The Angular Correlation of Three Nuclear Radiations

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An angular correlation formula for any three nuclear radiations is given, including the effects of other intermediate radiations which are not observed. Emission from aligned nuclei is also covered. Application to nuclear reactions is considered. In particular the correlation formula is given for γ rays following inelastic neutron scattering when continuum theory is applied to the compound nucleus state.

I. INTRODUCTION

THE triple angular correlation problem has been treated for three successive γ rays,[1] and for two γ rays following β decay of aligned nuclei[2] or a deuteron-stripping reaction.[3] The purpose of this paper is to give a general formula for the directional angular correlation of any three radiations from an arbitrary cascade, with intermediate radiations which are not observed. The formalism includes the correlation of two radiations from aligned nuclei, and nuclear reactions with aligned targets. For any particular set of transitions we merely insert the appropriate radiation parameters which have been fully tabulated elsewhere.[1,3] Previously published results are then included as special cases.

Not all three radiations need be emitted. Absorption from a beam is equivalent for the purpose of defining a direction. Hence the application to a nuclear reaction followed by several emissions [such as $(p,\gamma\gamma)$ or $(n,n'\alpha)$], provided there is no interference between compound nucleus states of different spin.

Mixtures of radiation multipoles or orbital angular momenta are considered but not polarization detection. The latter would increase the complexity of the results prohibitively.

II. CORRELATION FORMULAS

Consider the decay scheme $J_0(L_0)J_1(L_1)\cdots J_r(L_r)\times J_{r+1}\cdots J_s(L_s)J_{s+1}$, where the J are nuclear spins and the L are the total angular momenta carried away or absorbed at each transition. Using standard techniques[1] we find the angular correlation between radiations L_0, L_r, and L_s, taking the direction of L_0 as polar axis:

$$\begin{aligned}W(\theta_r\theta_s\phi)=\textstyle\sum_{\mu\nu\lambda} & A_\mu(J_0J_1)A_\lambda(J_{s+1}J_s)R_{\mu\nu\lambda}(J_rJ_{r+1})\\ &\times S_{\mu\nu\lambda}(\theta_r\theta_s\phi)U_\mu(L_1J_1J_2)U_\mu(L_2J_2J_3)\cdots\\ &\times U_\mu(L_{r-1}J_{r-1}J_r)U_\lambda(L_{r+1}J_{r+1}J_{r+2})\cdots\\ &\times U_\lambda(L_{s-1}J_{s-1}J_s), \qquad (1)\end{aligned}$$

where ϕ is the difference in azimuth of the directions of L_r and L_s. The A_μ depend only on the details of the first transition, the A_λ only on the final transition, and the coupling term $R_{\mu\nu\lambda}$ depends only on the intermediate radiation L_r.

All dependence on unobserved radiations is thrown into the (normalized) Racah functions:[4]

$$U_\mu(LJ_1J_2)=[(2J_1+1)(2J_2+1)]^{\frac{1}{2}}(-)^{J_1+J_2-1}W(J_1J_1J_2J_2;\mu L),$$

one for each transition. If any of these radiations is "mixed" (two or more L values) we have to sum over L with appropriate amplitudes, but the sum is incoherent and interference terms do not appear. The *nature* of unobserved radiations is irrelevant.

The $A_{\mu,\lambda}$ have been tabulated for all forms of radiation of interest. For example, for spin-$\frac{1}{2}$ particles,

$$A_\mu(J_0J_1)=\textstyle\sum_{LL'}B(L)B^*(L')\eta_\mu(LL'J_0J_1), \qquad (2)$$

$$\begin{aligned}\eta_\mu=(-)^{J_0-J_1-\frac{1}{2}}&[(2J_1+1)(2L+1)(2L'+1)]^{\frac{1}{2}}\\ &\times C(LL'\mu;\tfrac{1}{2}-\tfrac{1}{2})W(J_1J_1LL';\mu J_0),\end{aligned}$$

where η_μ has been tabulated,[3] and the $B(L)$ are reduced matrix elements so that the fraction of intensity with L is $|B(L)|^2$. $B(L)$ is real if Coulomb effects are neglected; these will introduce a phase factor $e^{i\delta_l}$, where l is the corresponding orbital momentum. If the only L's involved are those for a single l, i.e., $L=l\pm\frac{1}{2}$, this phase factor drops out.

