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# Variational Principles for the Acoustic Field*

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Variational principles are presented for the scattering amplitude in the general acoustic scattering problem, and, for spherically symmetric scatterers, for the phase shifts. Integral equations for the acoustic field are also given and the properties of the scattering matrix are developed. The entire formulation remains valid in the presence of discontinuities of density and/or velocity in the medium.

## I. INTRODUCTION

In recent years variational expressions have been developed for the total cross section and for the phase shifts of the partial waves, in a variety of scattering problems.[1] In many instances these variational expressions have led to accurate estimates of the scattering cross section.[1] To our knowledge, however, the scattering problems for which explicit variational formulations have been developed are confined to those in which the wave function and its normal derivative are both continuous across a surface of discontinuity. Consequently, these formulations are not applicable in acoustic scattering problems, where, using the customary definitions, the acoustic potential is discontinuous across a surface bounding two media of differing constant density.

It seemed worth while, therefore, to derive Schwinger-type[1] variational principles for acoustic scattering problems. We present a variational principle for the total scattering amplitude and, for spherically symmetric scatterers, variational principles for the phase shifts; the derivations of these variational principles are not trivial, as will be seen. We present also a general integral equation formulation of the acoustic scattering problem and determine explicitly the elements of the scattering matrix. These results do not seem to have been previously given and are required for proofs of the variational principles and of the cross-section theorem, Eq. (34), in the following.

Applications of these variational principles are not reported here, but are being considered. It seems likely that techniques similar to those we have used will lead to variational principles applicable to nonabsorptive electromagnetic scattering problems involving arbitrary variations of dielectric constant and magnetic permeability.

## II. THE ACOUSTIC FIELD

We start from the equations

$$\rho\partial\mathbf{v}/\partial t=-\mathrm{grad}p,$$
$$\rho\,\mathrm{div}\mathbf{v}=-(1/c^2)\partial p/\partial t. \tag{1}$$

Equation (1) describes the sound pressure $p$ and the sound velocity $\mathbf{v}$ at all points of space which are free of acoustic sources. The equilibrium density at any point $\mathbf{r}$ is $\rho(\mathbf{r})$ and $c(\mathbf{r})$ is the local velocity of sound. Equation (1) is valid for an ideal fluid, with $p$ and $\mathbf{v}$ small time-dependent increments about $p_0$ and zero, respectively, where $p_0$ is time independent.[2] The boundary conditions, that $p$ and $v_n$, the normal component of $\mathbf{v}$, remain continuous as $\rho$ and/or $c$ approach functions discontinuous across a surface, are implied by Eq. (1). The rate of flow of acoustic energy across the

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[1] B. A. Lippman and Julian Schwinger, Phys. Rev. **79**, 469 and 481 (1950), and references therein.

[2] In the presence of body forces, Eq. (1) is correct only at sufficiently high frequencies. In particular, in the gravitational field of the earth, we must have $\omega\gg g/c$, which is satisfied above a tenth of a cycle in air or water. See P. G. Bergmann, J. Acoust. Soc. Am. **17**, 329 (1946).

surface element $d\mathbf{S}\equiv dS\mathbf{n}$ is $p\mathbf{v}\cdot\mathbf{n}dS$. It is presumed that $p$ and $\mathbf{v}$ are real.

When the fields are harmonic, we write in Eq. (1)

$$p(\mathbf{r},t)=p(\mathbf{r})e^{-i\omega t}+p^*(\mathbf{r})e^{i\omega t},$$
$$\mathbf{v}(\mathbf{r},t)=\mathbf{v}(\mathbf{r})e^{-i\omega t}+\mathbf{v}^*(\mathbf{r})e^{i\omega t}, \tag{2}$$

the asterisk denoting complex conjugate. Then

$$\mathrm{grad}p-i\omega\rho\mathbf{v}=0,$$
$$(i\omega/c^2)p-\rho\,\mathrm{div}\mathbf{v}=0. \tag{3}$$

In Eq. (3) $p\equiv p(\mathbf{r})$ and $\mathbf{v}\equiv\mathbf{v}(\mathbf{r})$ of Eq. (2). The average rate of radiation across the surface element $dS\mathbf{n}$ is $(p^*\mathbf{v}+p\mathbf{v}^*)\cdot\mathbf{n}dS$. If $\rho$ and $c$ are everywhere constant, $\rho=\rho_0$ and $c=c_0$, Eq. (3) yields upon elimination of $\mathbf{v}$

$$\Delta p+k^2p=0,$$
$$\mathbf{v}=(1/i\omega\rho_0)\,\mathrm{grad}p, \tag{4}$$

where

$$k=\omega/c_0.$$

Equation (4) is entirely equivalent to the customary formulation in terms of the acoustic potential:

$$\mathbf{v}=-\mathrm{grad}\varphi,$$
$$p=-i\omega\rho_0\varphi, \tag{5}$$

together with

$$\Delta\varphi+k^2\varphi=0. \tag{6}$$

Equation (5) continues to be useful if $c$ varies, but no longer solves the first of Eq. (3) when $\rho$ is not constant. Nor does replacing the first of Eq. (5) by $\rho\mathbf{v}=-\rho_0\,\mathrm{grad}\varphi$ yield any particular simplification. Moreover, Eq. (5) simply makes $\varphi$ proportional to $p$. It is apparent therefore that for our purposes, developing generally applicable variational principles, there is no advantage to introducing an acoustic potential. Consequently, our results are given in terms of the directly observable quantities $p$ and $\mathbf{v}$.

The typical acoustic scattering problem is to find a solution of Eq. (3) of the form

$$p=p_0+p_s=\exp(ik\mathbf{n}_0\cdot\mathbf{r})+p_s,$$
$$\mathbf{v}=\mathbf{v}_0+\mathbf{v}_s=\frac{\mathbf{n}_0}{\rho_0c_0}\exp(ik\mathbf{n}_0\cdot\mathbf{r})+\mathbf{v}_s, \tag{7}$$

where $p_s$ and $\mathbf{v}_s$ are outgoing at infinity and represent the scattered wave, while $p_0$ and $\mathbf{v}_0$ represent a plane wave incident along $\mathbf{n}_0$ and are solutions of Eq. (4). The quantities $\rho_0$ and $c_0$ are the values of $\rho$ and $c$ at infinity. Of course, $p$ and the normal component of $\mathbf{v}$ are continuous across any surface of discontinuity of $\rho$ and/or $c$.

We now seek to replace Eq. (3) by a pair of integral equations incorporating the conditions (7). For this purpose, we first derive from Eq. (3),

$$\Delta p+k^2p=k^2(1-\mu^2)p+i\omega\mathbf{v}\cdot\mathrm{grad}\rho,$$
$$\Delta\mathbf{v}+k^2\mathbf{v}=k^2(1-\mu^2)\mathbf{v}+i\omega p\,\mathrm{grad}\frac{1}{\rho c^2}-\mathrm{curl}\left(\mathrm{grad}\frac{1}{\rho}\times\rho\mathbf{v}\right), \tag{8}$$

where

$$\mu=c_0/c.$$

Then as shown in Appendix A, using the outgoing free-space Green's function

$$G(\mathbf{r},\mathbf{r}')=-\frac{1}{4\pi}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}, \tag{9}$$

which satisfies

$$(\Delta+k^2)G(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}'), \tag{10}$$

we obtain, whether or not there are surfaces of discontinuity of $\rho$ and/or $c$,

$$p(\mathbf{r})=p_0+k^2\int d\mathbf{r}'G(\mathbf{r},\mathbf{r}')(1-\alpha'\mu'^2)p(\mathbf{r}')+i\omega\,\mathrm{div}\int d\mathbf{r}'G(\mathbf{r},\mathbf{r}')(\rho'-\rho_0)\mathbf{v}(\mathbf{r}'),$$

$$\mathbf{v}(\mathbf{r})=\mathbf{v}_0+k^2\int d\mathbf{r}'G(\mathbf{r},\mathbf{r}')\left(1-\frac{1}{\alpha'}\right)\mathbf{v}(\mathbf{r}') \tag{11}$$
$$-\frac{i\omega}{\rho_0c_0^2}\mathrm{grad}\int d\mathbf{r}'G(\mathbf{r},\mathbf{r}')(1-\alpha'\mu'^2)p(\mathbf{r}')$$
$$-\mathrm{curl\,curl}\int d\mathbf{r}'G(\mathbf{r},\mathbf{r}')\left(1-\frac{1}{\alpha'}\right)\mathbf{v}(\mathbf{r}'),$$

where

$$\alpha=\rho_0/\rho.$$

In Eq. (11) we have introduced the convention, which we shall use consistently, that functional dependence on $\mathbf{r}'$ is indicated by a prime. Thus $\mu'=c_0/c(\mathbf{r}')$ and $\alpha'=\rho_0/\rho(\mathbf{r}')$. As written, Eq. (11) is to be regarded as a pair of simultaneous integral equations for the unknown functions $p$ and $\mathbf{v}$, which functions, as verified in Appendix A, properly satisfy Eq. (3). An apparent simplification is readily obtained if $\mathbf{v}$ is eliminated from the first of Eq. (11) using the first of Eq. (3) since in this case one need consider only a single integral equation for the scalar function $p$. However, this equation explicitly involves grad$p$ and it turns out that in deriving the variational principles we seek, it would still be necessary to construct an expression for grad$p$ equivalent to the second of Eq. (11). Thus no real simplification is attained.

We prove[3] in Appendix B that the solutions to Eq.

[3] In Eq. (11) and in subsequent equations we assume that $\rho$, $c$, $p$, and $\mathbf{v}$ remain finite, and that $\rho$ and $c$ are different from zero, to keep $\alpha$ and $\mu$ finite. If these assumptions are not made, the question of the existence of the integrals in Eq. (11), and therefore of the existence of a solution to Eq. (11), has to be examined. In any

(11) satisfy the requirements of continuity of $p$ and $\mathbf{v}_n$. To establish the behavior of $p$ and $\mathbf{v}$ at infinity, we note that to terms of higher order

$$\lim_{r\to\infty} G(r\mathbf{n},\mathbf{r}')=-\frac{1}{4\pi}\frac{e^{ikr}}{r}\exp(-ik\mathbf{n}\cdot\mathbf{r}'), \tag{12}$$

whence Eq. (11) yields

$$\lim_{r\to\infty} p(r\mathbf{n})=p_0-\frac{k^2}{4\pi}\frac{e^{ikr}}{r}\int d\mathbf{r}'\exp(-ik\mathbf{n}\cdot\mathbf{r}')\times\left[(1-\alpha'\mu'^2)p'+\rho_0c_0\left(1-\frac{1}{\alpha'}\right)\mathbf{n}\cdot\mathbf{v}'\right],$$
$$\lim_{r\to\infty} \mathbf{v}(r\mathbf{n})=\mathbf{v}_0-\frac{1}{4\pi}\frac{k^2}{\rho_0c_0}\frac{e^{ikr}}{r}\mathbf{n}\int d\mathbf{r}'\exp(-ik\mathbf{n}\cdot\mathbf{r}')\times\left[(1-\alpha'\mu'^2)p'+\rho_0c_0\left(1-\frac{1}{\alpha'}\right)\mathbf{n}\cdot\mathbf{v}'\right]. \tag{13}$$

By comparison with Eq. (8), Eq. (13) demonstrates that the solutions $p(\mathbf{r})$, $\mathbf{v}(\mathbf{r})$ of Eq. (11) are such that $p_s$ and $\mathbf{v}_s$ are everywhere outgoing at infinity. This, together with Appendices A and B, proves that Eq. (11) is equivalent to Eq. (3) plus the boundary conditions $p$ and $\mathbf{v}_n$ continuous and $p_s$ and $\mathbf{v}_s$ outgoing at infinity. Incidentally Eq. (13) shows that the scattered field is longitudinal, $\mathbf{v}_s$ parallel to $\mathbf{n}$ at infinity, and that the acoustic impedance for the scattered field is $\rho_0c_0$, $|\mathbf{v}_s|=p_s/\rho_0c_0$ at infinity.

