# Extension of the Theory of the Junction Transistor

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An examination of the validity range of the Shockley theory reveals that it is applicable to the emitter and collector regions for all current values of interest, whereas it is valid in the base region only for very small currents. In the present paper the treatment of the base region is extended so as to apply to arbitrary injection level and to include the effect of surface recombination. Two predictions are made: (a) the surface recombination velocity should increase with injection level; and (b) the alpha cut-off frequency for a transistor with plane parallel junctions should increase with emitter current by a factor of two.

## I. INTRODUCTION

A SIMPLE physical theory of the junction transistor was presented by Shockley in 1949.[1] This theoretical treatment had the unusual merit of preceding by a period of two years the first experimental realization and study of the device.[2] In the course of reporting[2] these experiments, the theory was somewhat further elaborated. Since that time only minor extensions or modifications of the physical theory have been published; Steele[3] has restated the theory concisely with slight modification of the treatment of the frequency response; Early[4] has made the point that space charge widening of the collector junction with collector voltage has important consequences in the small-signal behavior; and Hall[5] has attempted to extend the theory to cover power transistors.[6]

The simplicity of the Shockley theory follows primarily from the restriction of the treatment to low injection levels (i.e., levels for which the density of minority carriers is much smaller than the density of majority carriers) and to one-dimensional flow of charge carriers. The first of these restrictions is presumed to permit neglect of all conduction currents relative to diffusion currents. The validity of this procedure is examined in some detail in Sec. III, based upon a general formulation[7,8] of the problem of the injection of minority carriers into a semiconductor (see Sec. II). Our conclusion is that Shockley's procedure is indeed valid provided that the dimension of the specimen is large compared to the diffusion length of the minority carrier, a condition fulfilled in the emitter and collector regions but not in the base. In the latter case the neglect of the electric field is justified only for very small injection levels. Moreover, on translating the injection level conditions into electric current terminology, it turns out that Shockley's theory is applicable to the emitter and collector regions for all realizable values of current, whereas it is applicable to the base region only for extremely small currents.[9]

In order to extend the theory to cover currents of the magnitude commonly employed we present in Sec. IV a steady-state solution in the base region for arbitrary injection level subject, however, to the restriction of no recombination. (An estimate of recombination effects is deferred to Sec. VI.) In Sec. V the solution in the base is combined with Shockley's solutions in the emitter and collector regions to yield the steady-state current-voltage relations of a *p-n-p* transistor, which are then briefly discussed.

Section VI is devoted to the problem of recombination in the base. First (paragraph A) we estimate the recombination current by a perturbation method and thereby learn that for carefully prepared base material surface recombination is much more important than volume recombination. Moreover, this calculation permits us to account for the magnitude of alpha experimentally observed as well as the initial increase in $\alpha$ with emitter current[2,10,11] resulting from the buildup of a field which aids the transit of minority carriers. The decrease in $\alpha$ with emitter current occurring at still higher values of the latter[10,11] is attributable mainly to a decrease in emitter efficiency as has already been pointed out by Webster.[6] Also, a secondary contribution to this decrease may result from an increase in surface recombination velocity with injection level, which is a consequence of a recent theory by Brattain and Bardeen[12] of the surface recombination process. Then in paragraph B we present a three-dimensional theory in the spirit of Shockley's approach, which is

[1] W. Shockley, Bell System Tech. J. 28, 435 (1949).
[2] Shockley, Sparks, and Teal, Phys. Rev. 83, 151 (1951).
[3] E. L. Steele, Proc. Inst. Radio Engrs. 40, 1424 (1952).
[4] J. M. Early, Proc. Inst. Radio Engrs. 40, 1401 (1952).
[5] R. N. Hall, Proc. Inst. Radio Engrs. 40, 1512 (1952).
[6] While this manuscript was in preparation a second paper by J. M. Early appeared [Bell System Tech. J. 32, 1271 (1953)] dealing extensively with small-signal design theory. In general, this treatment follows along the same lines as the Shockley theory with some extensions in the directions of high-frequency and base resistance effects. Also it has recently been called to our attention that an extension of the theory to include high level injection effects has been carried out by W. M. Webster. [Paper presented at Transistor Research Conference, Penn. State College (July 6, 1953); Proc. Inst. Radio Engrs. (to be published)].
[7] C. Herring, Bell System Tech. J. 28, 401 (1949).
[8] W. van Roosbroeck, Bell System Tech. J. 29, 560 (1950).
[9] It is worth mentioning that comparison between theory and experiment (reference 2) has been made with sufficiently small applied voltages so that the above condition applied.
[10] Law, Mueller, Pankove, and Armstrong, Proc. Inst. Radio Engrs. 40, 1352 (1952).
[11] D. A. Jenny, Proc. Inst. Radio Engrs. 41, 1728 (1953).
[12] W. H. Brattain and J. Bardeen, Bell System Tech. J. 32, 1 (1953).

TABLE I. Partial list of symbols used.

| Symbol | Meaning |
|---|---|
| $p_0, p_b$ | Equilibrium hole density in a semiconductor, in an $n$-type base region. |
| $n_0, n_i, n_e, n_c$ | Equilibrium electron density in a semiconductor, in an intrinsic semiconductor, in a $p$-type emitter region, in a $p$-type collector region. |
| $N_d, N_a$ | Density of donor centers, of acceptor centers. |
| $L_p, L_e, L_c$ | Diffusion length of holes in an $n$-type semiconductor, of electrons in a $p$-type emitter region, of electrons in a $p$-type collector region. |
| $w, 2a$ | Width of base region, length of side of square of transistor cross section. |
| $\tau_p, \tau_n, \tau_0$ | Volume lifetime for arbitrary injection of holes into $n$-type material, of electrons into $p$-type material, volume lifetime of minority carriers for low injection levels. |
| $\tau_p', \tau_e$ | Combined surface and volume lifetimes for arbitrary injection of holes into $n$-type material, of electrons into $p$-type emitter region. |
| $s, s_0$ | Surface recombination velocity for arbitrary injection level, for low injection level. |
| $J_{ne}, J_{nc}$; $J_{pe}, J_{pc}$ | Electron component of emitter current density, of collector current density; hole component of emitter current density, of collector current density. |
| $V_e, V_c$ | dc bias potential relative to base on emitter, on collector. |
| $I_e, I_c, I_r$ | Emitter current, collector current, recombination current in base. |
| $i_e, i_c, v_e, v_c$ | Small-signal emitter current, collector current, emitter voltage, collector voltage. |
| $A, A_s$ | Conduction area, area available for surface recombination. |

applicable in the base region for plane parallel junctions and for very small injection levels. This calculation permits precise numerical evaluation of the diminution factor $\beta$ as a function of the surface recombination velocity as well as an appraisal of the accuracy of the results of the perturbation calculation under very low injection level conditions.