For γ rays, in the notation of tabulated parameters,[1]

$$A_\mu(J_0J_1)=\textstyle\sum_{LL'}C(L)C(L')\lambda_\mu(LL'J_0J_1), \qquad (3)$$

$$\begin{aligned}\lambda_\mu&=F_\mu(LJ_0J_1)\quad \text{if}\quad L'=L,\\ &=(-)^{J_0-J_1-1}[(2J_1+1)(2L+1)(2L+3)]^{\frac{1}{2}}\\ &\qquad\times G_\mu(LJ_0J_1)\quad \text{if}\quad L'=L+1.\end{aligned}$$

Again the fraction of intensity with L is $[C(L)]^2$, with $C(L)$ real.

For conversion electrons, β and α rays, we obtain A_μ by multiplying the value for 2^L-pole γ rays, Eq. (3), by the appropriate tabulated[1,5,6] correction factors $b_\mu(L)$.

If the nucleus is initially aligned and of spin J_1, the density matrix describing its orientation may always be expressed as a series of statistical tensors,[7] $B_\mu(J_1)$. In this case we omit the L_0 transition and replace its

[1] L. C. Biedenharn and M. E. Rose, Revs. Modern Phys. **25**, 729 (1953).
[2] J. A. M. Cox and H. A. Tolhoek, Physica **18**, 359 (1952).
[3] G. R. Satchler, Proc. Phys. Soc. (London) **A66**, 1081 (1953).
[4] L. C. Biedenharn, Oak Ridge National Laboratory Report 1098 (unpublished).
[5] Rose, Biedenharn, and Arfken, Phys. Rev. **85**, 5 (1952).
[6] M. E. Rose and L. C. Biedenharn, Oak Ridge National Laboratory Report No. 1324 (unpublished).
[7] U. Fano, Natl. Bur. Standards Rept. No. 1214 (1951) (unpublished). Our $B_\mu(J)$ is his $\langle|(JJ)\mu 0\rangle$.

parameter $A_\mu(J_0J_1)$ by $B_\mu(J_1)$. (If aligned in sense as well as direction, odd values of μ may now appear.)

$$B_\mu(J)=\sum_M(-)^{J-M}(2J+1)^{\frac{1}{2}}C(JJ\mu;M-M)W_M,$$

where W_M is the relative population of the M magnetic substate[8] (for example, a Boltzmann factor $ae^{\beta M}$). Then $B_0=1$ if $\sum_M W_M=1$.

The coupling term $R_{\mu\nu\lambda}$ is given by

$$R_{\mu\nu\lambda}(J_aJ_b) =\sum_{LL'} r_\nu(LL')[(2J_a+1)(2J_b+1)(2L+1)(2L'+1)]^{\frac{1}{2}} \times X(J_aJ_a\mu;LL'\nu;J_bJ_b\lambda), \quad (4)$$

$$r_\nu(LL')=C(L)C(L')(-)^{L'-1}C(LL'\nu;1-1) \quad \text{for } \gamma \text{ rays}, \quad (4a)$$

or

$$r_\nu(LL')=\sum{}'_{ll'}S(Ll)S^*(L'l')C(ll'\nu;00)(i)^{l'-l} \times(-)^{L'-s}[(2l+1)(2l'+1)]^{\frac{1}{2}}W(LL'll';\nu s) =\sum{}'_{ll'}S(Ll)S^*(L'l')(-)^{L'-s}(-)^{\nu/2} \times Z(l'L'lL;s\nu), \quad (4b)$$

for particles of spin s and orbital momentum l. The Z functions are tabulated in reference 4. The primed sum over l, l' indicates we take only even or only odd values, according to the parity change. Again the $S(Ll)$ are reduced matrix elements, real except for any Coulomb phase factor $e^{i\delta_l}$. For $s=\frac{1}{2}$ (neutron or proton emission), (4b) reduces to

$$r_\nu=B(L)B^*(L')(-)^{L'-\frac{1}{2}}C(LL'\nu;\tfrac{1}{2}-\tfrac{1}{2}). \quad (4c)$$

$R_{\mu\nu\lambda}$ for pure dipole γ rays, and the Wigner coefficients $C(LL'\nu;\frac{1}{2}-\frac{1}{2})$, have been tabulated in reference 3, and the Fano recoupling symbol X[7,9] defined. As before, if the L_r radiation consists of conversion electrons, α or β rays, we multiply the r_ν for 2^{L_r}-pole γ rays by the appropriate $b_\nu(L_r)$.[1]

The angular factor is explicitly

$$S_{\mu\nu\lambda}=4\pi\sum_m C(\nu\lambda\mu;m-m)Y_\nu^m(\theta_r 0)Y_\nu^{-m}(\theta_s\phi), \quad (5)$$

where the Y_ν^m are Condon and Shortley spherical harmonics.

$R_{\mu\nu\lambda}S_{\mu\nu\lambda}$ assume very simple forms when all $(\mu\nu\lambda)\leqslant 2$ and the radiations are unmixed. These are given in reference 3 for γ rays; for spin-$\frac{1}{2}$ particles we just replace the F_ν occurring there by the corresponding η_ν.