actual problem $\rho$ and $c$ are bounded and different from zero. Thus, since in the scattering problems which we treat there are no sources at a finite distance from the origin, it is most reasonable on physical grounds to expect that $p(\mathbf{r})$ and $\mathbf{v}(\mathbf{r})$ are finite everywhere, although a general proof may be difficult to construct. For the proof of Appendix B it is necessary to assume that various derivatives of $\rho$, $c$, $p$, $\mathbf{v}$, $\alpha$, and $\mu$ remain finite as a surface of discontinuity; is approached from either side of the surface. In other words, except right at the discontinuity, the field variables $p$ and $\mathbf{v}$, and the medium itself, are well behaved. Again these assumptions are reasonable for any actual problem and are made to simplify the argument in Appendix B, which is already complicated enough. If finiteness of the derivatives is not assumed Eq. (11) may remain valid in various circumstances, but another proof that the boundary conditions are satisfied will have to be devised. We stress that we are not assuming the derivatives exist at the discontinuity; this would be an unreasonable assumption. To illustrate, as the everywhere continuous $\rho(\mathbf{r})$ becomes in the limit discontinuous with a finite jump across a surface, grad$\rho$ becomes infinite at the surface, although it remains finite on either side. Because the integrands in Eq. (11) do not involve derivatives the integrals may be and are extended over all space. If derivatives or other expressions which cannot be assumed finite at the discontinuity are included in the integrand, the volume of integration must omit thin strips on either side of the surfaces of discontinuity, generally with the consequent complication of adding compensating surface integrals to the integral equation, as in Eqs. (1B) and (7B). It should be added that in the event that $\rho$ and/or $c$ are idealized to be zero in an extended region, as for example in the scattering of sound from a bubble in water, then $p$ and $\mathbf{v}$ are zero within this region, and a reformulation in terms of surface integrals is necessary.

## III. THE SCATTERING AMPLITUDE. RECIPROCITY

We define $A(\mathbf{n},\mathbf{n}_0)$ as the pressure amplitude at infinity of the outgoing spherical wave proceeding along $\mathbf{n}$, which results from an incident plane wave of unit amplitude along $\mathbf{n}_0$. From Eq. (13),

$$A(\mathbf{n},\mathbf{n}_0)=-\frac{k^2}{4\pi}\int d\mathbf{r}'\exp(-ik\mathbf{n}\cdot\mathbf{r}')\left[(1-\alpha'\mu'^2)p(\mathbf{r}',\mathbf{n}_0)+\rho_0c_0\left(1-\frac{1}{\alpha'}\right)\mathbf{n}\cdot\mathbf{v}(\mathbf{r}',\mathbf{n}_0)\right], \tag{14}$$

where $p(\mathbf{r},\mathbf{n}_0)$ and $\mathbf{v}(\mathbf{r},\mathbf{n}_0)$ represent the fields resulting from an incident unit plane wave along $\mathbf{n}_0$. The average rate of flow of acoustic energy across the surface element $dS\mathbf{n}=r^2d\Omega\mathbf{n}$ at infinity is, from Eqs. (13) and (14) and the definition following Eq. (3),

$$I(\mathbf{n})d\Omega=\frac{2|A(\mathbf{n},\mathbf{n}_0)|^2}{\rho_0c_0}d\Omega. \tag{15}$$

The incident intensity is $2/\rho_0c_0$, so the differential cross section is $\sigma(\mathbf{n})=|A(\mathbf{n},\mathbf{n}_0)|^2$.

Replacing $\mathbf{n}$ and $\mathbf{n}_0$ in Eq. (14) by $-\mathbf{n}_0$ and $-\mathbf{n}$, respectively, we obtain an expression for $A(-\mathbf{n}_0,-\mathbf{n})$ in terms of $p(\mathbf{r},-\mathbf{n})$ and $\mathbf{v}(\mathbf{r},-\mathbf{n})$, where $p(\mathbf{r},-\mathbf{n})$, the pressure resulting from an incident wave along $-\mathbf{n}$, satisfies, according to Eq. (11),

$$p(\mathbf{r},-\mathbf{n})=\exp(-ik\mathbf{n}\cdot\mathbf{r})+k^2\int d\mathbf{r}'G(\mathbf{r},\mathbf{r}')(1-\alpha'\mu'^2)p(\mathbf{r}',-\mathbf{n})-i\omega\rho_0\,\mathrm{div}\int d\mathbf{r}'G(\mathbf{r},\mathbf{r}')\left(1-\frac{1}{\alpha'}\right)\mathbf{v}(\mathbf{r}',-\mathbf{n}). \tag{16}$$

Multiplying Eq. (16) by $(1-\alpha\mu^2)p(\mathbf{r},\mathbf{n}_0)$, and the corresponding equation for $\mathbf{v}(\mathbf{r},-\mathbf{n})$ by $\rho_0c_0(1-\alpha^{-1})\mathbf{v}(\mathbf{r},\mathbf{n}_0)$, adding and integrating, we get, by virtue of Eq. (14), another expression for $A(\mathbf{n},\mathbf{n}_0)$, namely,

$$\begin{aligned}A(\mathbf{n},\mathbf{n}_0)=&\frac{-1}{4\pi}\int d\mathbf{r}[\beta p(\mathbf{n}_0)p(-\mathbf{n})-\omega\rho_0\gamma\mathbf{v}(\mathbf{n}_0)\cdot\mathbf{v}(-\mathbf{n})]\\&-\frac{i}{4\pi}\int d\mathbf{r}\beta p(\mathbf{n}_0)\,\mathrm{div}\int d\mathbf{r}'G\gamma'\mathbf{v}'(-\mathbf{n})\\&+\frac{i}{4\pi}\int d\mathbf{r}\gamma\mathbf{v}(\mathbf{n}_0)\cdot\mathrm{grad}\int d\mathbf{r}'G\beta'p'(-\mathbf{n})\\&+\frac{1}{4\pi}\int d\mathbf{r}\gamma\mathbf{v}(\mathbf{n}_0)\cdot\mathrm{curl\,curl}\int d\mathbf{r}'G\gamma'\mathbf{v}'(-\mathbf{n})\\&+\frac{1}{4\pi}\int d\mathbf{r}d\mathbf{r}'\left[p(\mathbf{n}_0)\beta G\beta'p'(-\mathbf{n})-k^2\gamma G\gamma'\mathbf{v}(\mathbf{n}_0)\cdot\mathbf{v}'(-\mathbf{n})\right],\end{aligned} \tag{17}$$

where we have used $G \equiv G(\mathbf{r},\mathbf{r}')$ and

$$\beta(\mathbf{r}) = k^2(1-\alpha\mu^2),\qquad \gamma(\mathbf{r}) = \omega\rho_0\left(1-\frac{1}{\alpha}\right). \tag{18}$$

From Eq. (17) and the corresponding expression for $A(-\mathbf{n}_0,-\mathbf{n})$ follows the principle of reciprocity,

$$A(\mathbf{n},\mathbf{n}_0) = A(-\mathbf{n}_0,-\mathbf{n}). \tag{19}$$

The proof is given in Appendix C.

## IV. THE SCATTERING MATRIX

In Eq. (11) $p_0$ and $\mathbf{v}_0$ need not be a plane wave but may be any solution of the homogeneous equations, namely Eq. (3) with $\rho=\rho_0$ and $c=c_0$. Presumably the solution to Eq. (11) remains unique for any such choice of $p_0$ and $\mathbf{v}_0$. It follows therefore from Eq. (11) that the amplitudes at infinity of the outgoing spherical waves are wholly determined by the incoming waves. In fact, since $\mathbf{v}_0 = (\text{grad}p_0)/i\omega\rho_0$, $p_s$ and $\mathbf{v}_s$ at infinity are solely determined by the incoming spherical waves in $p_0$. Consequently, since the solutions are linear, we infer[4] the existence of a scattering matrix $S(\mathbf{n},\mathbf{n}')$ defined as follows. At infinity $p_0$ is composed of incoming and outgoing spherical waves. Hence, by Eq. (13), neglecting terms of order $1/r^2$

$$\lim_{r\to\infty} p(r\mathbf{n}) = F_1(\mathbf{n})\frac{e^{ikr}}{r} + F_2(\mathbf{n})\frac{e^{-ikr}}{r}, \tag{20}$$

where

$$F_1(\mathbf{n}) = -\int d\Omega' S(\mathbf{n},\mathbf{n}')F_2(-\mathbf{n}'). \tag{21}$$

$\mathbf{n}'$ is the unit vector along the element of solid angle $d\Omega'$. As will be seen this definition, Eq. (21), of the scattering matrix has the virtue that $S(\mathbf{n},\mathbf{n}')$ reduces to the unit matrix when there is no scattering.

We now derive some general properties of the scattering matrix. Since energy conservation is a ready consequence of Eq. (3), as it must be, we have, for any solution of that equation

$$\int dS\mathbf{n}\cdot(p^*\mathbf{v}+p\mathbf{v}^*) = 0, \tag{22}$$

where the integration extends over the sphere at infinity. Recalling that $\mathbf{v} = (\text{grad}p)/i\omega\rho_0$ at infinity, substitution of Eq. (20) in Eq. (22) yields

$$\frac{2}{\rho_0 c_0}\int d\Omega[|F_1(\mathbf{n})|^2 - |F_2(\mathbf{n})|^2] = 0. \tag{23}$$

[4] Although the assertion that the outgoing waves are determined by the incoming waves is intuitively obvious, a satisfactory proof is not easily obtained. This question is discussed by Friedrichs, Marcuvitz, and John in Sec. III of "Recent developments in the theory of wave propagation," lecture notes, New York University Institute for Mathematics and Mechanics, 1949–1950 (unpublished).

Since $F_2(\mathbf{n})$ may be chosen arbitrarily, Eq. (21) implies

$$\int d\Omega S(\mathbf{n},\mathbf{n}')S^*(\mathbf{n},\mathbf{n}'') = \delta(\mathbf{n}'-\mathbf{n}''). \tag{24}$$

Next we observe that Eq. (3) is invariant when its complex conjugate is taken and $\mathbf{v}$ is simultaneously replaced by $-\mathbf{v}$. That is, if $p$ and $\mathbf{v}$ are any solutions of Eq. (3), then so are $p^*$ and $-\mathbf{v}^*$. Thus from Eq. (20) we infer the behavior of $p^*$ at infinity, namely,

$$\lim_{r\to\infty} p^*(r\mathbf{n}) = F_2^*(\mathbf{n})\frac{e^{ikr}}{r} + F_1^*(\mathbf{n})\frac{e^{-ikr}}{r}, \tag{25}$$

whence, from the definition (21) of the scattering matrix,

$$F_2^*(\mathbf{n}) = -\int d\Omega' S(\mathbf{n},\mathbf{n}')F_1^*(-\mathbf{n}'). \tag{26}$$

Substitution of the conjugate of Eq. (26) into Eq. (21) then yields, after rearrangement of the dummy variables,

$$\int d\Omega S(-\mathbf{n}',-\mathbf{n})S^*(\mathbf{n},\mathbf{n}'') = \delta(\mathbf{n}'-\mathbf{n}''). \tag{27}$$