Finally, in Sec. VII we take up the problem of the frequency response to a very small sinusoidal signal in the presence of either an extremely small or a large steady injection level. In the first of these cases we show that the injection factor, $\gamma$, has a negligible influence on the alpha cut-off frequency. In the second case we first neglect the injection factor but take into account the possibility of transit time dispersion resulting from surface recombination effects at high emitter currents. For values of the surface recombination velocity not exceeding a known upper limit this transit time dispersion proves to be negligible. Under these circumstances the alpha cut-off frequency should increase monotonically with emitter current reaching a limiting value at high currents just twice the low current value. We then consider the possible effects of the injection factor and find that this might cause the cut-off frequency to decrease slightly with increasing emitter current.

The scheme that has been adopted with respect to notation is illustrated by the partial list of symbols shown in Table I. In addition, the symbol $\mu$ has been employed for charge carrier mobility, $D$ for diffusion constant, $b$ for the ratio of electron to hole mobility, and $q$ for the electronic charge; other symbols are defined in the text as required.

A compilation of assumed numerical values of geometric and material parameters, which have been used throughout the paper for illustrative computation, is given in Table II.

## II. FORMULATION OF THE PROBLEM OF INJECTION OF MINORITY CARRIERS INTO A SEMICONDUCTOR

The problem of the injection of minority carriers into a semiconductor has been studied theoretically by Herring[7] and by van Roosbroeck.[8] In this section we shall recapitulate the main results of their formulation in a form suitable for subsequent use in this article.

The equations governing the behavior of injected minority carriers and the excess majority carriers drawn in to neutralize space charge are as follows:

$$\partial p/\partial t = -(p-p_0)/\tau_p - q^{-1}\,\mathrm{div}J_p, \quad \text{(continuity equation for holes)} \tag{1}$$

$$\partial n/\partial t = -(n-n_0)/\tau_n + q^{-1}\,\mathrm{div}J_n, \quad \text{(continuity equation for electrons)} \tag{2}$$

$$J_p = q\mu_p pE - qD_p\,\mathrm{grad}p, \quad \text{(definition of current density of holes)} \tag{3}$$

$$J_n = q\mu_n nE + qD_n\,\mathrm{grad}n, \quad \text{(definition of current density of electrons)} \tag{4}$$

$$J = J_p + J_n, \quad \text{(definition of total current density)} \tag{5}$$

$$\mathrm{div}E = (4\pi q/\epsilon)(p-n+N_d-N_a) \quad \text{(Poisson equation for semiconductor with completely ionized impurities).} \tag{6}$$

These six equations constitute an exact formulation of the problem.

To simplify the problem it is assumed that:

(a) electrons and holes disappear by mutual recombination at identical rates, i.e.,

$$(p-p_0)/\tau_p = (n-n_0)/\tau_n; \tag{7}$$

TABLE II. Assumed values of geometric and material parameters (appropriate to germanium) for illustrative computation.

| |
|---|
| $w = 5.0\times 10^{-3}$ cm |
| $a = 1.76\times 10^{-2}$ cm[a] |
| $A = 4a^2 = 1.24\times 10^{-3}$ cm$^2$ |
| $A_s = 8aw = 7.04\times 10^{-4}$ cm$^2$ |
| $D_p = 44$ cm$^2$/sec |
| $D_n = 93$ cm$^2$/sec |
| $n_i = 2.5\times 10^{13}$/cm$^3$ |
| $p_b = 1.25\times 10^{12}$/cm$^3$ |
| $n_e = n_c = 4.0\times 10^{8}$/cm$^3$ |
| $s_0 = 400$ cm/sec |
| $\tau_0 = 5\times 10^{-4}$ sec |
| $L_e = L_c = 10^{-3}$ cm |

[a] This seemingly weird choice greatly facilitates the determination of the roots of Eq. (87).

(b) space charge neutrality is preserved at every point,

$$p-n+N_d-N_a=0, \tag{8}$$

which has as a consequence:

$$\partial p/\partial t=\partial n/\partial t, \tag{9}$$

$$\mathrm{grad}p=\mathrm{grad}n. \tag{10}$$

(Although Herring[7] has presented a convincing argument that the approximation represented by Eq. (8) is an excellent one, it is perhaps useful to demonstrate that div$E$ is indeed negligible for each solution based upon its use.)

It follows from the preceding equations that

$$\mathrm{div}J=0, \tag{11}$$

and also for an $n$-type semiconductor with $N_a \ll N_d$ that

$$E=\frac{J-qD_p(b-1)\,\mathrm{grad}p}{q\mu_p\{p(b+1)+bN_d\}}, \tag{12}$$

$$J_p=\frac{pJ-qbD_p(2p+N_d)\,\mathrm{grad}p}{p(b+1)+bN_d}, \tag{13}$$

$$\frac{\partial p}{\partial t}+\frac{p-p_0}{\tau_p}=-\mathrm{div}\left[\frac{pJ-qD_pb(2p+N_d)\,\mathrm{grad}p}{q\{p(b+1)+bN_d\}}\right]. \tag{14}$$

This last equation reduces for the one-dimensional case to:

$$-\{\partial p/\partial t+(p-p_0)/\tau_p'\}=\frac{1}{qbN_d\{1+p(b+1)/bN_d\}^2}\times[J(\partial p/\partial x-qbD_pN_d\{1+p(b+1)/bN_d\}\times(1+2p/N_d)(\partial^2p/\partial x^2)-qD_p(b-1)(\partial p/\partial x)^2]. \tag{15}$$

Here $\tau_p'$ is written instead of $\tau_p$ to take account of the effect of surface recombination in the one-dimensional theory.

In general, the procedure is to find a solution to (14) or (15) satisfying the boundary conditions and to put this back into (13) in order to obtain the current-voltage relations. This is usually an extremely difficult feat owing to the nonlinearity of the differential equation unless further simplification can be achieved with the use of additional restrictions.

### III. VALIDITY RANGE OF SHOCKLEY'S THEORY

Shockley[1,2] has greatly simplified the problem for the case of low-level injection ($p/N_d \ll 1$) and one-dimensional flow of carriers by dealing only with Eqs. (1) and (3) for injection into an $n$-type semiconductor (or with (2) and (4) for injection into $p$-type material) and by assuming that the conduction current is negligible relative to the diffusion current, i.e., $E=0$. In this case the resulting differential equation for $p(x,t)$ is

$$\partial p/\partial t+(p-p_0)/\tau_p'=D_p\partial^2p/\partial x^2, \tag{16}$$

and the expression for the current density is simply

$$J_p=-qD_p\partial p/\partial x. \tag{17}$$

Although Eq. (13) readily reduces to (17) for the condition $p/N_d \ll 1$, this is not the case with respect to Eq. (15), which does not reduce to (16) for this condition but rather to

$$\frac{\partial p}{\partial t}+\frac{p-p_0}{\tau_p'}=\frac{-J\partial p/\partial x+qbD_pN_d\partial^2p/\partial x^2+qD_p(b-1)(\partial p/\partial x)^2}{qbN_d}. \tag{18}$$

Indeed, Eq. (16) cannot be valid unless the terms in (18) involving $(\partial p/\partial x)$ are negligible. It therefore becomes of interest to check whether Shockley's solutions satisfy Eq. (18) to a good approximation without the imposition of additional conditions.