Integration over the direction of any one of the three radiations immediately reduces Eq. (1) to the general double correlation formula (Appendix IV of reference 3).

The angular complexity of the correlation is restricted by the largest values of μ, ν, and λ. These are limited by the usual requirements for the nonvanishing of Racah functions.[4] They are always even (except for nuclear alignment, when μ may be odd) and have to satisfy triangular inequalities in the following triads: $(\mu\nu\lambda)$, $(\mu L_0L_0')$, (μJ_1J_1), $(\nu L_rL_r')$, $(\lambda L_sL_s')$, (λJ_sJ_s). In addition, the unobserved radiations do not limit the complexity, but the intervening nuclear spins still act as "gates" for angular information, so that

$$\mu\leqslant 2J_2,\, 2J_3\cdots 2J_r, \quad \lambda\leqslant 2J_{r+1},\, 2J_{r+2}\cdots 2J_s.$$

To avoid possible confusion we should point out that if two unobserved radiations (e.g., L_1 and L_2) are replaced by a crossover transition in a competing cascade, one of the nuclear spins (J_2) is then omitted and does not restrict the angular complexity. The observed correlation, of course, is then an incoherent superposition of the two competing cascades.

Also, it has recently been shown[10] that a cascade of "basic" transitions between nuclear states whose spins form a monotonic sequence gives angular correlations which are independent of the nuclear spins and the number of intervening unobserved radiations. That is to say, if $J_1=J_2+L_1$, $J_2=J_3+L_2\cdots$ or $J_1=J_2-L_1$, $J_2=J_3-L_2\cdots$ for the unobserved part of our cascade, the angular correlation is the same as if these transitions were replaced by a single crossover with $J_1=J_r+L$ or $J_1=J_r-L$, respectively, where $L=L_1+L_2\cdots+L_{r-1}$. In this case, the two competing cascades mentioned above give the same correlation.

If the three observed transitions also form part of the monotonic sequence ($J_0=J_1+L_0$, etc.) reference 10 is easily extended to show that the *triple* correlation is now independent of nuclear spins and unobserved radiations; it depends only on L_0, L_r, and L_s. In fact, *any* multiple correlation from such a sequence depends only on the nature of the observed radiations.

Practical interest centers on the correlation when only one angular momentum is concerned in each transition (e.g., pure multipole γ rays). The correlation function (1) then simplifies and gives unambiguous results which depend only on angular momenta. All dependence on unknown matrix elements $B(L)$, $C(L)$, etc., disappears.

III. NUCLEAR REACTIONS

Blatt and Biedenharn[11] have given expressions for the angular distribution of nuclear reaction products ("double correlation"). It is clear from the complexity of their results that little is to be gained by generalizing to reactions followed by cascades of radiation. However, our formulas are easily applied when there are no

[8] N. R. Steenberg, Proc. Phys. Soc. (London) A66, 399 (1953).

[9] H. A. Jahn and J. Hope, Phys. Rev. 93, 318 (1954). There X is called the "Wigner $9j$ symbol."

[10] J. Weneser and D. R. Hamilton, Phys. Rev. 92, 321 (1953). While the conclusions they draw are correct, the formulas contain a misprint; j_f should be j_n everywhere except in the last column of their paper. Their j_f is defined by $j=L_n+j_f$ and is not the same as j_n. The products of Racah functions representing unobserved radiations reduce (in their notation) to $W(j_1j_1j_{n-1}j_{n-1};\nu L)$, where $L=L_1+L_2\cdots+L_{n-1}$; i.e., all unobserved radiations may be replaced by a single unobserved crossover with angular momentum L without observable effects.

[11] J. M. Blatt and L. C. Biedenharn, Revs. Modern Phys. 25, 258 (1953).

interference terms between compound nucleus states of different spin and parity. This occurs

(i) at resonance when the reaction proceeds through a compound state of definite spin J_1 and parity. The correlation formula (1) is used as it stands, the L_0 transition representing the capture process. Values of the matrix elements $B(L)$ for neutrons or protons when the initial and compound nuclear states obey **j-j** or **L-S** coupling rules have been given in reference 3.