Comparison of Eqs. (24) and (27) implies, as can be shown by multiplying Eq. (27) by $S^*(-\mathbf{n}',-\mathbf{n}''')$, and integrating over $\mathbf{n}'$, that

$$S(-\mathbf{n}',-\mathbf{n}) = S(\mathbf{n},\mathbf{n}'), \tag{28}$$

a result which can be regarded as an extension of the reciprocity relation (19). Equations (27) and (28) yield

$$\int d\Omega S(\mathbf{n}',\mathbf{n})S^*(\mathbf{n}'',\mathbf{n}) = \delta(\mathbf{n}'-\mathbf{n}''), \tag{29}$$

which, together with Eq. (24), expresses the fact that $S$ is unitary.[5]

Finally, we relate the scattering amplitude $A(\mathbf{n},\mathbf{n}_0)$, which is defined only for plane wave excitation, to the general scattering matrix. For plane wave excitation, the field at infinity has the form

$$\begin{aligned}\lim_{r\to\infty} p(r\mathbf{n}) &= \exp(ik\mathbf{n}_0\cdot r\mathbf{n}) + A(\mathbf{n},\mathbf{n}_0)\frac{e^{ikr}}{r}\\ &= \frac{2\pi i}{k}\sum_{l,m}(-1)^l Y_l^m(\mathbf{n}_0)Y_l^{m*}(\mathbf{n})\frac{e^{-ikr}}{r}\\ &\quad + \left[A(\mathbf{n},\mathbf{n}_0) - \frac{2\pi i}{k}\sum_{l,m} Y_l^m(\mathbf{n}_0)Y_l^{m*}(\mathbf{n})\right]\frac{e^{ikr}}{r},\end{aligned} \tag{30}$$

where the $Y_l^m$ are normalized spherical harmonics. Since these form a complete orthonormal set, we thus

[5] As in reference 1, for a spherically symmetric scatter $S$ is expressible in terms of the usual phase shifts and is diagonal in a representation in which the spherical harmonics are the basis vectors.

have

$$\lim_{r\to\infty} p = \frac{2\pi i}{k}\delta(\mathbf{n}_0+\mathbf{n})\frac{e^{-ikr}}{r} + \left[A(\mathbf{n},\mathbf{n}_0) - \frac{2\pi i}{k}\delta(\mathbf{n}_0-\mathbf{n})\right]\frac{e^{ikr}}{r}. \quad (31)$$

If we regard Eq. (31) as that special case of Eq. (20) for which

$$\begin{aligned} F_1(\mathbf{n}) &= A(\mathbf{n},\mathbf{n}_0) - (2\pi i/k)\delta(\mathbf{n}_0-\mathbf{n}), \\ F_2(\mathbf{n}) &= (2\pi i/k)\delta(\mathbf{n}_0+\mathbf{n}), \end{aligned} \quad (32)$$

Equation (21) yields at once

$$A(\mathbf{n},\mathbf{n}_0) = (2\pi i/k)\delta(\mathbf{n}_0-\mathbf{n}) - (2\pi i/k)S(\mathbf{n},\mathbf{n}_0), \quad (33)$$

which is the desired relation.[6] Equation (33) and the unitary character of $S$ lead directly to the cross-section theorem[1]

$$\sigma = \int d\Omega |A(\mathbf{n},\mathbf{n}_0)|^2 = \frac{4\pi}{k}\,\mathrm{Im}A(\mathbf{n}_0,\mathbf{n}_0). \quad (34)$$

## V. VARIATIONAL PRINCIPLE FOR $A(n,n_0)$

Suppose $A_1$, $A_2$, and $A_3$ are three functions of pressure and velocity which are exactly equal for correctly chosen pressure and velocity, but are not necessarily equal when the pressure and velocity are varied about their correct values. Then for correct $p$, $\mathbf{v}$

$$A_1 = A_2 = A_3 = B = A_1A_2/A_3, \quad (35)$$

and

$$\begin{aligned} \delta B &= (A_1/A_3)\delta A_2 + (A_2/A_3)\delta A_1 - (A_1A_2/A_3^2)\delta A_3 \\ &= \delta A_2 + \delta A_1 - \delta A_3. \end{aligned} \quad (36)$$

From Eq. (14) introduce

$$A(-\mathbf{n}_0,-\mathbf{n}) = \frac{-k^2}{4\pi}\int d\mathbf{r} \times \exp(ik\mathbf{n}_0\cdot\mathbf{r})\left[(1-\alpha\mu^2)p(\mathbf{r},-\mathbf{n}) - \rho_0c_0\left(1-\frac{1}{\alpha}\right)\mathbf{n}_0\cdot\mathbf{v}(\mathbf{r},-\mathbf{n})\right]. \quad (37)$$

Denote $A(\mathbf{n},\mathbf{n}_0)$ from Eq. (14) by $A_1$, $A(-\mathbf{n}_0,-\mathbf{n})$ from Eq. (37) by $A_2$, and $A(\mathbf{n},\mathbf{n}_0)$ from Eq. (17) by $A_3$. Then, as shown in Appendix D, B defined by Eq. (35) is stationary for arbitrary variations $\delta p(\mathbf{r},\mathbf{n}_0)$, $\delta\mathbf{v}(\mathbf{r},\mathbf{n}_0)$, $\delta p(\mathbf{r},-\mathbf{n})$, $\delta\mathbf{v}(\mathbf{r},-\mathbf{n})$. We mention specifically that these variations need not satisfy the boundary conditions, although of course they must be so chosen that the integrals converge.

Since the right side of Eq. (35) is stationary, Eq. (36) shows $B' = A_1 + A_2 - A_3$ is also stationary. If we define $A_4 = A(-\mathbf{n}_0,-\mathbf{n})$ of Eq. (17) then by symmetry the expressions

$$\begin{aligned} B &= A_1A_2/A_4, \\ B' &= A_1 + A_2 - A_4, \end{aligned} \quad (38)$$

are stationary for arbitrary variations $\delta p(\mathbf{r},\mathbf{n}_0)$, $\delta\mathbf{v}(\mathbf{r},\mathbf{n}_0)$, $\delta p(\mathbf{r},-\mathbf{n})$, $\delta\mathbf{v}(\mathbf{r},-\mathbf{n})$.

Since $B$ and $B'$ are stationary for arbitrary variations they are stationary for the special choice

$$\begin{aligned} \delta\mathbf{v}(\mathbf{r},\mathbf{n}_0) &= (1/i\omega\rho)\,\mathrm{grad}\delta p(\mathbf{r},\mathbf{n}_0), \\ \delta\mathbf{v}(\mathbf{r},-\mathbf{n}) &= (1/i\omega\rho)\,\mathrm{grad}\delta p(\mathbf{r},-\mathbf{n}). \end{aligned} \quad (39)$$

Hence, comparing with Eq. (3), the above expressions for $B$ and $B'$ are stationary for arbitrary $\delta p(\mathbf{r},\mathbf{n}_0)$ and $\delta p(\mathbf{r},-\mathbf{n})$ when, in $B$ and $B'$, $\mathbf{v}$ is everywhere replaced by $(\mathrm{grad}p)/i\omega\rho$. Similarly the expressions are stationary for arbitrary $\delta\mathbf{v}(\mathbf{r},\mathbf{n}_0)$ and $\delta\mathbf{v}(\mathbf{r},-\mathbf{n})$ when $p$ is everywhere replaced by $(\rho c^2\,\mathrm{div}\mathbf{v})/i\omega$.

## VI. ABSTRACT FORMULATION OF THE VARIATIONAL PRINCIPLE

The difficulties involved in obtaining a variational principle for $A(\mathbf{n},\mathbf{n}_0)$ are more readily recognized when the problem is given a more general abstract formulation,[4] with a condensed notation. Suppose

$$Kx = z, \quad (40)$$

where $K$ is a given operator, $z$ is known, and $x$ is unknown. Suppose further that we are interested in determining

$$A = z'^T Lx, \quad (41)$$

where $L$ is a given operator and $z'$ is known. If $K$ and $L$ are matrices, then $x$ and $z'$ are single column matrices, and $z'^T$, the transpose of $z'$, is a single row matrix. If now we can determine operators $K'$ and $L'$ such that

$$K'L = L'K, \quad (42)$$

and for which there exists a solution $y$, not necessarily known, to the equation

$$y^TK' = z'^T, \quad (43)$$

then it is easily verified, by virtue of Eqs. (40)–(43), that

$$A = y^TK'Lx = y^TL'Kx = y^TL'z, \quad (44)$$

and moreover that either of the expressions

$$\begin{aligned} B' &= y^TL'z + z'^TLx - y^TK'Lx, \\ B' &= y^TL'z + z'^TLx - y^TL'Kx, \end{aligned} \quad (45)$$

is stationary for arbitrary $\delta x$ and $\delta y^T$. We note that Eqs. (40), (43), and (44) alone, without Eq. (42), are not sufficient to prove $B'$ of Eq. (45) stationary. It is sufficient to replace Eq. (42) by the requirements

$$\begin{aligned} K'Lx &= L'Kx, \\ y^TL'K &= y^TK'L. \end{aligned} \quad (46)$$

[6] With Eq. (19), Eq. (33) provides an alternative proof of Eq. (28).

When $\rho'=\rho_0$ in Eq. (11) derivatives of the Green's function do not appear in the integral equation for $p(\mathbf{r})$, and the problem reduces to the type which has been treated previously.[1] In this event in Eq. (40) we identify $x$ with $p(\mathbf{r},\mathbf{n}_0)$, $z$ with $p_0(\mathbf{n}_0)$. Also, in conformity with matrix notation,

$$K=1-GM, \tag{47}$$

where $\mathbf{r}$ and $\mathbf{r}'$ index, respectively, the rows and columns of the continuous matrix $G(\mathbf{r},\mathbf{r}')$, $1\equiv\delta(\mathbf{r}-\mathbf{r}')$ is the unit matrix, and from Eq. (11), $M$ is the diagonal matrix

$$M(\mathbf{r},\mathbf{r}')=k^2(1-\mu^2)\delta(\mathbf{r}-\mathbf{r}'), \tag{48}$$

if we remember that we are presently assuming $\alpha=1$. We then identify $A(\mathbf{n},\mathbf{n}_0)$ with $A$ of Eq. (41). Because of Eq. (12) and the way in which $A(\mathbf{n},\mathbf{n}_0)$ is defined, we see that $z'$ of Eq. (41) must be identified with $p_0(-\mathbf{n})$, and $L\equiv -M/4\pi$. The only equation involving $p_0(-\mathbf{n})$ which is immediately at hand is, again by Eq. (11),

$$(1-GM)p(-\mathbf{n})=p_0(-\mathbf{n}). \tag{49}$$

Hence, in Eq. (43),

$$K'=(1-GM)^T=1-M^TG^T=1-MG, \tag{50}$$

since $M$ and $G$ are both symmetric. Consequently $L'=L=-M/4\pi$ satisfies Eq. (42). The resultant variational principle, corresponding to Eq. (45), is the usual one.[1]