For a semi-infinite ($0<x<\infty$) $n$-type semiconductor (representing either emitter or collector regions) and for steady-state conditions the solution to (16) is

$$p-p_0=(p_1-p_0)e^{-x/L_p}, \tag{19}$$

where $p_1$ is the appropriate boundary value of $p$ at $x=0$ and $L_p=(D_p\tau_p')^{\frac{1}{2}}$. Equations (17) and (19) yield

$$J_p=qD_p(p_1-p_0)/L_p. \tag{20}$$

The condition $p/N_d \ll 1$ thus requires that

$$J_p \ll qN_dD_p/L_p. \tag{21}$$

Now, trying (19) as a steady-state solution of (18), we obtain

$$\frac{p-p_0}{\tau_p'}=\frac{J(p-p_0)/L_p+qbD_pN_d(p-p_0)/L_p^2+qD_p(b-1)\{(p-p_0)/L_p\}^2}{qbN_d}. \tag{22}$$

Thus, we find that (19) is indeed a solution of (18) provided that

$$(b-1)(p-p_0)/bN_d \ll 1, \tag{23}$$

$$J \ll qbN_dD_p/L_p. \tag{24}$$

Equation (23) is obviously satisfied by the low-level injection condition; however, Eq. (24) represents a more severe restriction than (21) owing to the fact that the total currents flowing through the emitter and collector regions are much larger than the currents injected from the base into the emitter or from the collector into the base. Nevertheless, numerical evaluation of (24) with the use of values of $N_d$ and $L_p$ appropriate to well-doped emitter and collector regions (see

Table II) reveals that this condition is satisfied provided that the total emitter (or collector) current is small compared to about 25 amperes. Hence, it is clear that Shockley's treatment of the emitter and collector regions represents a valid approximation for all realizable values of current. That this is the case is in part a consequence of the fact that the terms involving the first derivative,

$$dp/dx = -(p-p_0)/L_p, \tag{25}$$

are very small.

However, the situation in the base region is quite different. For an $n$-type semiconductor of finite width $w$ the steady-state solution to (16) may be written in the form

$$p-p_0 = \frac{(p_1-p_0)\sinh((w-x)/L_p) + (p_2-p_0)\sinh(x/L_p)}{\sinh(w/L_p)}, \tag{26}$$

where $p_2$ is the boundary value of $p$ at $x=w$. The corresponding current density expression (evaluated at $x=0$) is

$$J_p = (qD_p/L_p)\{(p_1-p_0)\coth(w/L_p) - (p_2-p_0)\operatorname{csch}(w/L_p)\} \tag{27}$$

$$\cong qD_p(p_1-p_0)/L_p \quad (\text{for } w/L_p \ll 1,\ p_1 \gg p_0 \gg p_2). \tag{28}$$

Thus, again the condition $p/N_d \ll 1$ leads to condition (21).

Now, trying (26) as an approximate solution of (18), we find that the solution is valid provided that

$$J \ll qbN_dD_pw/L_p^2, \tag{29}$$

$$(b-1)(p-p_0)L_p^2/bN_dw^2 \ll 1. \tag{30}$$

Both conditions require smaller current densities than is called for by (21). Equation (30) is the more stringent of the two conditions and implies that Eq. (26) can be justified only for extremely small injection levels such that $p/N_d \ll (w/L_p)^2$, corresponding to

$$J \ll qN_dD_pw^2/L_p^3. \tag{31}$$

The reason for this is that the terms involving the first derivative,

$$dp/dx \cong -(p-p_0)/w, \tag{32}$$

are no longer negligible when $w$ becomes small compared to the diffusion length $L_p$.

Putting numbers into (31) appropriate to relatively pure base material (see Table II) we find that Shockley's solution in the base is valid only for total emitter currents small compared to about $4\times10^{-6}$ ampere. Thus, it is important that existing theory be extended so as to apply to currents of the order employed technically.

## IV. STEADY-STATE SOLUTION IN THE BASE REGION FOR ARBITRARY INJECTION LEVEL NEGLECTING RECOMBINATION

It is clear from the fact that the current amplification factor in junction transistors is quite close to unity that recombination cannot be a very important process in the base region. If recombination is completely neglected, then a solution to the steady-state problem of the flow of minority carriers in the base region may be readily obtained for arbitrary injection level.

Although it is possible to derive the desired result starting with the general equation (15), it proves to be more convenient to return to the fundamental equations (1)–(6) inclusive. With the neglect of recombination and for the steady-state case (1) and (2) become

$$\operatorname{div} J_p = 0, \tag{33}$$

$$\operatorname{div} J_n = 0. \tag{34}$$

Furthermore, since neglecting recombination renders the carrier flow strictly one-dimensional, Eqs. (33) and (34) lead to

$$J_p = \text{constant}, \tag{35}$$

$$J_n = \text{constant}. \tag{36}$$

The value of $J_n$ is known immediately from the boundary conditions to be

$$J_n = J_{ne} - J_{nc}. \tag{37}$$

Since Shockley's theory is valid in the emitter and collector regions, we may express (37) for $p$-type material as[13]

$$J_n = qD_n(n_1-n_e)/L_e + qD_n(n_1-n_c)/L_c, \tag{38}$$

where the index 1 refers to boundary values at the emitter-base and at the base-collector junctions, respectively. Returning now to the fundamental equations it follows from (3), (4), (8), and (10) that

$$E = \frac{J_p + qD_p dp/dx}{q\mu_p p} = \frac{J_n - qbD_p dp/dx}{qb\mu_p(p+N_d)}. \tag{39}$$

Solving (39) for $dx$, we obtain:

$$dx = -\frac{qD_p}{J_p}\left\{\frac{(2p+N_d)dp}{p(1-J_n/bJ_p)+N_d}\right\}. \tag{40}$$

Direct integration yields

$$x+K = -\frac{2qD_p\{N_d + p(1-J_n/bJ_p) - N_d\ln\{N_d+p(1-J_n/bJ_p)\}}{J_p(1-J_n/bJ_p)^2} - \frac{qD_pN_d\ln\{N_d+p(1-J_n/bJ_p)\}}{J_p(1-J_n/bJ_p)}, \tag{41}$$

[13] The sign of the currents has been so chosen that a conventional current flowing from the emitter into the base or from the base into the collector is considered positive.