(ii) if we apply continuum theory[12,13] the interference terms disappear on taking a statistical average, and the outgoing partial waves are incoherent. We may then replace the matrix elements by "transmission coefficients"[12] for the incoming and outgoing particles at energy E:

$$S(Ll)S^*(L'l')\to\delta(ll')T_l(E).$$

For spin-$\frac{1}{2}$ particles each L implies a definite orbital $=L+\frac{1}{2}$ or $l=L-\frac{1}{2}$, but not both by parity considerations. Then the matrix elements $B(L)\equiv B(Ll)$ are a special case of $S(Ll)$, and we put

$$B(L)B^*(L')\to\delta(ll')T_l(E).$$

The sums over L now reduce to two values $L=l\pm\frac{1}{2}$. The final angular correlation is then an incoherent sum of functions like (1) for different J_1, parity, l_0, and l_1. All unknown matrix elements have disappeared; assuming a definite nuclear model (the statistical model) we have been able to replace them by calculable transmission coefficients.

An example of interest is the (n',γ) correlation following inelastic scattering of neutrons. Making the above assumptions, we find for emission of a 2^L-pole γ ray:

$$W(\theta_1\theta_2\phi)=\sum T_{l_0}(E_0)T_{l_1}(E_1)(2J_1+1)\times\sum \eta_\mu(L_0L_0'J_0J_1)F_\lambda(LJ_3J_2)\times R_{\mu\nu\lambda}(J_2J_1)S_{\mu\nu\lambda}(\theta_1\theta_2\phi),\quad(6)$$

summed over l_0, l_1, J_1, μ, ν, and λ, and L_0, $L_0'=l_0\pm\frac{1}{2}$, where now

$$R_{\mu\nu\lambda}=\sum(-)^{L_1'-\frac{1}{2}}[(2J_1+1)(2J_2+1)(2L_1+1)(2L_1'+1)]^{\frac{1}{2}}\times C(L_1L_1'\nu;\tfrac{1}{2}-\tfrac{1}{2})X(J_1J_1\mu;L_1L_1'\nu;J_2J_2\lambda),$$

summed over L_1, $L_1'=l_1\pm\frac{1}{2}$, where E_0, E_1 are the energies of the incident and scattered neutrons, respectively. Expressions for $T_l(E)$ are given in Blatt and Weisskopf.[12]

Equation (6) simplifies if the target nucleus is even-even, so that $J_0=0$, $(+)$, and $J_1=L_0$. For fairly low E_0 (say about 2 Mev and below) we need only consider s, p, and d waves. If the first state $J_2=2$, $(+)$, is excited, followed by $E2$ emission $(L=2)$, the $n'-\gamma$ correlation (6) becomes

$$\begin{aligned}W(\theta_1\theta_2\phi)&=T_0(E_0)T_2(E_1)[2+1.421P_2(\cos\gamma)\\&-0.571P_4(\cos\gamma)]+T_1(E_0)T_1(E_1)\{5+P_2(\cos\theta_2)\\&-1.200P_2(\cos\theta_1)+0.500P_2(\cos\gamma)\\&-0.004[\cos2\phi P_2^2(\cos\theta_1)P_2^2(\cos\theta_2)\\&-2\cos\phi P_2^1(\cos\theta_1)P_2^1(\cos\theta_2)-12P_2(\cos\theta_1)P_2(\cos\theta_2)]\\&-0.020[\cos2\phi P_2^2(\cos\theta_1)P_4^2(\cos\theta_2)\\&+12\cos\phi P_2^1(\cos\theta_1)P_4^1(\cos\theta_2)\\&+72P_2(\cos\theta_1)P_4(\cos\theta_2)]\}+T_2(E_0)T_0(E_1)\\&\times[2+2.713P_2(\cos\theta_2)-1.716P_4(\cos\theta_2)].\end{aligned}$$

The P_ν^m are (un-normalized) associated Legendre polynomials,[14] $P_1^1=\sin\theta$, etc.$\cdots$.

Since the compound state is formed isotropically by S waves, the first bracket depends only on the angle γ between the scattered neutron and the γ ray:

$$\cos\gamma=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\phi.$$

The constant terms and terms in θ_2 alone are together the same as the angular distribution of the γ rays alone, when the scattered neutrons are not observed, as may be seen by integrating over θ_1, the neutron angle. Proton scattering would only mean different T_l's to account for the Coulomb barrier.

Finally we note that reactions involving aligned nuclei may also be considered, provided again we have no interference between different compound nucleus spins. If we replace the L_0 transition by alignment according to the prescription given above, and take the L_r radiation to be captured, not emitted, we have immediately the angular distribution of the reaction products when an unpolarized beam is fired at a target of aligned nuclei.

The author wishes to thank Dr. J. A. Spiers for advice on this problem.

[12] J. M. Blatt and V. W. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), Chapter VIII.

[13] O. Hittmair, Phys. Rev. **87**, 375 (1952). The transmission coefficient for the outgoing neutrons, $Tl_1(E_1)$, and a factor $W(j_2s_2j_2s_2;i_2\nu)$, appear to have been omitted.

[14] E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, New York, 1945).