We observe that when derivatives of the Green's function do not appear in the integral equation, i.e., when $p$ and $(\text{grad}p)_n$ are continuous, the variational principle is a necessary and natural consequence of the simple relationship between the form of $K$, Eq. (47), and the expression for $A(\mathbf{n},\mathbf{n}_0)$, which makes $L=-M/4\pi$. This simple relationship no longer obtains in the more general case when $\rho$ is not constant. In the first place there are now four independent quantities in Eq. (11), $p(\mathbf{r})$ and the three components of $\mathbf{v}(\mathbf{r})$, corresponding to the elements $x_i$, $i=1$ to 4, in Eq. (40). Consequently, $K$ in Eq. (40) is a supermatrix of continuous submatrices $K_{ij}(\mathbf{r},\mathbf{r}')$, $i$, $j=1$ to 4. Further, since derivative terms enter, off diagonal elements of the continuous submatrices enter in a complicated way. Moreover, because of these derivative terms, alternative expressions are possible, by virtue of partial integrations, for $K$ and for $A(\mathbf{n},\mathbf{n}_0)$. However, the integral equations (11) and Eq. (14) appear to give the only reasonably simply obtained formulation of the problem in terms of volume integrals which extend over all space and assuredly remain well defined even when discontinuities of $\rho$ and/or $c$ exist, thereby obviating any necessity for the inclusion of surface integrals. Thus it is interesting to note that this formulation is also the only one from which we have been able to derive a variational principle for $A(\mathbf{n},\mathbf{n}_0)$, even when surfaces of discontinuity are in fact absent. For example, when all functions and their derivatives are well behaved, derivatives of $G(\mathbf{r},\mathbf{r}')$ may be avoided by using Eqs. (1A) and (4A), yielding the integral equation

$$p(\mathbf{r},\mathbf{n}_0)-\int d\mathbf{r}'G(\mathbf{r},\mathbf{r}')[k^2(1-\mu'^2)p'(\mathbf{n}_0)+i\omega\mathbf{v}'(\mathbf{n}_0)\cdot\text{grad}'(\rho'-\rho_0)]=\exp(ik\mathbf{n}_0\cdot\mathbf{r}), \tag{51}$$

for $p(\mathbf{n}_0)$, with corresponding equations for $\mathbf{v}(\mathbf{n}_0)$, $p(-\mathbf{n})$, and $\mathbf{v}(-\mathbf{n})$ and, for $A(\mathbf{n},\mathbf{n}_0)$, the equation

$$A(\mathbf{n},\mathbf{n}_0)=-\frac{1}{4\pi}\int d\mathbf{r}'\exp(-ik\mathbf{n}\cdot\mathbf{r}')[k^2(1-\mu'^2)p'(\mathbf{n}_0)+i\omega\mathbf{v}'(\mathbf{n}_0)\cdot\text{grad}'(\rho'-\rho_0)], \tag{52}$$

with a corresponding equation for $A(-\mathbf{n}_0,-\mathbf{n})$. However, Eqs. (51) and (52), which correspond to Eqs. (40) and (41), do not lead to a variational principle for $A(\mathbf{n},\mathbf{n}_0)$, as is readily verified. This does not mean that a variational principle based on Eqs. (51) and (52) does not exist, but demonstrates that the obvious equations involving $\exp(-ik\mathbf{n}\cdot\mathbf{r})$, which corresponds to $z'$ of Eq. (41), do not satisfy Eq. (42) or Eq. (46). Similar remarks pertain to the other combinations of integral equations and expressions for $A(\mathbf{n},\mathbf{n}_0)$ which we have tried,[7] excepting of course the combination of Eqs. (11) and (14). Evidently it would be most helpful to have a deeper comprehension of the kinds of formulations of the scattering problem which can yield variational principles for $A(\mathbf{n},\mathbf{n}_0)$.

We add that the two distinct variational principles obtained from Eqs. (35) and (38) correspond to the pair of stationary expressions of Eq. (45), and that the content of Appendix D is essentially the demonstration of Eq. (42).

## VII. SPHERICALLY SYMMETRIC SCATTERER

When $\rho$ and $c$ are spherically symmetric, the pressure $p$ of Eq. (8) can be written in the form

$$p(\mathbf{r})=\sum_l R_l(r)P_l(\cos\theta), \tag{53}$$

where $P_l$ are the Legendre polynomials and $\theta$ is the angle between $\mathbf{r}$ and $\mathbf{n}_0$. By Eqs. (3) and (8) $R_l$ satisfies

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{dR_l}{dr}-\frac{l(l+1)}{r^2}R_l+k^2R_l=k^2(1-\mu^2)R_l+\frac{1}{\rho}\frac{d\rho}{dr}\frac{dR_l}{dr}. \tag{54}$$

In Eq. (54) we have used $\rho=\rho(r)$, $c=c(r)$.

[7] Nor have we succeeded in finding any different variational principles when the condition that the variations be unrestricted is relaxed. More precisely, the restrictions on the variations turn out to be complicated and difficult to interpret or achieve, at least in the cases we have examined.

In Appendix E we derive an integral equation for $R_l(r)$, namely

$$R_l(r)=i^l(2l+1)j_l(kr)+\frac{1}{r}\int_0^\infty dr'r'G_l\left\{\left[(1-\alpha')\frac{l(l+1)}{r'^2}-k^2(1-\alpha'\mu'^2)\right]R_l'-(1-\alpha')\frac{\dot{R}_l'}{r'}\right\}+\frac{1}{r}\int_0^\infty dr'r'(1-\alpha')\dot{R}_l'\frac{dG_l}{dr'}. \quad (55)$$

In Eq. (55) the dot signifies the derivative, the prime continuing to denote functional dependence on $r'$. Thus $\dot{R}_l=dR_l/dr$, $\dot{R}_l'=\dot{R}_l(r')$. $G_l(r,r')$ is defined by

$$\begin{aligned}G_l(r,r')&=ikrr'j_l(kr)h_l^{(1)}(kr'), \quad r<r',\\ G_l(r,r')&=ikrr'j_l(kr')h_l^{(1)}(kr), \quad r'<r,\end{aligned} \quad (56)$$

where $j_l$ and $h_l^{(1)}$ are spherical Bessel functions.[8] Equation (55) is valid whether or not there are discontinuities in $\rho$ and/or $c$. As in Eq. (11), derivatives of $\rho(r)$ and $c(r)$ have been eliminated in Eq. (55). As explained in Sec. II, $\rho$ and $\mathbf{v}$ are presumed finite, so that Eq. (53) and the first of Eq. (3) imply $\dot{R}_l$ is finite. The integrals in Eq. (55) extend over all values of $r'$ from 0 to $\infty$, with the understanding that $\alpha'$, $\mu'$, $R_l'$, and $\dot{R}_l'$ may be assigned arbitrary finite values at discontinuities of $\rho'$ and/or $c'$. The boundary conditions on $R_l$ are inferred from the boundary conditions on $p$ and $v_n$, using Eqs. (3) and (8), and remembering that in the present case the surfaces of discontinuity are spheres centered at the origin. We conclude that $R_l$ and $\dot{R}_l/\rho$ are continuous at the discontinuities $r=r_d$, and that except for its incoming part, obtained from the expansion of $p_0$ in Eq. (8), $R_l$ is outgoing at infinity.[9]

From Eqs. (55) and (65) and the asymptotic forms[8] of $j_l(kr)$ and $h_l^{(1)}(kr)$ we find that for large $r$

$$R_l(r)\sim\frac{i^l(2l+1)e^{i\delta_l}}{kr}\sin\left(kr-\frac{l\pi}{2}+\delta_l\right), \quad (57)$$

where

$$\frac{i^l(2l+1)e^{i\delta_l}\sin\delta_l}{k}=X_l=\int_0^\infty dr'r'^2\times\left[\sigma'R_l'j_l(kr')+\tau'\dot{R}_l'\frac{d}{dr'}j_l(kr')\right], \quad (58)$$

and we use the abbreviations

$$\sigma=\frac{l(l+1)(1-\alpha)}{r^2}-k^2(1-\alpha\mu^2), \qquad \tau=1-\alpha. \quad (59)$$

As usual,[10] by virtue of Eqs. (8), (53), (57), and the definition of $A(\mathbf{n},\mathbf{n}_0)$,

$$A(\mathbf{n},\mathbf{n}_0)=k^{-1}\sum_l(2l+1)e^{i\delta_l}\sin\delta_lP_l(\cos\theta)=\sum_l(-i)^lX_lP_l(\cos\theta). \quad (60)$$

In order to eliminate the inconsequential imaginary part of $R_l$, we introduce

$$S_l(r)=\frac{R_l(r)}{i^l(2l+1)e^{i\delta_l}\cos\delta_l}. \quad (61)$$

Then Eq. (57) shows that for large $r$,

$$S_l(r)\sim\frac{\sin(kr-\frac{1}{2}l\pi)}{kr}+\tan\delta_l\frac{\cos(kr-\frac{1}{2}l\pi)}{kr}; \quad (62)$$

and, from Eqs. (58) and (61),

$$\frac{1}{k}\tan\delta_l=\int_0^\infty drr^2\left[\sigma S_lj_l(kr)+\tau\dot{S}_l\frac{d}{dr}j_l(kr)\right]. \quad (63)$$

$S_l(r)$ satisfies Eq. (54) and the same boundary conditions at a discontinuity $r=r_d$ as does $R_l(r)$, since it is merely proportional to $R_l(r)$.

Define $\bar{G}_l(r,r)$ by

$$\begin{aligned}\bar{G}_l(r,r')&=-krr'j_l(kr)n_l(kr'), \quad r<r';\\ \bar{G}_l(r,r')&=-krr'j_l(kr')n_l(kr), \quad r'<r.\end{aligned} \quad (64)$$

$\bar{G}_l(r,r')$ satisfies the same differential Eq. (2E) as does $G_l(r,r')$, but behaves like $[\cos(kr-\frac{1}{2}l\pi)]/kr$ as $r$ approaches infinity, i.e., like the second or scattering term in Eq. (62). Consequently, just as in Appendix *E*, $S_l(r)$ can be shown to obey the integral equation,

$$S_l(r)=j_l(kr)+\frac{1}{r}\int_0^\infty dr'r'\bar{G}_l\left[\sigma'S_l'-\tau'\frac{\dot{S}_l'}{r'}\right]+\frac{1}{r}\int_0^\infty dr'r'\tau'\dot{S}_l'\frac{d\bar{G}_l}{dr'}. \quad (65)$$

Equation (65) also can be demonstrated directly from Eq. (55).

[8] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), pp. 404–406.

[9] Since in Appendix E Eq. (55) is shown to follow directly from Eq. (54), we expect, and can prove without much difficulty, that the solutions to Eq. (55) do in fact satisfy Eq. (54). A proof directly from Eq. (55) that its solutions satisfy the boundary conditions can be furnished along the lines of Appendix F. However neither of these proofs is necessary here in view of the proportionality between $R_l(r)$ and the function $S_l(r)$ which we define immediately below, and for which we do give proofs.

[10] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 105.

Differentiating Eq. (65), we obtain

$$\dot{S}_l(r)=\frac{d}{dr}j_l(kr)-\frac{1}{r^2}\int_0^\infty dr'\bar{G}_l[r'\sigma'S_l'-\tau'\dot{S}_l']+\frac{1}{r}\frac{d}{dr}\int_0^\infty dr'\bar{G}_l[r'\sigma'S_l'-\tau'\dot{S}_l']-\frac{1}{r^2}\int_0^\infty dr'r'\tau'\dot{S}_l'\frac{d\bar{G}_l}{dr'}+\frac{1}{r}\frac{d}{dr}\int_0^\infty dr'r'\tau'\dot{S}_l'\frac{d\bar{G}_l}{dr'}. \quad (66)$$

Equations (65) and (66) may now be regarded as a pair of coupled integral equations in the independent variables $S_l(r)$, $\dot{S}_l(r)$. As is proved in Appendix F, the solutions to Eqs. (65) and (66), with $S_l(r)$ and $\dot{S}_l(r)$ regarded as independent, satisfy the requisite boundary conditions and

$$\dot{S}_l(r)=\frac{d}{dr}S_l(r), \quad (67)$$

$$\frac{d\dot{S}_l}{dr}+\frac{2}{r}\dot{S}_l-\frac{l(l+1)}{r^2}S_l+k^2S_l=k^2(1-\mu^2)S_l+\frac{1}{\rho}\dot{\rho}\dot{S}_l. \quad (68)$$

Equation (68) implies $S_l$ satisfies the original differential Eq. (54). Therefore Eqs. (65) and (66), with $S_l$ and $\dot{S}_l$ regarded as independent quantities, solve the scattering problem for the $l$th partial wave. We may therefore infer also that $S_l(r)$ of Eq. (65), with $\dot{S}_l$ defined as the derivative of $S_l$, satisfies Eq. (54) and the boundary conditions, and similarly for $R_l(r)$ of Eq. (55). Evidently Eqs. (65) and (66) are the analogs, for the $l$th partial wave, of Eq. (11) which held for the total fields $p(\mathbf{r})$ and $\mathbf{v}(\mathbf{r})$.