which is an equation for $p(x)$ containing two unknown constants $J_p$ and $K$ (a constant of integration). Since the equation is transcendental, it cannot be solved directly for $J_p$. We can obtain an approximate solution by neglecting the term $J_n/bJ_p$ relative to unity, a step which we shall justify later. In this manner we obtain

$$x+K\cong(qD_pN_d/J_p)\ln(p+N_d)\\ -2qD_p(p+N_d)/J_p. \quad (42)$$

The two constants may now be evaluated with the use of the usual boundary conditions:

$$x=0, \quad p=p_1=p_b e^{qV_e/kT}, \quad (43)$$

$$x=w, \quad p=p_2=p_b e^{-qV_c/kT}. \quad (44)$$

Eliminating from (42) the expression for $K$ obtained in this manner, we arrive at

$$x=\frac{qD_pN_d}{J_p}\left\{\frac{2(p_1-p)}{N_d}-\ln\left(\frac{1+p_1/N_d}{1+p/N_d}\right)\right\}, \quad (45)$$

$$J_p=\frac{qD_pN_d}{w}\left\{\frac{2(p_1-p_2)}{N_d}-\ln\left(\frac{1+p_1/N_d}{1+p_2/N_d}\right)\right\}. \quad (46)$$

Note that for small injection levels ($p/N_d\ll 1$) Eq. (45) indicates a linear dependence of $p$ on $x$ and Eq. (46) reduces to

$$J_p=qD_p(p_1-p_2)/w. \quad (47)$$

Similarly for high injection levels, $p$ is again a linear function of $x$ and Eq. (46) reduces to

$$J_p=2qD_p(p_1-p_2)/w. \quad (48)$$

We are now in a position to check the magnitude of $J_n/bJ_p$. Equations (38) and (46) indicate that this quantity is of the order of $(1-\gamma)$ and therefore its neglect relative to unity is well justified.

Hence, Eq. (46) represents the desired result for the $J$-$V$ characteristics of the base region for arbitrary injection level and should yield correct values of the conductance parameters apart from a small error of the order of $(1-\alpha)$.

There still remains the problem of demonstrating that $dE/dx\cong 0$ in accordance with the assumption of space charge neutrality [Eq. (8)]. For both low and high injection levels, since $p$ is a linear function of $x$ and since $J_n$ is small, it follows from (39) and the Einstein relation that

$$dE/dx\cong kT(dp/dx)^2/q(p+N_d)^2. \quad (49)$$

For $p/N_d\ll 1$, Eq. (49) becomes

$$dE/dx\cong kTp_1^2/qN_d^2w^2, \quad (50)$$

which quantity must be small compared to the total charge density expressed in appropriate units,

$$\rho=4\pi qN_d/\epsilon. \quad (51)$$

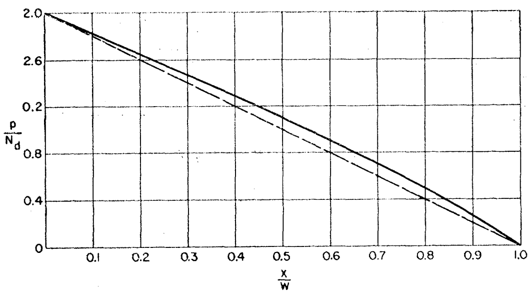


FIG. 1. Concentration of holes in $n$-type base region as a function of distance for intermediate injection level, illustrating largest deviations from linearity (see dashed curve).

Equations (50) and (51) lead to the condition:

$$p_1/N_d\ll 2qw(\pi N_d/\epsilon kT)^{\frac{1}{2}}. \quad (52)$$

Since for relatively pure base material the right-hand side of (52) is of the order of 20 (see Table II) and since $p/N_d\ll 1$ for this case, condition (52) is well fulfilled.

For $p/N_d\gg 1$, Eq. (49) becomes

$$dE/dx\cong kT/qw^2, \quad (53)$$

which must be small compared to the total charge density,

$$\rho=4\pi qp/\epsilon. \quad (54)$$

Equations (53) and (54) lead to the condition:

$$p/N_d\gg\epsilon kT/4\pi q^2w^2N_d. \quad (55)$$

Since numerical evaluation of the right-hand side of (55) yields the value $2\times 10^{-3}$ and since $p/N_d\gg 1$ for this case, condition (55) is also well fulfilled.

For intermediate injection levels (i.e., $p/N_d\sim 1$) the dependence of $p$ on $x$ may be obtained by plotting Eq. (45). We thereby find (see Fig. 1) that at worst only small deviations from linearity occur. Hence $d^2p/dx^2$, while no longer zero, is small and $dE/dx$ will likewise be small for this case as well.

## V. TRANSISTOR CURRENT-VOLTAGE CHARACTERISTICS

### A. General Relations

With the neglect of recombination in the junctions, an excellent approximation for abrupt impurity transitions, the total emitter and collector current densities are given by

$$J_e=J_{pe}+J_{ne}, \quad (56)$$

$$J_c=J_{pc}+J_{nc}. \quad (57)$$

As we have shown in Sec. IV, the larger components of these current densities are given to a good approximation by the expression:

$$J_{pe}=J_{pc}=\frac{qD_pN_d}{w}\left\{\frac{2(p_1-p_2)}{N_d}-\ln\left(\frac{1+p_1/N_d}{1+p_2/N_d}\right)\right\}, \quad (46)$$

where $p_1 = p_b e^{qV_e/kT}$ and $p_2 = p_b e^{-qV_c/kT}$. Also, since Shockley's theory represents an excellent approximation in the emitter and collector regions, the smaller components of these current densities may be written as

$$J_{ne} = qD_n(n_1 - n_e)/L_e \tag{58}$$

where $n_1 = n_e e^{qV_e/kT}(1 + p_1/N_d)$,

$$J_{nc} = -qD_n(n_1 - n_c)/L_c \tag{59}$$

where $n_1 = n_c e^{-qV_c/kT}$. Note that the boundary condition at the emitter-base junction differs from Shockley's by a correction term which becomes important as the injection level in the base region becomes high. Equations (56), (57), (46), (58), and (59) represent the desired result for the $J$-$V$ characteristics of the junction transistor. If emitter and collector regions are well doped, these equations are applicable for any values of emitter current realizable in practice. The only essential feature missing from the simple theory leading to these equations is the small amount of recombination in the base region, which we shall discuss in detail in Sec. VI.

