.1) may be written after integration by parts as

$$-\delta I=\int[(R^{\mu\nu*}-\tfrac{1}{2}g^{\mu\nu*}R+\kappa\theta^{\mu\nu*})\delta g_{\mu\nu}{}^*-\kappa\theta^{\mu\nu*}{}_{;\nu}\delta x_\mu{}^*$$
$$+2\kappa(\rho^{0*}S^{0*}u^{\sigma*})_{;\sigma}\delta\alpha^*]\surd(-g^*)dudvdwds \quad (4.3)$$

for variations which vanish on the three-dimensional hypersurface bounding the region of integration.

The Euler equations for $\delta I=0$ for variations which vanish on the bounding hypersurface are then

$$R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R+\kappa\theta^{\mu\nu}=0, \quad (4.4)$$

$$\theta^{\mu\nu}{}_{;\nu}=0, \quad (4.5)$$

$$(\rho^0S^0u^\sigma)_{;\sigma}=0, \quad (4.6)$$

where we have written the unstarred quantities since the equations are tensor equations. The last of these equations may be written as

$$\rho^0S^0{}_{;\sigma}u^\sigma=0, \quad (4.6')$$

in view of Eq. (1.6).

Equation (4.5) may be written as

$$\rho^0u^\nu(\mu u^\mu)_{;\nu}=p_{,\nu}g^{\mu\nu}, \quad (4.7)$$

as a consequence of (4.2) and (1.6). The quantity μ entering into this equation must be determined so that (1.4) is satisfied. Multiplying Eq. (4.7) by u_μ and summing, we obtain

$$2\rho^0\mu d(g_{\mu\nu}u^\mu u^\nu)=dp-\rho^0g_{\mu\nu}u^\mu u^\nu d\mu, \quad (4.8)$$

where $df=f_{,\sigma}u^\sigma$.

It follows from Eq. (2.3) that

$$dp/\rho^0=d\epsilon^0+d(p/\rho^0)-T^0dS^0.$$

Hence,

$$2\rho^0\mu d(g_{\mu\nu}u^\mu u^\nu)=\rho^0[d\epsilon^0+d(p/\rho^0)]-\rho^0g_{\mu\nu}u^\mu u^\nu d\mu,$$

where we have used (4.6′). Hence, (1.4) will be satisfied if

$$d\mu=d\epsilon^0+d(p/\rho^0).$$

That is,

$$\mu=c^2+\epsilon^0+p/\rho^0,$$

where the constant of integration has been chosen to be c^2 in order that the stress-energy tensor be correctly given when $p=0$. If this is done, then Eqs. (4.4), (4.2), and (4.5) become (1.1), (1.2), and (1.5), respectively. Equation (4.6′) is then a consequence of Eqs. (1.2) to (1.6), which may be seen as follows: Eq. (4.7) with μ given by (1.3) follows from Eqs. (1.2), (1.5), and (1.6). Multiplying (4.7) by u_μ, summing, and using (1.4), we obtain

$$\rho^0d\mu=dp, \quad (4.9)$$

that is,

$$d\epsilon^0+d(1/\rho^0)=0;$$

but in view of Eq. (2.3) this may be written as (4.6).

If Eq. (4.9) is substituted into (4.7), the latter equation may be written as

$$\rho^0\mu u^\nu u^\mu{}_{,\nu}=p_{,\nu}(g^{\mu\nu}-u^\mu u^\nu). \quad (4.10)$$

Equations (1.6), (4.10), and (4.6) are a set of five linearly independent equations equivalent to the five equations (1.5) and (1.6).

Thus we have shown that the variational principle $\delta I=0$, where the field variables ρ^0, T^0, $g_{\mu\nu}$ and the particle paths are varied so that (1.6) is always satisfied and the Lagrange multiplier is chosen so that (1.4) is satisfied, has as Euler equations the field equations for the gravitational field created by the fluid and the equations of motion of the fluid. This variational principle is an extension of a nonrelativistic one previously given[1] for compressible fluids alone. In the previous work it was shown that the Rankine-Hugoniot equations could be derived from considerations involving the existence of singular surfaces in the fluid and the variations of these surfaces. A derivation of the relativistic Rankine-Hugoniot equations should follow from the variational principle given above.

[1] A. H. Taub, in *Proceedings of First Symposium of Applied Mathematics* (American Mathematical Society, New York, 1949), pp. 148–57.