We are now finally in a position to formulate the variational principle which we have been seeking. Denote the right side of Eq. (63) by $A_{1l}$. Define

$$A_{2l}=\int_0^\infty drr^2\left[\sigma S_l^Tj_l(kr)+\tau\dot{S}_l^T\frac{d}{dr}j_l(kr)\right], \quad (69)$$

where $S_l^T$ and $\dot{S}_l^T$ are independent functions whose variations are independent of the variations of $S_l$ and $\dot{S}_l$, but such that the unvaried (correct) values of $S_l^T$ and $\dot{S}_l^T$ satisfy the same Eqs. (65) and (67) as do $S_l$ and $\dot{S}_l$. In other words, for $\delta S_l=\delta S_l^T=\delta\dot{S}_l=\delta\dot{S}_l^T=0$, $S_l=S_l^T$, $\dot{S}_l=\dot{S}_l^T$, $A_{1l}=A_{2l}$. From Eqs. (63), (65), and (66), and the corresponding equations for $S_l^T$, $\dot{S}_l^T$, we obtain directly[11]

$$\frac{1}{k}\tan\delta_l=A_{3l}=\int_0^\infty drr^2(\sigma S_l^TS_l+\tau\dot{S}_l^T\dot{S}_l)-\int_0^\infty drr\sigma S_l^T\int_0^\infty dr'\bar{G}_l[r'\sigma'S_l'-\tau'\dot{S}_l']-\int_0^\infty drr\sigma S_l^T\int_0^\infty dr'r'\tau'\dot{S}_l'\frac{d\bar{G}_l}{dr'}+\int_0^\infty dr\tau\dot{S}_l^T\int_0^\infty dr'\bar{G}_l[r'\sigma'S_l'-\tau'\dot{S}_l']-\int_0^\infty drr\tau\dot{S}_l^T\frac{d}{dr}\int_0^\infty dr'\bar{G}_l[r'\sigma'S_l'-\tau'\dot{S}_l']+\int_0^\infty dr\tau\dot{S}_l^T\int_0^\infty dr'r'\tau'\dot{S}_l'\frac{d\bar{G}_l}{dr'}-\int_0^\infty drr\tau\dot{S}_l^T\frac{d}{dr}\int_0^\infty dr'r'\tau'\dot{S}_l'\frac{d\bar{G}_l}{dr'}. \quad (70)$$

Moreover the quantities

$$B_l=A_{1l}A_{2l}/A_{3l},\qquad B_l'=A_{1l}+A_{2l}-A_{3l} \quad (71)$$

are stationary for arbitrary variations $\delta S_l$, $\delta S_l^T$, $\delta\dot{S}_l$, $\delta\dot{S}_l^T$. These variations need not satisfy the boundary conditions. By symmetry, $S_l$, $\dot{S}_l$ and $S_l^T$, $\dot{S}_l^T$ can be interchanged in Eq. (70), yielding in Eq. (71) the second, though equivalent, variational principle we have learned to expect from Eq. (45) and the discussion in Sec. VI.

The proof that $B_l'$ in Eq. (71) is stationary is given in Appendix G. Since it is stationary for arbitrary variations it is stationary for $\delta S_l=\delta S_l^T$ and $\delta\dot{S}_l=\delta\dot{S}_l^T$, as well as for $\delta\dot{S}_l=d(\delta S_l)/dr$. That Eqs. (65) and (66) extend over the entire interval $0\leq r'<\infty$, without endpoint sums over $r'=r_d$, whether or not there are discontinuities, has already been remarked. As in the preceding section, we have not been able to construct variational principles from various alternative equations [to Eqs. (63), (65), and (66) for $\tan\delta_l$, $S_l$ and/or $\dot{S}_l$] which can be shown to hold even when discontinuities are absent.

## APPENDIX A

The simplest way to derive Eq. (11) is to assume first that there are no discontinuities in $\rho$ and/or $c$. Let $p=p_0+p_s$ on the left side of Eq. (8) and, for convenience in the manipulations which follow, replace grad$\rho$

[11] We introduce this seemingly awkward distinction between $S_l$ and $S_l^T$ merely to emphasize that in applications of the variational principle it is legitimate to use different trial functions for $S_l$ and $S_l^T$, a procedure which might be useful in obtaining tractable integrals.

by grad$(\rho-\rho_0)$ on the right side of Eq. (8). Then

$$p_s(\mathbf{r})=\int d\mathbf{r}'G(\mathbf{r},\mathbf{r}')\times[k^2(1-\mu'^2)p'+i\omega\mathbf{v}'\cdot\text{grad}'(\rho'-\rho_0)],\quad (1\text{A})$$

as may be verified by direct application of $(\Delta+k^2)$ to $p_s$, by using Eq. (10). In Eq. (1A)

$$G(\mathbf{r},\mathbf{r}')\mathbf{v}'\cdot\text{grad}'(\rho'-\rho_0)=\text{div}'G(\mathbf{r},\mathbf{r}')(\rho'-\rho_0)\mathbf{v}'-(\rho'-\rho_0)\mathbf{v}'\cdot\text{grad}'G(\mathbf{r},\mathbf{r}')-\frac{i\omega}{c'^2}G(\mathbf{r},\mathbf{r}')(1-\alpha')p',\quad (2\text{A})$$

by use of Eq. (3). The integral of $\text{div}'G(\rho'-\rho_0)\mathbf{v}'$ vanishes since $\rho'-\rho_0$ is zero at infinity. The other terms yield

$$p_s(\mathbf{r})=k^2\int d\mathbf{r}'G(\mathbf{r},\mathbf{r}')(1-\alpha'\mu'^2)p(\mathbf{r}')-i\omega\int d\mathbf{r}'(\rho'-\rho_0)\mathbf{v}'\cdot\text{grad}'G(\mathbf{r},\mathbf{r}').\quad (3\text{A})$$

By noting $\text{grad}'G=-\text{grad}G$, Eq. (3A) and Eq. (7) are seen to be identical with Eq. (11) for $p(\mathbf{r})$.

Similarly, from Eq. (8)

$$\mathbf{v}_s(\mathbf{r})=\int d\mathbf{r}'G(\mathbf{r},\mathbf{r}')\left[k^2(1-\mu'^2)\mathbf{v}'+i\omega p'\,\text{grad}'\left(\frac{1}{\rho'c'^2}-\frac{1}{\rho_0c_0^2}\right)-\text{curl}'\left\{\text{grad}'\left(\frac{1}{\rho'}-\frac{1}{\rho_0}\right)\times\rho'\mathbf{v}'\right\}\right].\quad (4\text{A})$$

The derivative terms in Eq. (4A) can be in effect integrated by parts by manipulations similar to those of Eq. (2A), remembering always that, because the surface integrals at infinity vanish, any integral of a pure divergence, a pure curl, or a pure gradient, is zero. Thus, using Eq. (3) to eliminate $\text{grad}'p'$, Eq. (4A) becomes

$$\mathbf{v}_s(\mathbf{r})=k^2\int d\mathbf{r}'G(\mathbf{r},\mathbf{r}')\left(1-\frac{1}{\alpha'}\right)\mathbf{v}'-\frac{i\omega}{\rho_0c_0^2}\text{grad}\int d\mathbf{r}'G(\mathbf{r},\mathbf{r}')(1-\alpha'\mu'^2)p'-\text{curl}\int d\mathbf{r}'G(\mathbf{r},\mathbf{r}')\,\text{grad}'\left(\frac{1}{\rho'}-\frac{1}{\rho_0}\right)\times\rho'\mathbf{v}'.\quad (5\text{A})$$

Because of Eq. (3), $\text{curl}\rho\mathbf{v}=0$, so that, in Eq. (5A),

$$\text{grad}'\left(\frac{1}{\rho'}-\frac{1}{\rho_0}\right)\times\rho'\mathbf{v}'=\text{curl}'\left(\frac{1}{\rho'}-\frac{1}{\rho_0}\right)\rho'\mathbf{v}'.\quad (6\text{A})$$

With the aid of Eq. (6A), and of course another integration by parts, Eq. (5A) becomes identical with Eq. (11) for $\mathbf{v}(\mathbf{r})$.

The integrals in Eq. (11) do not involve derivatives of $\rho'$ and/or $c'$. Hence Eq. (11) remains valid as the continuous functions $\rho(\mathbf{r})$ and/or $c(\mathbf{r})$ are permitted to have sharper and sharper gradients, becoming, in the limit, discontinuous with a finite jump across one or more surfaces. Consequently Eq. (11) holds whether or not such surfaces of discontinuity exist. Of course Eq. (11) can be established directly, but also more awkwardly, without making the initial assumption that $\rho$ and/or $c$ are everywhere continuous. Green's theorem and Eqs. (8) and (10) imply that

$$\int dS\mathbf{n}\cdot[G(\mathbf{r},\mathbf{r}')\,\text{grad}p_s(\mathbf{r})-p_s\,\text{grad}G(\mathbf{r},\mathbf{r}')]=\int d\mathbf{r}G(\mathbf{r},\mathbf{r}')[k^2(1-\mu^2)p(\mathbf{r})+i\omega\mathbf{v}\cdot\text{grad}(\rho-\rho_0)].\quad (7\text{A})$$

The volume integral in Eq. (7A) runs over the space exterior to a small sphere about the point $\mathbf{r}=\mathbf{r}'$, not including an infinitesimal strip on either side of any surface of discontinuity of $\rho$ and/or $c$. The surface integral in Eq. (7A) is over the sphere at infinity, the sphere around $\mathbf{r}=\mathbf{r}'$, and both sides of every strip; the normal $\mathbf{n}$ points out of the volume of integration of the right-hand side. The surface integral at infinity vanishes because $G$ and $p_s$ are both outgoing so that, shrinking the radius $\rho$ of the small sphere to zero,

$$p_s(\mathbf{r}')+\int dS\mathbf{n}\cdot\{G(\mathbf{r},\mathbf{r}')[\text{grad}p_{si}-\text{grad}p_{se}]-\text{grad}G(\mathbf{r},\mathbf{r}')[p_{si}-p_{se}]\}=\int d\mathbf{r}G(\mathbf{r},\mathbf{r}')[k^2(1-\mu^2)p+i\omega\mathbf{v}\cdot\text{grad}(\rho-\rho_0)].\quad (8\text{A})$$

The surface integral in Eq. (8A) runs only over the surfaces of discontinuity. These are assumed to be finite closed surfaces, with $\mathbf{n}$ in Eq. (8A) directed out of the surface, toward the exterior, and $i$ and $e$ referring respectively to the interior and exterior sides. The volume integral in Eq. (8A) is over all space excluding infinitesimal strips on either side of the surfaces of discontinuity. Since $p_0$ and $\mathbf{v}_0$, Eq. (8), are everywhere continuous, the subscript $s$ may be dropped in Eq. (8A). Also, according to the boundary conditions, $p$ and $\mathbf{v}_n$ are continuous across $S$. Hence, using Eq. (3), Eq. (8A) becomes

$$p_s(\mathbf{r}')=\int d\mathbf{r}G(\mathbf{r},\mathbf{r}')[k^2(1-\mu^2)p+i\omega\mathbf{v}\cdot\text{grad}(\rho-\rho_0)]-i\omega\int dS\mathbf{n}\cdot G(\mathbf{r},\mathbf{r}')\mathbf{v}(\mathbf{r})(\rho_i-\rho_e).\quad (9\text{A})$$

By interchanging $\mathbf{r}$ and $\mathbf{r}'$, and employing Eq. (2A), the volume integral in Eq. (9A) is integrated by parts just as previously, except that the integral of $\mathrm{div}'G(\rho'-\rho_0)\mathbf{v}'$ gives a surface integral on either side of $S$ which just cancels the surface integral already in Eq. (9A). We are left with Eq. (3A), integrated over all of space exterior to the strips. In Eq. (3A) the integrands remain finite on $S$, since derivatives of $\rho$ and $c$, and $p$ and $\mathbf{v}$ for that matter, do not appear. Consequently, in Eq. (3A) the integral over an infinitesimal strip surrounding $S$ is infinitesimal, and Eq. (9A) is seen to yield Eq. (11) for $p(\mathbf{r})$. With much more labor Eq. (11) for $\mathbf{v}(\mathbf{r})$ can be similarly deduced directly, without assuming $\rho$ and $c$ everywhere continuous. However it is apparent from a comparison of the above two derivations of $p(\mathbf{r})$ that our original simpler derivation of Eq. (11) was quite legitimate.