### B. Small-Signal Relations

The small-signal equations are readily derived from the general $J$-$V$ equations by differentiation. The important point made by Early,[4] namely that the base width $w$ is a function of the collector voltage owing to space charge broadening, is readily introduced into the theory at this point by noting that both $J_e$ and $J_c$ are functions of $w$ and that

$$\begin{aligned} dw &= (\partial w/\partial V_c)dV_c + (\partial w/\partial V_e)dV_e \\ &\cong (\partial w/\partial V_c)dV_c. \end{aligned} \tag{60}$$

On carrying out the differentiation, converting from current densities to currents with the use of the conduction area $A$, and neglecting all terms containing $e^{-qV_c/kT}$ since $qV_c/kT \gg 1$, we obtain

$$i_e = g_{11}v_e + g_{12}v_c, \tag{61}$$

$$i_c = g_{21}v_e + g_{22}v_c, \tag{62}$$

where

$$g_{11} = \frac{q^2 A}{kT} e^{qV_e/kT} \left\{ \frac{2D_p p_b}{w} - \frac{D_p p_b}{w} \left( \frac{1}{1 + (p_b/N_d)e^{qV_e/kT}} \right) + \frac{D_n n_e}{L_e} \left( 1 + \frac{2p_b}{N_d} e^{qV_e/kT} \right) \right\}, \tag{63}$$

$$g_{12} = g_{22} = \left( -\frac{\partial w}{\partial V_c} \right) \left( \frac{qAD_p}{w^2} \right) \times \{ 2p_b e^{qV_e/kT} - N_d \ln(1 + (p_b/N_d)e^{qV_e/kT}) \}, \tag{64}$$

$$g_{21} = \frac{q^2 A}{kT} e^{qV_e/kT} \left\{ \frac{2D_p p_b}{w} - \frac{D_p p_b}{w} \times \left( \frac{1}{1 + (p_b/N_d)e^{qV_e/kT}} \right) \right\}. \tag{65}$$

(Expressions for the quantity $(-\partial w/\partial V_c)$ may be obtained from reference 4.) Equations (61) and (62) are the starting point for transistor circuit theory.

### C. Current Amplification Factor

The current amplification factor is defined by

$$\alpha = \left( \frac{\partial I_c}{\partial I_e} \right)_{V_c = \text{const.}} = \left( \frac{i_c}{i_e} \right)_{v_c = 0} = \frac{g_{21}}{g_{11}}. \tag{66}$$

Thus, it follows from (63) and (65) that

$$\alpha = \frac{2D_p p_b/w - (D_p p_b/w)(1 + (p_b/N_d)e^{qV_e/kT})^{-1}}{2D_p p_b/w - (D_p p_b/w)(1 + (p_b/N_d)e^{qV_e/kT})^{-1} + (D_n n_e/L_e)(1 + (2p_b/N_d)e^{qV_e/kT})}. \tag{67}$$

For small injection levels, (67) reduces to

$$\alpha \cong \{1 + (D_n n_e w/D_p p_b L_e)\}^{-1}, \tag{68}$$

which is just the injection factor,

$$\gamma = J_{pe}/(J_{pe} + J_{ne}), \tag{69}$$

similarly reduced for $p/N_d \ll 1$, $qV_e/kT \gg 1$, and $qV_c/kT \gg 1$.[14] For large injection levels, Eq. (67) becomes

$$\alpha \cong \{1 + (D_n n_e w/D_p N_d L_e)e^{qV_e/kT}\}^{-1}. \tag{70}$$

Thus $\alpha$ decreases with emitter current at high injection levels because of a decrease in injection efficiency.

Since the theory of Sec. IV is based on the assumption of no recombination in the base, the diminution factor,

$$\beta = J_{pc}/J_{pe}, \tag{71}$$

is of course equal to unity in this approximation.

## VI. RECOMBINATION IN THE BASE REGION

### A. Approximate Solution for Arbitrary Injection Level

The volume recombination rate (No./cm$^3$ sec) is obviously given by the expression $(p - p_b)/\tau_p$ while the surface recombination rate (No./cm$^2$ sec) is represented by $s(p - p_b)$. The total recombination current in the base is then evidently given to a good approximation by

$$I_r = qA_s\langle s(p - p_b)\rangle_{\text{Av}} + qAw\langle (p - p_b)/\tau_p\rangle_{\text{Av}}, \tag{72}$$

where the quantities in brackets are to be averaged so

[14] Shockley's interpretation of $\gamma$ as the injection factor and $\beta$ as the diminution factor in the equation $\alpha = \beta\gamma$ tacitly involves these same assumptions.

as to take into account the spatial dependence of the injected carriers. It should be kept in mind that both $s$ and $\tau_p$ may be concentration dependent.

We shall base our estimate of $I_r$ on the solution in the base region without recombination (see Sec. IV), since the recombination process may be considered to be only a small perturbation upon the main flow of minority carriers from emitter to collector. Thus, the value of $p$ to be put into Eq. (72) is that given by Eq. (45). Note that to a good approximation (see Fig. 1 for the worst case) $p$ may be taken to be a linear function of $x$, thereby simplifying greatly the task of averaging.

Equation (72) may then be employed to obtain a first order correction to the hole current arriving at the collector, as the latter current will be smaller than the current injected at the emitter junction by the recombination current, i.e.,

$$J_{pc}=J_{pe}-I_r/A. \tag{73}$$

Thus, Eqs. (72), (73), (45), (46), (58), and (66) permit a calculation of the current amplification factor including the effects of recombination, provided that the concentration dependences of $s$ and $\tau_p$ are known.

In order to bring out clearly the essential features associated with alpha we shall first compute its value on the supposition that the surface recombination velocity and the volume lifetime are independent of minority carrier concentration and have their low injection level values $s_0$ and $\tau_0$, respectively. It is then readily shown with the use of the afore-mentioned equations that

$$1-\alpha=\frac{s_0A_sp_b/2A+wp_b/2\tau_0+(D_nn_e/L_e)(1+(2p_b/N_d)e^{qV_e/kT})}{2D_pp_b/w-(D_pp_b/w)(1+(p_b/N_d)e^{qV_e/kT})^{-1}+(D_nn_e/L_e)(1+(2p_b/N_d)e^{qV_e/kT})}. \tag{74}$$

For low injection levels and for a well-doped emitter region, (74) reduces to

$$1-\alpha\cong\frac{s_0A_sw}{2AD_p}+\frac{w^2}{2D_p\tau_0}+\frac{D_nn_ew}{D_pp_bL_e}, \tag{74a}$$

the terms on the right-hand side evidently representing contributions from surface recombination, volume recombination, and injection, respectively.

Numerical evaluation of the terms of (74a), with the use of the values assumed in Table II, yields

$$\frac{s_0A_sw}{2AD_p}=1.3\times10^{-2},\quad \frac{w^2}{2D_p\tau_0}=5.7\times10^{-4},\quad \frac{D_nn_ew}{D_pp_bL_e}=3.4\times10^{-3}.$$

It is evident that the surface recombination term is much larger than the volume recombination term[15] and is of the correct magnitude to account for observed values of $\alpha$ at low levels.

Plots of $1/(1-\alpha)$ *vs* $I_e$, computed with the use of Eq. (74), are shown in Fig. 2. Curve 1 was obtained with the use of the values of the parameters assumed in Table II; curve 2 results from increased doping of the emitter region relative to case 1 by an order of magnitude; while curve 3 corresponds to such high doping of the emitter that the terms arising from $J_{ne}$ may be neglected. The initial increase in $\alpha$ with emitter current is a consequence of the buildup in the base region of a field which assists the transit of minority carriers. At higher emitter currents $\alpha$ passes through a maximum and then declines with increasing emitter current owing to the decrease in injection efficiency, except for case 3 where the injection efficiency has been deliberately taken to be unity in order to show just the field-aided transit effect. Curves 1 and 2 are sufficiently similar to those experimentally observed[10,11] as to suggest that the theory is adequate in its present simple form. It is of some interest, however, to examine the consequences of the concentration dependences of $s$ and $\tau$.