From Eq. (11), using Eq. (7), Eq. (10), and curl curl $\equiv$grad div$-\Delta$,

$$\begin{aligned}\mathrm{grad}p-i\omega\rho_0\mathbf{v}&=-i\omega(\rho_0-\rho)\mathbf{v},\\(i\omega/c_0^2)p-\rho_0\,\mathrm{div}\mathbf{v}&=(i\omega/c_0^2)(1-\alpha\mu^2)p.\end{aligned}\tag{10A}$$

Equation (10A) is identical with Eq. (3), proving that the solutions $p$ and $\mathbf{v}$ to Eq. (11) satisfy the original differential relations Eq. (3).

## APPENDIX B

To prove $p(\mathbf{r})$ of Eq. (11) is continuous across a surface of discontinuity $S$ we first eliminate the derivatives of the Green's function in Eq. (11). Differentiating under the integral sign, Eq. (11) for $p(\mathbf{r})$ yields Eq. (3A). The integrals in Eq. (3A) exist for all $\mathbf{r}$. Consequently for any point $\mathbf{r}$ not actually in $S$, exclusion from the region of integration in Eq. (3A) of an arbitrarily thin strip on either side of $S$ modifies $p(\mathbf{r})$ by an arbitrarily small amount. Excluding such a region then, the last integral in Eq. (3A) can be integrated by parts, and we obtain

$$\begin{aligned}p(\mathbf{r})=p_0(\mathbf{r})&+k^2\int d\mathbf{r}'G(\mathbf{r},\mathbf{r}')(1-\alpha'\mu'^2)p'\\&+i\omega\int d\mathbf{r}'G(\mathbf{r},\mathbf{r}')\,\mathrm{div}'(\rho'-\rho_0)\mathbf{v}'\\&-i\omega\int dS'\mathbf{n}'\cdot G[(\rho_i'-\rho_0)\mathbf{v}_i'-(\rho_e'-\rho_0)\mathbf{v}_e'].\end{aligned}\tag{1B}$$

In Eq. (1B) $\mathbf{n}'$, the normal to $dS'$, and the subscripts $i$ and $e$ are defined as in Eq. (8A). The volume integrals in Eq. (1B) are over all space except for a thin strip on either side of $S$. However these volume integrals may be extended over all space by assigning an arbitrary finite value to $\mathrm{div}'(\rho'-\rho_0)\mathbf{v}'$ on $S'$.

Let $\mathbf{r}_0$ lie in $S$. We wish to show that given any $\epsilon$ it is possible to choose $\delta$ sufficiently small that $|p(\mathbf{r}_0+z_1\mathbf{n})-p(\mathbf{r}_0-z_2\mathbf{n})|<\epsilon$ whenever $0<z_1<\delta$ and $0<z_2<\delta$. $p_0(\mathbf{r})$, Eq. (8), is a continuous function of $\mathbf{r}$. There is no difficulty in demonstrating this inequality for the volume integrals in Eq. (1B), since the factors multiplying $G(\mathbf{r},\mathbf{r}')$ in the integrands are assumed bounded. There remains to be proved continuous only the surface integral of Eq. (1B). Draw a sphere of radius $\alpha\epsilon$ about $\mathbf{r}_0$, $\alpha$ to be determined, which sphere, for small $\alpha\epsilon$, intersects $S$ in some nearly circular closed curve. (We assume of course that $S$ possesses all the usual attributes of a well-behaved surface, e.g., a tangent plane.) The surface integral of Eq. (1B), integrated over points $\mathbf{r}'$ outside the sphere for $\mathbf{r}=\mathbf{r}_0+z_1\mathbf{n}$, can be made as nearly equal as desired to the surface integral integrated over the same $\mathbf{r}'$ for $\mathbf{r}=\mathbf{r}_0-z_2\mathbf{n}$. This is accomplished by choosing $\delta$ sufficiently small, since the integrand is finite and continuous as $\delta\to 0$ for points $\mathbf{r}'$ outside the sphere. We now prove that for sufficiently small $\alpha\epsilon$ and $\delta$ the contribution to the surface integral from points on $S'$ within the sphere vanishes. The factors multiplying $G(\mathbf{r},\mathbf{r}')$ in the integrand are bounded, so that, within the sphere,

$$\left|\int dS'\mathbf{n}'\cdot G[(\rho_i'-\rho_0)\mathbf{v}_i'-(\rho_e'-\rho_0)\mathbf{v}_e']\right|\leq\beta\int dS'\frac{1}{|\mathbf{r}-\mathbf{r}'|},\tag{2B}$$

where $\beta$ is some upper bound to $|\mathbf{n}'\cdot[(\rho_i'-\rho_0)\mathbf{v}_i'-(\rho_e'-\rho_0)\mathbf{v}_e']|$ within the sphere. Write

$$\mathbf{r}'-\mathbf{r}_0=x'\mathbf{t}'+y'\mathbf{n},\tag{3B}$$

with $\mathbf{t}'$ a unit vector in the tangent plane. Then with $z=z_1$ or $z_2$

$$|\mathbf{r}'-\mathbf{r}|=(x'^2+z^2)^{\frac{1}{2}}\left[1+\frac{y'^2\pm 2zy'}{x'^2+z^2}\right]^{\frac{1}{2}}.\tag{4B}$$

Since $\mathbf{t}'$ lies in the tangent plane, $y'\sim x'^2$ for sufficiently small $\alpha\epsilon$. For sufficiently small $\alpha\epsilon$ therefore

$$\left|\frac{y'^2\pm 2zy'}{x'^2+z^2}\right|\sim\frac{x'^4}{x'^2+z^2}+\frac{2zx'^2}{x'^2+z^2}=\frac{x'^2}{1+z^2/x'^2}+\frac{2z}{1+z^2/x'^2},\tag{5B}$$

which can be made arbitrarily small by choosing $\alpha\epsilon$ and $\delta$ sufficiently small, for any $x'$ within the circle of radius $\alpha\epsilon$ in the tangent plane, and for all $0<z<\delta$. In Eq. (2B), therefore, we may write

$$\beta\int dS'\frac{1}{|\mathbf{r}-\mathbf{r}'|}=2\pi\beta\int_0^{\alpha\epsilon}dx'\frac{x'}{(x'^2+z^2)^{\frac{1}{2}}}=2\pi\beta[(z^2+\alpha^2\epsilon^2)^{\frac{1}{2}}-z].\tag{6B}$$

The right side of Eq. (6B) can be made $<\epsilon$ by choosing $\delta$ arbitrarily small and $\alpha<1/2\pi\beta$. We conclude that $p(\mathbf{r})$ is continuous across $S$.

Equation (11) for $\mathbf{v}(\mathbf{r})$ yields, corresponding to Eq. (1B),

$$\begin{aligned}\mathbf{v}(\mathbf{r})=&\mathbf{v}_0(\mathbf{r})+k^2\int d\mathbf{r}'G(\mathbf{r},\mathbf{r}')\left(1-\frac{1}{\alpha'}\right)\mathbf{v}'\\&-\frac{i\omega}{\rho_0c_0^2}\int d\mathbf{r}'G(\mathbf{r},\mathbf{r}')\,\mathrm{grad}'(1-\alpha'\mu'^2)p'\\&-\int d\mathbf{r}'G(\mathbf{r},\mathbf{r}')\,\mathrm{curl}'\,\mathrm{curl}'\left(1-\frac{1}{\alpha'}\right)\mathbf{v}'\\&+\frac{i\omega}{\rho_0c_0^2}\int dS'\mathbf{n}'G(\mathbf{r},\mathbf{r}')\\&\times[(1-\alpha'\mu'^2)_ip_i'-(1-\alpha'\mu'^2)_ep_e']\\&+\int dS'\mathbf{n}'G(\mathbf{r},\mathbf{r}')\\&\times\left[\mathrm{curl}'\left(1-\frac{1}{\alpha'}\right)_i\mathbf{v}_i'-\mathrm{curl}'\left(1-\frac{1}{\alpha'}\right)_e\mathbf{v}_e'\right]\\&-\int dS'\,\mathrm{grad}'G(\mathbf{r},\mathbf{r}')\\&\times\left\{\mathbf{n}'\times\left[\left(1-\frac{1}{\alpha'}\right)_i\mathbf{v}_i'-\left(1-\frac{1}{\alpha'}\right)_e\mathbf{v}_e'\right]\right\}.\end{aligned}\tag{7B}$$

In Eq. (7B) it is possible that $[1-(1/\alpha')]\mathbf{v}'$ is discontinuous or nondifferentiable on a curve $C$ on one or both sides of $S$, in which case $\mathrm{curl}'[1-(1/\alpha')]\mathbf{v}'$ would not exist on $C$ on one or both sides of $S$. For example, $\mathbf{v}_e$ might be discontinuous on the rim of a plane piston source. However the value of the surface integral is not changed by assigning arbitrary finite values to $\mathrm{curl}'[1-(1/\alpha')]\mathbf{v}'$ on $C$, so that such discontinuities do not affect the validity of Eq. (7B). Moreover, in deriving the boundary conditions it is assumed that $p(\mathbf{r}_0)$ and $\mathbf{v}_n(\mathbf{r}_0)$ on either side of $S$ differ infinitesimally from the values of $p(\mathbf{r})$ and $\mathbf{v}_n(\mathbf{r})$ at points $\mathbf{r}$ on the same side of $S$ in the neighborhood of $\mathbf{r}_0$. This assumption need not be correct at points $\mathbf{r}_0$ where $[1-(1/\alpha')]\mathbf{v}'$ is discontinuous or nondifferentiable on either side of the surface. Consequently we are required to demonstrate the boundary conditions only at points $\mathbf{r}_0$ where $[1-(1/\alpha')]_i\mathbf{v}_i$ and $[1-(1/\alpha')]_e\mathbf{v}_e$ are well behaved. It can now be seen that in Eq. (7B) all integrals except the surface integral involving $\mathrm{grad}'1/|\mathbf{r}-\mathbf{r}'|$ (arising from $\mathrm{grad}'G$) are of the same types as those in Eq. (1B) and can be proved continuous in a like manner. As before, drawing a sphere of radius $\alpha\epsilon$ about $\mathbf{r}_0$, we observe that the contribution to the $\mathrm{grad}'1/|\mathbf{r}-\mathbf{r}'|$ surface integral from points $\mathbf{r}'$ outside the sphere is continuous on $\mathbf{r}$. It follows that the continuity of $\mathbf{v}_n$ will be assured if

$$\left|\mathbf{n}\cdot\int dS'\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\times\left\{\mathbf{n}'\times\left[\left(1-\frac{1}{\alpha'}\right)_i\mathbf{v}_i'-\left(1-\frac{1}{\alpha'}\right)_e\mathbf{v}_e'\right]\right\}\right|<\epsilon,\tag{8B}$$

for $\mathbf{r}=\mathbf{r}_0\pm z\mathbf{n}$ and sufficiently small $\alpha$ and $\delta$, the integral running over points on $S'$ lying within the sphere, and the integrand well behaved, i.e., differentiable, at $\mathbf{r}_0$.

Replace $\mathbf{r}'$ by $\mathbf{r}_0$ in the integrand of Eq. (8B), except in $(\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|^3$, which is equivalent to making a Taylor expansion with remainder term of order of magnitude

$$\int dS'\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\times\{(\mathbf{r}'-\mathbf{r}_0)\cdot\mathrm{grad}'\}\left\{\exp ik|\mathbf{r}-\mathbf{r}'|\times\mathbf{n}'\left[\left(1-\frac{1}{\alpha'}\right)_i\mathbf{v}_i'-\left(1-\frac{1}{\alpha'}\right)_e\mathbf{v}_e'\right]\right\}.\tag{9B}$$

By using Eqs. (3B)–(5B) and remembering $y'\sim x'^2$, each of the components of the expression (9B) can be seen to have an upper bound involving sums of integrals such as

$$\int_0^{\alpha\epsilon}dx'\frac{zx'^2}{(x'^2+z^2)^{\frac{3}{2}}},\quad\int_0^{\alpha\epsilon}dx'\frac{zx'^3}{(x'^2+z^2)^{\frac{3}{2}}},\quad\int_0^{\alpha\epsilon}dx'\frac{x'^3}{(x'^2+z^2)^{\frac{3}{2}}},\tag{10B}$$

each of which can be made as small as desired by choosing $\alpha\epsilon$ and $\delta$ sufficiently small. Consequently the remainder term Eq. (9B) is negligible, and, again using Eqs. (3B)–(5B) and examining integrals like those of (10B), Eq. (8B) is seen to be equivalent to the condition that

$$\left|\mathbf{n}\cdot\int_0^{\alpha\epsilon}dx'\int_0^{2\pi}d\theta'x'\frac{e^{\pm ikz}}{(x'^2+z^2)^{\frac{3}{2}}}\times[x'\mathbf{t}'+(y'\pm z)\mathbf{n}]\times\{\mathbf{n}\times\mathbf{q}(\mathbf{r}_0)\}\right|<\epsilon,\tag{11B}$$

where

$$\mathbf{q}(\mathbf{r}_0)=\left(1-\frac{1}{\alpha'}\right)_i\mathbf{v}_i'-\left(1-\frac{1}{\alpha'}\right)_e\mathbf{v}_e',\tag{12B}$$

evaluated at $\mathbf{r}'=\mathbf{r}_0$. In Eq. (11B) $y'$ and $\mathbf{t}'$ are the only quantities depending on $\theta'$. The region of integration is a circle in the tangent plane, centered at $\mathbf{r}_0$, with $\theta'$ the azimuth of $\mathbf{t}'$. Thus the terms involving $\mathbf{t}'$ in Eq. (11B) vanish when integrated over $\theta'$. The remaining terms in the integrand, in $(y'\pm z)\mathbf{n}$, are perpendicular to $\mathbf{n}$. We conclude therefore that the normal component of any $\mathbf{v}(\mathbf{r})$ solving Eq. (11) is continuous at $\mathbf{r}=\mathbf{r}_0$ on $S$.

We remark that our procedure in referring the demonstration of the continuity of $\mathbf{v}_n(\mathbf{r})$ to the preceding proof of the continuity of $p(\mathbf{r})$ means that the rather lengthy argument used in proving $p(\mathbf{r})$ continuous really did not involve any extra work. Actually $p(\mathbf{r})$ can be readily proved continuous directly from Eq. (3A), but this proof cannot be extended to proving $\mathbf{v}_n(\mathbf{r})$ continuous. We see no obvious way to avoid the detailed examination of the surface integrals in Eq. (7B).

## APPENDIX C

Corresponding to Eq. (17) we have

$$\begin{aligned}A(-\mathbf{n}_0,-\mathbf{n}) &= -\frac{1}{4\pi}\int d\mathbf{r}[\beta p(-\mathbf{n})p(\mathbf{n}_0)-\omega\rho_0\gamma\mathbf{v}(-\mathbf{n})\cdot\mathbf{v}(\mathbf{n}_0)] \\ &-\frac{i}{4\pi}\int d\mathbf{r}\beta p(-\mathbf{n})\,\mathrm{div}\int d\mathbf{r}'G\gamma'\mathbf{v}'(\mathbf{n}_0) \\ &+\frac{i}{4\pi}\int d\mathbf{r}\gamma\mathbf{v}(-\mathbf{n})\cdot\mathrm{grad}\int d\mathbf{r}'G\beta'p'(\mathbf{n}_0) \\ &+\frac{1}{4\pi}\int d\mathbf{r}\gamma\mathbf{v}(-\mathbf{n})\cdot\mathrm{curl\,curl}\int d\mathbf{r}'G\gamma'\mathbf{v}'(\mathbf{n}_0) \\ &+\frac{1}{4\pi}\int d\mathbf{r}d\mathbf{r}'[p(-\mathbf{n})\beta G\beta'p'(\mathbf{n}_0) \\ &\qquad -k^2\gamma G\gamma'\mathbf{v}(-\mathbf{n})\cdot\mathbf{v}'(\mathbf{n}_0)], \quad (1\mathrm{C})\end{aligned}$$

where $G$ is symmetric, $G(\mathbf{r},\mathbf{r}')=G(\mathbf{r}',\mathbf{r})$, and $\mathrm{grad}'G=-\mathrm{grad}G$. Moreover

$$\begin{aligned}\int d\mathbf{r}\beta p(\mathbf{n}_0)\,\mathrm{div}\int d\mathbf{r}'G\gamma'\mathbf{v}'(-\mathbf{n}) &= \int d\mathbf{r}d\mathbf{r}'\beta p(\mathbf{n}_0)\gamma'\mathbf{v}'(-\mathbf{n})\cdot\mathrm{grad}G \\ &= \int d\mathbf{r}d\mathbf{r}'\beta'p'(\mathbf{n}_0)\gamma\mathbf{v}(-\mathbf{n})\cdot\mathrm{grad}'G, \quad (2\mathrm{C})\end{aligned}$$

interchanging the dummy variables $\mathbf{r}$ and $\mathbf{r}'$. It follows that the second integral on the right of Eq. (17) equals the third integral on the right of Eq. (1C). Similarly, the second integral on the right of Eq. (1C) equals the third integral on the right of Eq. (17). Consequently, comparing Eqs. (17) and (1C), to prove Eq. (19) it is only necessary to show

$$\int d\mathbf{r}\gamma\mathbf{v}(\mathbf{n}_0)\cdot\mathrm{curl\,curl}\int d\mathbf{r}'G\gamma'\mathbf{v}'(-\mathbf{n}) = \int d\mathbf{r}\gamma\mathbf{v}(-\mathbf{n})\cdot\mathrm{curl\,curl}\int d\mathbf{r}'G\gamma'\mathbf{v}'(\mathbf{n}_0). \quad (3\mathrm{C})$$

By elementary manipulations using Stokes' theorem, it can be seen that the left side of Eq. (3C) equals

$$\begin{aligned}&\int dS\boldsymbol{\nu}\cdot\gamma\mathbf{v}(\mathbf{n}_0)\times\int dS'\boldsymbol{\nu}'\times G\gamma'\mathbf{v}'(-\mathbf{n}) \\ &-\int dS\boldsymbol{\nu}\cdot\gamma\mathbf{v}(\mathbf{n}_0)\times\int d\mathbf{r}'G\,\mathrm{curl}'\gamma'\mathbf{v}'(-\mathbf{n}) \\ &-\int d\mathbf{r}[\mathrm{curl}\gamma\mathbf{v}(\mathbf{n}_0)]\cdot\int dS'\boldsymbol{\nu}'\times G\gamma'\mathbf{v}'(-\mathbf{n}) \\ &+\int d\mathbf{r}[\mathrm{curl}\gamma\mathbf{v}(\mathbf{n}_0)]\cdot\int d\mathbf{r}'G[\mathrm{curl}'\gamma'\mathbf{v}'(-\mathbf{n})]. \quad (4\mathrm{C})\end{aligned}$$

In (4C) $\boldsymbol{\nu}$ is the normal to $dS$, and the surface integrals run over the interior and exterior of each surface of discontinuity, with $\boldsymbol{\nu}_i=\boldsymbol{\nu}$ as previously defined in Eq. (8A) on the interior surface and $\boldsymbol{\nu}_e=-\boldsymbol{\nu}$ on the exterior surface. A similar expression for the integral on the right of Eq. (3C) is obtained by interchanging $\mathbf{n}_0$ and $-\mathbf{n}$ in (4C). But interchanging the dot and first cross in each of the first two integrals on the right side of (4C), and then interchanging the dummy variables $\mathbf{r}$ and $\mathbf{r}'$, it is seen that the right side of (4C) is symmetric in $\mathbf{n}_0$ and $-\mathbf{n}$. It follows that Eq. (3C) is true, and therefore that Eq. (19) is proved.

## APPENDIX D

From Eqs. (14), (17), and (37):

$$\delta A_1=\frac{-k^2}{4\pi}\int d\mathbf{r}\,\exp(-ik\mathbf{n}\cdot\mathbf{r})\left[(1-\alpha\mu^2)\delta p(\mathbf{n}_0)+\rho_0c_0\left(1-\frac{1}{\alpha}\right)\mathbf{n}\cdot\delta\mathbf{v}(\mathbf{n}_0)\right], \quad (1\mathrm{D})$$

$$\delta A_2=\frac{-k^2}{4\pi}\int d\mathbf{r}\,\exp(ik\mathbf{n}_0\cdot\mathbf{r})\left[(1-\alpha\mu^2)\delta p(-\mathbf{n})-\rho_0c_0\left(1-\frac{1}{\alpha}\right)\mathbf{n}_0\cdot\delta\mathbf{v}(-\mathbf{n})\right], \quad (2\mathrm{D})$$

$$\delta A_3=\frac{-1}{4\pi}\int d\mathbf{r}[\beta p(\mathbf{n}_0)\delta p(-\mathbf{n})+\beta p(-\mathbf{n})\delta p(\mathbf{n}_0)$$
$$-\omega\rho_0\gamma\mathbf{v}(\mathbf{n}_0)\cdot\delta\mathbf{v}(-\mathbf{n})-\omega\rho_0\gamma\mathbf{v}(-\mathbf{n})\cdot\delta\mathbf{v}(\mathbf{n}_0)]$$
$$-\frac{i}{4\pi}\int d\mathbf{r}\beta\delta p(\mathbf{n}_0)\,\mathrm{div}\int d\mathbf{r}'G\gamma'\mathbf{v}'(-\mathbf{n})$$
$$-\frac{i}{4\pi}\int d\mathbf{r}d\mathbf{r}'[\beta p(\mathbf{n}_0)\gamma'\,\mathrm{grad}G]\cdot\delta\mathbf{v}'(-\mathbf{n})$$
$$+\frac{i}{4\pi}\int d\mathbf{r}\gamma\delta\mathbf{v}(\mathbf{n}_0)\cdot\mathrm{grad}\int d\mathbf{r}'G\beta'p'(-\mathbf{n})$$
$$+\frac{i}{4\pi}\int d\mathbf{r}d\mathbf{r}'\gamma\beta'[\mathbf{v}(\mathbf{n}_0)\cdot\mathrm{grad}G]\delta p'(-\mathbf{n})$$
$$+\frac{1}{4\pi}\int d\mathbf{r}\gamma\delta\mathbf{v}(\mathbf{n}_0)\cdot\mathrm{curl\,curl}\int d\mathbf{r}'G\gamma'\mathbf{v}'(-\mathbf{n})$$
$$+\frac{1}{4\pi}\int d\mathbf{r}\gamma\mathbf{v}(\mathbf{n}_0)\cdot\mathrm{curl\,curl}\int d\mathbf{r}'G\gamma'\delta\mathbf{v}'(-\mathbf{n})$$
$$+\frac{1}{4\pi}\int d\mathbf{r}d\mathbf{r}'[\delta p(\mathbf{n}_0)\beta G\beta'p'(-\mathbf{n})$$
$$+p(\mathbf{n}_0)\beta G\beta'\delta p'(-\mathbf{n})$$
$$-k^2\gamma G\gamma'\mathbf{v}'(-\mathbf{n})\cdot\delta\mathbf{v}(\mathbf{n}_0)$$
$$-k^2\gamma G\gamma'\mathbf{v}(\mathbf{n}_0)\cdot\delta\mathbf{v}'(-\mathbf{n})]. \quad (3D)$$