The dependence of $\tau_p$ on $p$ has already been derived by Shockley and Read[16] based upon a recombination mechanism involving trapping states near the middle of the forbidden band. They find an expression of the form

$$\tau_p=\tau_0\{1+c(p-p_0)\}/\{1+d(p-p_0)\}, \tag{75}$$

where $c$ and $d$ are constants, the latter having the

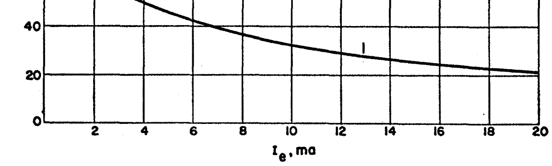


FIG. 2. Current amplification factor *vs* emitter current. Curve 1, assuming values of the parameters given in Table II; curve 2, increasing the doping of the emitter region by a factor of 10; curve 3, assuming an injection efficiency of unity for all currents.

[15] The dominance of surface recombination relative to volume recombination in carefully prepared, relatively pure germanium base material is assured even for the smallest values of $s_0$ that have been thus far realized (see reference 12).

[16] W. Shockley and W. T. Read, Jr., Phys. Rev. 87, 835 (1952).

simple value $1/(n_0+p_0)$.[17] For $c<d$, a condition which is likely to occur in relatively pure material, the lifetime decreases monotonically with injection level. Although this dependence tends to produce a decrease in $\alpha$, it is probably a very small effect relative to those already discussed.

No published information appears to be available concerning the concentration dependence of $s$. However, it is interesting to note that a recent theory of the surface recombination process by Brattain and Bardeen[12] leads to a decrease in $s$ with injection level, somewhat analogous to the case for $\tau$. Their theory is based upon the assumptions that there exist donor-type surface traps near the conduction band and acceptor-type surface traps near the full band, that the recombination rate is limited by the rate of trapping, and that the traps are largely unoccupied (which restricts the validity range to relatively low injection levels). The following expression is derived for the surface recombination rate $U$,

$$U=C(pn-p_0n_0), \tag{76}$$

where $C$ is a constant involving recombination cross sections, trap concentrations, etc.

We may rewrite Eq. (76) in the form

$$U=C\{n_0+p_0+(p-p_0)\}(p-p_0), \tag{77}$$

and then define an effective surface recombination velocity as follows:

$$s=U/(p-p_0). \tag{78}$$

We thereby obtain an expression for $s$ displaying a marked dependence on injection level, namely,

$$s=C\{n_0+p_0+(p-p_0)\}. \tag{79}$$

It is convenient to express Eq. (79) in terms of the low injection level value of the recombination velocity $s_0$; for an $n$-type base region (79) becomes

$$s=s_0\{1+(p-p_b)/N_d\}. \tag{80}$$

If the quantity,

$$\langle s(p-p_b)\rangle_{\mathrm{Av}}=\frac{1}{w}\int_0^w s(p-p_b)dx, \tag{81}$$

is evaluated with the use of (80) subject to the assumptions that $p$ is a linear function of $x$, $qV_e/kT\gg1$, and $qV_c/kT\gg1$, one obtains

$$\langle s(p-p_b)\rangle_{\mathrm{Av}}=(s_0p_1/2)(1+2p_1/3N_d). \tag{82}$$

If (82) is used to evaluate the dependence of $1/(1-\alpha)$ on $I_e$, it is found that the computed curve displays a maximum at much too small a value of current and then falls well below the experimental curve even if an injection efficiency of unity is assumed for all values of $I_e$. This is a consequence of the fact that $s$ is overestimated by the above theory when the injection level becomes appreciable. It is intuitively obvious that the traps must become completely filled at sufficiently high injection levels so that $s$ will reach a limiting value and will not increase indefinitely with $I_e$ as is implied by Eq. (80).

Therefore the possible importance of the concentration dependency of $s$ on the $\alpha$ vs $I_e$ characteristic cannot be assessed until experimental information bearing on this point becomes available. The similarity between the curves of Fig. 2 and the experimental data suggests that it may play only a secondary role.

### B. Three-Dimensional Theory for Very Small Injection Level[18]

For sufficiently small injection levels (see Sec. III), the electric field and the $n$-type currents in the base region become negligible and it suffices to deal only with Eqs. (1) and (3), which yield, for the steady-state case,

$$\operatorname{div}\operatorname{grad}p-(p-p_b)/D_p\tau_p=0. \tag{83}$$

For convenience let us consider a rectangular parallelepiped bounded by the planes $x=0$, $x=w$, $y=\pm a$, $z=\pm a$. The boundary conditions are

$$x=0,\quad p=p_1; \tag{43}$$

$$x=w,\quad p=p_2; \tag{44}$$

$$y=\pm a,\quad \partial p/\partial y\pm s_0(p-p_b)/D_p=0; \tag{84}$$

$$z=\pm a,\quad \partial p/\partial z\pm s_0(p-p_b)/D_p=0. \tag{85}$$

The solution to (83) satisfying these boundary conditions is

$$p-p_b=\sum_{i,j=0}^{\infty}\frac{a_{ij}[(p_1-p_b)\sinh((w-x)/L_{ij})+(p_2-p_b)\sinh(x/L_{ij})]\cos\beta_iy\cos\beta_jz}{\sinh(w/L_{ij})}, \tag{86}$$

where $\beta_ia=\theta_i$ are the roots of

$$\beta_ia\tan\beta_ia=s_0a/D_p, \tag{87}$$

and

$$1/\tau_{ij}=D_p(\beta_i^2+\beta_j^2)+1/\tau_p, \tag{88}$$

$$L_{ij}=(D_p\tau_{ij})^{\frac{1}{2}}, \tag{89}$$

$$a_{ij}=\frac{4\sin\theta_i\sin\theta_j}{(\theta_i+\frac{1}{2}\sin2\theta_i)(\theta_j+\frac{1}{2}\sin2\theta_j)}. \tag{90}$$

[17] It is worth mentioning that Hall's lifetime measurements[5] as a function of injection level for pure germanium may be well fitted by Eq. (75) with $c=8\times10^{-16}$ cm$^3$ and $d=2\times10^{-14}$ cm$^3$, both of which values are entirely reasonable from the point of view of the theory.

[18] The argument of this paragraph represents an extension to a semiconductor of finite width of the theory presented in Appendix V of Shockley's paper (see reference 1).

The current densities of holes may be determined from (86) as follows:

$$J_{pe}=-(qD_p/4a^2)\int_{-a}^{a}\int_{-a}^{a}(\partial p/\partial x)_{x=0}dydz$$
$$=qD_pp_b\sum_{i,j=0}^{\infty}\frac{A_{ij}}{L_{ij}}\left\{(e^{qV_e/kT}-1)\coth\left(\frac{w}{L_{ij}}\right)-(e^{-qV_c/kT}-1)\operatorname{csch}\left(\frac{w}{L_{ij}}\right)\right\}, \quad (91)$$

$$J_{pc}=-(qD_p/4a^2)\int_{-a}^{a}\int_{-a}^{a}(\partial p/\partial x)_{x=w}dydz$$
$$=qD_pp_b\sum_{i,j=0}^{\infty}\frac{A_{ij}}{L_{ij}}\left\{(e^{qV_e/kT}-1)\operatorname{csch}\left(\frac{w}{L_{ij}}\right)-(e^{-qV_c/kT}-1)\coth\left(\frac{w}{L_{ij}}\right)\right\}, \quad (92)$$

where

$$A_{ij}=\frac{4\sin^2\theta_i\sin^2\theta_j}{\theta_i^2\theta_j^2[1+(\sin2\theta_i)/2\theta_i][1+(\sin2\theta_j)/2\theta_j]}. \quad (93)$$

Note that for $s_0=0$, $\theta_i=0$, $A_{ij}=1$, $\tau_{ij}=\tau_p$, and hence Eqs. (91) and (92) reduce properly to the well known expressions of the simple theory.[1–3]

The diminution factor is given approximately ($qV_e/kT\gg1$, $qV_c/kT\gg1$)[14] by

$$\beta\cong\frac{\sum_{i,j=0}^{\infty}(A_{ij}/L_{ij})\operatorname{csch}(w/L_{ij})}{\sum_{i,j=0}^{\infty}(A_{ij}/L_{ij})\coth(w/L_{ij})}. \quad (94)$$

Equation (94) has been evaluated numerically for various values of $s_0$ with the results shown in Table III. In these computations $\tau_p$ has been taken to be infinite in order to permit comparison with the surface recombination term of the perturbation theory [first term of (74a)], which for the particular geometry considered here, assumes the form $s_0w^2/aD_p$. It may readily be seen from the table that the one-dimensional perturbation theory yields quite acceptable results for values of $\beta$ encountered in present day practice.

## VII. FREQUENCY RESPONSE

Despite the fact that it is usually a valid procedure to neglect recombination[19] in the base region in computing the response to a periodic signal of varying frequency, nevertheless an enormous simplification similar to that resulting from omission of the recombination term in the steady-state case does not occur in the time dependent case. The reason for this is that in the latter case Eqs. (1) and (2) become

$$\partial p/\partial t=-q^{-1}\operatorname{div}J_p, \quad (95)$$
$$\partial n/\partial t=q^{-1}\operatorname{div}J_n. \quad (96)$$

Hence, it is no longer true that $J_p$ and $J_n$ are constants and as a consequence we are confronted with the formidable differential equation (14) or (15).

We shall content ourselves with a solution to the problem of the alpha cut-off frequency only for the limiting cases of very small and very large injection levels in a transistor with plane parallel junctions. Since it may readily be shown that the magnitude of the field in the base region varies monotonically with injection level between zero and a value of the order of $kT/qw$, it is to be expected that the alpha cut-off frequency should also vary monotonically between the values found for these limiting cases.

### A. Very Small Injection

For sufficiently small injection levels (see Sec. III) the field and the $n$-type currents in the base become negligible, and Eqs. (1) and (3) yield, for the one-dimensional case (including recombination),

$$\partial^2p/\partial x^2-(p-p_b)/L_p^2=D_p^{-1}\partial p/\partial t. \quad (97)$$

If it is assumed that the voltage applied at the emitter consists of an extremely small sinusoidal signal superimposed on a very small steady bias,

$$V=V_e+V_ae^{i\omega t}\quad(V_a\ll kT/q), \quad (98)$$

the boundary condition for the base region at $x=0$ becomes

$$p=p_b\exp[(q/kT)(V_e+V_ae^{i\omega t})]$$
$$\cong p_1+Pe^{i\omega t}, \quad (99)$$

where $p_1=p_be^{qV_e/kT}$ [by Eq. (43)] and

$$P=qV_ap_1/kT. \quad (100)$$

Since we are interested only in computing alpha, the short circuit current amplification factor, the boundary condition at $x=w$ remains simply

$$p=p_2=p_be^{-qV_c/kT}. \quad (44)$$

TABLE III. Diminution factor *vs* surface recombination velocity and comparison of perturbation theory with three-dimensional theory.

| $s_0$, cm/sec | $1-\beta$ | $s_0w^2/aD_p$ |
|---|---|---|
| 100 | $3.10\times10^{-3}$ | $3.23\times10^{-3}$ |
| 250 | $7.86\times10^{-3}$ | $8.07\times10^{-3}$ |
| 500 | $1.54\times10^{-2}$ | $1.61\times10^{-2}$ |
| 1000 | $3.03\times10^{-2}$ | $3.23\times10^{-2}$ |
| 2500 | $6.96\times10^{-2}$ | $8.07\times10^{-2}$ |
| 5000 | $1.22\times10^{-1}$ | $1.61\times10^{-1}$ |

[19] Unless $s$ becomes too large at very high injection levels; see paragraph B.

The solution of (97) satisfying (44) and (99) is

$$p-p_b=\frac{(p_1-p_b)\sinh((w-x)/L_p)+(p_2-p_b)\sinh(x/L_p)}{\sinh(w/L_p)}+\frac{Pe^{i\omega t}\sinh\{(1+i\omega\tau_p')^{\frac{1}{2}}(w-x)/L_p\}}{\sinh\{(1+i\omega\tau_p')^{\frac{1}{2}}w/L_p\}}. \quad (101)$$

The analogous solution in the emitter ($-\infty<x<0$) is

$$n-n_e=(n_1-n_e)e^{x/L_e}+Qe^{i\omega t}\exp[(1+i\omega\tau_e)^{\frac{1}{2}}x/L_e], \quad (102)$$

where

$$n_1=n_e e^{qV_e/kT},\quad Q=qV_a n_1/kT. \quad (103)$$

Since we are interested only in the ac component of the currents, it is unnecessary to consider the steady electron current in the collector and the first term on the right in each of Eqs. (101) and (102). The resulting expressions for the ac current densities are

$$J_e(\sim)=Pe^{i\omega t}(qD_p/L_p)(1+i\omega\tau_p')^{\frac{1}{2}}\times\coth\{(1+i\omega\tau_p')^{\frac{1}{2}}w/L_p\}+Qe^{i\omega t}(qD_n/L_e)(1+i\omega\tau_e)^{\frac{1}{2}}, \quad (104)$$

$$J_c(\sim)=P_e{}^{i\omega t}(qD_p/L_p)(1+i\omega\tau_p')^{\frac{1}{2}}\times\mathrm{csch}\{(1+i\omega\tau_p')^{\frac{1}{2}}w/L_p\}. \quad (105)$$

Evaluation of $\alpha$ from (104) and (105) yields

$$\alpha=\left[\cosh\left\{\frac{(1+i\omega\tau_p')^{\frac{1}{2}}w}{L_p}\right\}+\frac{D_n n_e L_p}{D_p p_b L_e}\frac{(1+i\omega\tau_e)^{\frac{1}{2}}}{(1+i\omega\tau_p')^{\frac{1}{2}}}\times\sinh\left\{\frac{(1+i\omega\tau_p')^{\frac{1}{2}}w}{L_p}\right\}\right]^{-1}. \quad (106)$$

For frequencies approaching the cut-off value it is not a very good approximation to expand the hyperbolic terms for small arguments; however, we may neglect unity relative to $i\omega\tau_p'$ (which is equivalent to neglecting recombination) and $i\omega\tau_e$ relative to unity.