By substituting Eqs. (1D)–(3D) in Eq. (36) and collecting terms, it is found that the terms in $\delta p(\mathbf{n}_0)$ vanish by virtue of Eq. (16). The terms in $\delta\mathbf{v}(\mathbf{n}_0)$ vanish by virtue of the equation for $\mathbf{v}(-\mathbf{n})$. The terms in $\delta p'(-\mathbf{n})$ vanish because of the first of Eq. (11), after interchanging the dummy variables $\mathbf{r}$ and $\mathbf{r}'$ in the double integrals of Eq. (3D). We similarly interchange $\mathbf{r}$ and $\mathbf{r}'$ in the terms in $\delta\mathbf{v}'(-\mathbf{n})$, with the exception of the curl curl term where this interchange is of no utility. However, by following the procedure used to demonstrate Eq. (3C), it is seen that

$$\int d\mathbf{r}\gamma\mathbf{v}(\mathbf{n}_0)\cdot\mathrm{curl\,curl}\int d\mathbf{r}'G\gamma'\delta\mathbf{v}'(-\mathbf{n})$$
$$=\int d\mathbf{r}\gamma\delta\mathbf{v}(-\mathbf{n})\cdot\mathrm{curl\,curl}\int d\mathbf{r}'G\gamma'\mathbf{v}'(\mathbf{n}_0). \quad (4D)$$

Moreover the proof of Eq. (4D) does not necessitate using the boundary conditions. With the aid of Eq. (4D) the terms in $\delta\mathbf{v}'(-\mathbf{n})$ are thereby all converted to integrals over $\delta\mathbf{v}(-\mathbf{n})$, and at once are observed to vanish because of the second of Eq. (11). This completes the desired proof.

## APPENDIX E

The simplest way to derive Eq. (55) is to assume first that $\rho$ and $c$ are continuous. Introduce $u_l=rR_l$ into Eq. (54), which yields

$$\frac{d^2u_l}{dr^2}-\frac{l(l+1)}{r^2}u_l+k^2u_l=k^2(1-\mu^2)u_l+\frac{1}{\rho}\dot{\rho}\left(\dot{u}_l-\frac{u_l}{r}\right). \quad (1E)$$

The function $G_l(r,r')$ of Eq. (56) is the outgoing Green's function satisfying

$$\frac{d^2G_l}{dr^2}-\frac{l(l+1)}{r^2}G_l+k^2G_l=-\delta(r-r'), \quad (2E)$$

where $\delta(r-r')$ is a one-dimensional $\delta$ function over the interval $0\leq r<\infty$. It may then be directly verified that

$$u_l=u_{l0}-\int_0^\infty dr'G_l(r,r')\times\left[k^2(1-\mu'^2)u_l'+\frac{1}{\rho'}\left(\dot{u}_l'-\frac{u_l'}{r'}\right)\dot{\rho}'\right], \quad (3E)$$
$$u_{l0}=i^l(2l+1)rj_l(kr),$$

satisfies Eq. (1E) and consequently that

$$R_l=i^l(2l+1)j_l(kr)-\frac{1}{r}\int^\infty dr'r'G_l(r,r')\left[k^2(1-\mu'^2)R_l'+\frac{\dot{R}_l'}{\rho'}\dot{\rho}'\right], \quad (4E)$$

satisfies Eq. (54). Now use the identity,

$$r'G_l\dot{\rho}'\frac{\dot{R}_l'}{\rho'}=\frac{d}{dr'}\left[r'G_l(\rho'-\rho_0)\frac{\dot{R}_l'}{\rho'}\right]-(\rho'-\rho_0)\frac{\dot{R}_l'}{\rho'}\frac{d}{dr'}(r'G_l)-r'G_l(\rho'-\rho_0)\frac{d}{dr'}\left(\frac{\dot{R}_l'}{\rho'}\right), \quad (5E)$$

and substitute in Eq. (4E). The total derivative term contributes nothing since $r'G_l$ vanishes at the origin while $\rho'-\rho_0$ vanishes at infinity. Thus there results

$$R_l=i^l(2l+1)j_l(kr)-\frac{1}{r}\int_0^\infty drr'G_l(r,r')\times\left[k^2(1-\mu'^2)R_l'-(\rho'-\rho_0)\frac{d}{dr'}\frac{\dot{R}_l'}{\rho'}\right]+\frac{1}{r}\int^\infty dr'(1-\alpha')\dot{R}_l'\frac{d}{dr'}(r'G_l). \quad (6E)$$

Equation (54) implies

$$\frac{d}{dr}\left(\frac{\dot{R}_l}{\rho}\right)=-\frac{2}{r}\frac{\dot{R}_l}{\rho}+\frac{l(l+1)}{r^2}\frac{\Gamma}{\rho}-\mu^2k^2\frac{R_l}{\rho}. \quad (7E)$$

Substitution of Eq. (7E) into Eq. (6E) then yields Eq. (55). Just as in Appendix A, we now argue that since Eq. (55) does not involve derivatives of $\rho'$ and/or $c'$, it remains valid in the limit in which $\rho'$ and/or $c'$ becomes discontinuous across one or more surfaces. Eq. (55) can also be deduced without assuming $\rho'$ and/or $c'$ continuous, either directly from Eq. (54) or via the integral equation (9A), in either case with rather more labor.

## APPENDIX F

We regard $S_l$ and $\dot{S}_l$ as independent quantities in Eqs. (65) and (66). Since Eq. (66) was obtained by differentiating Eq. (65), Eq. (67) is evidently true. To demonstrate Eq. (68), differentiate Eq. (66) to get $d\dot{S}_l/dr$ and, combining this with Eqs. (65) and (66), evaluate the left side of Eq. (68). By using the differential equation[4] for $j_l(kr)$, the results obtained are

$$\frac{d\dot{S}_l}{dr}+\frac{2}{r}\dot{S}_l-\frac{l(l+1)}{r^2}S_l+k^2S_l=\frac{1}{r}\left[\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}+k^2\right]\times\left[\int_0^\infty dr'\bar{G}_l(r'\sigma'S_l'-\tau'\dot{S}_l')+\int_0^\infty dr'r'\tau'\dot{S}_l'\frac{d\bar{G}_l}{dr'}\right]. \quad (1F)$$

We integrate by parts the second integral in Eq. (1F). Thereby we convert the right side of Eq. (1F) to integrals to which Eq. (2E) can be applied under the integral sign, plus a sum over $r'=r_d$ which vanishes after application of Eq. (2E), since the delta function vanishes for the only points in which we are now interested, namely points $r$ not equal to any $r_d$. Thus we find

$$\frac{d\dot{S}_l}{dr}+\frac{2}{r}\dot{S}_l-\frac{l(l+1)}{r^2}S_l+k^2S_l=-\sigma S_l+\frac{\tau\dot{S}_l}{r}+\frac{1}{r}\frac{d}{dr}(r\tau\dot{S}_l), \quad (2F)$$

which, after some manipulation, reduces to Eq. (68).

In Eq. (65) the first integrand is bounded. The second integral when integrated by parts yields an integral excluding infinitesimal intervals surrounding the points $r'=r_d'$, and therefore one for which the integrand is again bounded. The integration by parts also yields a well-behaved sum over $r'=r_d'$, and we infer that $S_l(r)$ is continuous. We similarly integrate by parts the integrals involving $d\bar{G}_l/dr$ in Eq. (66), perform the $d/dr$ operation indicated in the last integral of Eq. (66), and observe that discontinuities in $\dot{S}_l$ can arise only from the terms in $d\bar{G}_l/dr$ in the sum over $r'$. In this way we find that

$$\dot{S}_{le}(r_d)-\dot{S}_{li}(r_d)=\frac{1}{r_d}\left[\frac{d}{dr}G_l(r_d+\epsilon,r_d)-\frac{d}{dr}G_l(r_d-\epsilon,r_d)\right]\{[r'\tau_i'\dot{S}_{li}'-r'\tau_e'\dot{S}_{le}']\}_{r'=r_d}. \quad (3F)$$

Since from Eq. (2E)

$$\frac{dG_l(r_d+\epsilon,r_d)}{dr}-\frac{dG_l(r_d-\epsilon,r_d)}{dr}=-1, \quad (4F)$$

we get $\alpha_e\dot{S}_{le}=\alpha_i\dot{S}_{li}$, which is the desired boundary condition.

## APPENDIX G

The proof that $B_l'$, Eq. (71), is stationary is straightforward. Appendix $D$ is a guide. The coefficients of $\delta S_l^T$ and $\delta\dot{S}_l^T$ are readily seen to vanish by virtue of Eqs. (65) and (66). Similarly the coefficient of $\delta S_l'$ vanishes. There is a complication in the coefficient of $\delta\dot{S}_l$. It is seen that this coefficient will not vanish by Eq. (66) unless it is true that

$$\int_0^\infty drr\tau\dot{S}_l^T\frac{d}{dr}\int_0^\infty dr'r'\tau'\frac{d\bar{G}_l(r,r')}{dr'}\delta\dot{S}_l'=\int_0^\infty dr'r'\tau'\delta\dot{S}_l'\frac{d}{dr'}\int_0^\infty drr\tau\dot{S}_l^T\frac{d\bar{G}_l(r,r')}{dr}. \quad (1G)$$

Equation (1G) can be proved however. The left side of Eq. (1G) is

$$\begin{aligned}&-k\int_0^\infty drr\tau\dot{S}_l^T\frac{d}{dr}\left\{\int_0^r dr'r'\tau'\delta\dot{S}_l'rn_l(kr)\frac{d}{dr'}r'j_l(kr')+\int_r^\infty dr'r'\tau'\delta\dot{S}_l'rj_l(kr)\frac{d}{dr'}r'n_l(kr')\right\}\\&=-k\int_0^\infty drr\tau\dot{S}_l^Tr\tau\delta\dot{S}_l\left[rn_l(kr)\frac{d}{dr}rj_l(kr)-rj_l(kr)\frac{d}{dr}rn_l(kr)\right]\\&\quad-k\int_0^\infty drr\tau\dot{S}_l^T\left\{\int_0^r dr'r'\tau'\delta\dot{S}_l'\frac{d}{dr}rn_l(kr)\frac{d}{dr'}r'j_l(kr')+\int_r^\infty dr'r'\tau'\delta\dot{S}_l'\frac{d}{dr}rj_l(kr)\frac{d}{dr'}r'n_l(kr')\right\}.\end{aligned} \quad (2G)$$

We evaluate the right side of Eq. (1G) as we did the left side and observe the expression so obtained is identical with the result of interchanging the order of integration in the double integrals on the right side of Eq. (2G). Hence Eq. (1G) is true.