On defining the following quantities:

$$\eta=(w/L_p)(\omega\tau_p'/2)^{\frac{1}{2}}, \quad (107)$$

$$\zeta=(D_n n_e L_p)/D_p p_b L_e(2\omega\tau_p')^{\frac{1}{2}}, \quad (108)$$

expanding the hyperbolic products, collecting terms, and extracting the modulus, we find that cutoff occurs when

$$2=(\cosh\eta\cos\eta+\zeta\sinh\eta\cos\eta+\zeta\cosh\eta\sin\eta)^2+(\sinh\eta\sin\eta+\zeta\cosh\eta\sin\eta-\zeta\sinh\eta\cos\eta)^2. \quad (109)$$

We shall derive an approximate solution to (109) by neglecting the $\zeta$ terms since $\zeta$ is obviously a small quantity, thus obtaining

$$2=\cosh^2\eta\cos^2\eta+\sinh^2\eta\sin^2\eta=(\cos2\eta+\cosh2\eta)/2. \quad (110)$$

The solution to (110) is $\eta=1.103$ which leads, with the use of (107), to a cut-off frequency,[20,21]

$$\omega_0=2.434D_p/w^2. \quad (111)$$

This approximate root permits the evaluation of $\zeta$ at the cut-off frequency; one finds that

$$\zeta=0.462D_n n_e w/D_p p_b L_e \cong 2\times10^{-3}.$$

Since all of the other terms in (109) are of the order unity, neglect of the $\zeta$ terms is justifiable to an excellent approximation, which means that the influence of the injection factor on the cut-off frequency is negligible.

### B. High Steady Injection Level

It follows from Eqs. (3), (4), (8), (10), and the condition $N_a\ll N_d$ that

$$E=\frac{J_n-qbD_p\,\mathrm{grad}p}{qb\mu_p(p+N_d)}, \quad (112)$$

$$J_p=\frac{J_n-qbD_p(2+N_d/p)\,\mathrm{grad}p}{b(1+N_d/p)}. \quad (113)$$

For a high steady injection level ($p/N_d\gg1$), Eqs. (112) and (113) become

$$E=\frac{J_n-qbD_p\,\mathrm{grad}p}{qb\mu_p p}, \quad (114)$$

and

$$J_p=J_n/b-2qD_p\,\mathrm{grad}p\cong-2qD_p\,\mathrm{grad}p, \quad (115)$$

since as shown in Sec. IV, $J_n/bJ_p\ll1$.

Consider now the superposition on the high-level steady bias an extremely small ac signal. This will not have a significant influence on the electric field (114), but it will introduce a time dependence of $p$. Combining (1) and (115) we obtain

$$2D_p\,\mathrm{div\,grad}p-(p-p_b)/\tau_p=\partial p/\partial t, \quad (116)$$

an equation which will be employed in the three-dimensional form in order to take into account possible transit time dispersion effects due to surface recombination, since $s$ may become large at high injection levels. The boundary conditions are given by Eqs. (99), (44), (84), and (85) with $s_0$ replaced by $s$.

The solution to (116) satisfying the boundary conditions contains two terms, a dc term given by Eq. (86)

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[20] This result, which has already been obtained by Pritchard (reference 21), is 22 percent higher than the commonly cited value (see references 2 and 3).

[21] R. L. Pritchard, Proc. Inst. Radio Engrs. **40**, 1476 (1952).

plus an ac term represented by the expression

$$Pe^{i\omega t}\sum_{i,j=0}^{\infty}\times\frac{a_{ij}\sinh\{(1+i\omega\tau_{ij})^{\frac{1}{2}}(w-x)/L_{ij}\}\cos\beta_i y\cos\beta_j z}{\sinh\{(1+i\omega\tau_{ij})^{\frac{1}{2}}w/L_{ij}\}}, \quad (117)$$

where the symbols are defined by Eq. (87) with $s_0$ replaced by $s$ and by Eqs. (88), (89), and (90) with $D_p$ replaced by $2D_p$.

Since the influence of the injection factor on the frequency response will be small, we shall defer for a moment consideration of the analogous solution in the emitter. Also we need be concerned only with the ac component of the current density at the emitter and collector junctions, which may readily be derived from (117) with the use of (115). Thus, we obtain finally

$$\alpha\cong\frac{\sum_{i,j=0}^{\infty}(A_{ij}/L_{ij})\operatorname{csch}\{(1+i\omega\tau_{ij})^{\frac{1}{2}}w/L_{ij}\}}{\sum_{i,j=0}^{\infty}(A_{ij}/L_{ij})\coth\{(1+i\omega\tau_{ij})^{\frac{1}{2}}w/L_{ij}\}}, \quad (118)$$

with $A_{ij}$ defined by (93).

For values of $s$ which are not too high (i.e., up to $s=2500$ cm/sec for the cross-sectional area assumed in Table II) the quantity $i\omega\tau_{ij}$ will be much greater than unity for all of the terms of the series that contribute significantly to the sums. In this case Eq. (118) reduces to

$$\alpha\cong\operatorname{sech}(i\omega w^2/2D_p)^{\frac{1}{2}}, \quad (119)$$

which leads to a cut-off frequency,

$$\omega_0=4.868D_p/w^2, \quad (120)$$

just twice the value found for very small injection.

If the asymptotic value of $s$ at very high emitter currents proves to be much higher than the above limit, then (118) must be evaluated numerically and the cut-off frequency may become somewhat smaller than the value indicated by (120).

A somewhat more important correction to $\omega_0$ is likely to result from a fall-off in injection efficiency at high currents. Neglecting three dimensional effects in the base, it may readily be shown that an expression for $\alpha$ results given by Eq. (106) except that $D_p$ becomes multiplied by 2, $L_p$ by $\sqrt{2}$, and $n_e$ by the factor

$$1+p_b e^{qV_e/kT}/N_d.$$

The cut-off frequency may then be obtained from this equation by the method outlined in paragraph A. The result for $\omega_0$ is the value given by Eq. (120) slightly diminished by a correction factor which increases with increasing emitter current.

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