For these processes it was found that for small accelerations the only effect produced by the meson fields is a change in the effective mass of the particle. It appears that it is this changed mass which has to be identified with the mass as determined experimentally, and thus for any processes involving only small accelerations there is no distinction between the predictions of the two classical points of view. In particular, this will always be the case for the motion of a charge in the macroscopic electromagnetic fields available in practice. However, there is such a distinction in the case of scattering of electromagnetic radiation for frequencies comparable to or higher than χ, although for such high frequencies the approximations used in deriving Eq. (35) are not very good. Here, as well as in all other cross sections calculated in Secs. IV and V, the two points of view differ only in the terms $M+g^2\chi P$. The values for Q and the angular distribution of any mesonic radiation emitted, although different for the vector and the scalar case, do not introduce any difference between field and action-at-a-distance theory.

If the electromagnetic radiation damping terms are neglected, our results for meson scattering [Eqs. (54)-(56)] agree with those obtained earlier by several authors from both points of view.[20] The radiation damping due to mesonic (and also to electromagnetic) radiation has the effect of producing a maximum in these as well as in all other cross sections calculated in this paper. This maximum occurs at energies too high to agree with that observed for photomeson production; however, much better agreement appears to be possible if other forms of meson theory are used. This will be discussed in the second part of this paper.

[20] These are summarized in Table I of MH; however, a factor 2 was omitted there erroneously in all cross sections for incoming longitudinal mesons.

PHYSICAL REVIEW VOLUME 93, NUMBER 4 FEBRUARY 15, 1954

Ingoing Waves in Final State of Scattering Problems

G. Breit, *Sloane Physics Laboratory, Yale University,* New Haven, Connecticut*

AND

H. A. Bethe, *Institute for Nuclear Studies, Cornell University, Ithaca, New York*

(Received November 9, 1953)

The employment of the ingoing wave modification of plane waves in final states is justified by a general argument.

The fact that the calculation of transition probabilities by means of matrix elements with modified plane waves should be made employing the ingoing rather than outgoing wave modification of the final state has been known for some time and proofs are available in special cases.[1] This fact has assumed more pronounced importance in connection with recent work on bremsstrahlung.[2] In the last-mentioned publication there is an indication of a way of understanding the reason for employing the ingoing wave modification in terms of a temporal sequence of events. The present note is a presentation of relations which give a general reason for the choice of the final state function.

The related and simpler case of scattering by a central field has a sufficient bearing on the problem to justify a preliminary brief discussion. In this case it is customary to employ the boundary condition that at large distances from the center r the time-independent wave function $\psi_{\mathbf{k}}$ have the asymptotic form

$$\psi_{\mathbf{k}}\sim e^{i\mathbf{k}\cdot\mathbf{r}}+f(\theta)e^{ikr}/r, \tag{1}$$

with θ and $\mathbf{k}$ standing, respectively, for colatitude and propagation vector. Among the various ways of justifying the outgoing wave modification represented by the last term in Eq. (1), the one closest to that used below for the main problem consists in the consideration of a wave packet for a time-dependent ψ. At the time $t=0$ this wave packet will be supposed to be moving along the z axis toward the scattering center. The time-dependent function can be represented as

$$\Psi=\int C_{\mathbf{k}}\psi_{\mathbf{k}}e^{-iEt/\hbar}d\mathbf{k}, \tag{2}$$

* Assisted by the joint program of the U. S. Office of Naval Research and U. S. Atomic Energy Commission and the Air Research and Development Command of the U. S. Air Force.

[1] A. Sommerfeld, *Atombau und Spektrallinien* (F. Vieweg and Son, Braunschweig, 1939), Vol. 2, pp. 457 and 502; N. F. Mott and H. S. W. Massey, *Atomic Collisions* (Oxford University Press, London, 1949), second edition, pp. 111–113; also first edition (1933) pp. 82–84; Rose, Biedenharn, and Arfken, Phys. Rev. **85**, 5 (1952), see Sec. III, p. 9. One of the authors (GB) would like to thank Dr. L. C. Biedenharn for a helpful and stimulating discussion of the last reference. General arguments resulting in employment of ingoing waves along lines differing from those of the present note have been given by K. M. Watson, Phys. Rev. **88**, 1163 (1952) as well as by M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953). Watson works out the probability of a spin direction by means of Wigner's time reversal transformation.

[2] L. Maximon and H. A. Bethe, Phys. Rev. **87**, 156 (1952); Bethe, Maximon and Low, Phys. Rev. **91**, 417 (1953).

where E is the energy corresponding to k so that nonrelativistically $E=\hbar^2k^2/2m$, where $m=$mass. Since for $t<0$ the wave packet has not yet hit the field, it behaves like a free space packet and can be made to look as in "Condition I" of Fig. 1. On the other hand, after the collision with the scattering field there is also a scattered wave as in "Condition F" of Fig. 1.

The wave packet is taken to be so far from $r=0$ at $t=0$ that the difference between plane and spherical wave fronts in the packet is negligible. Similarly, for condition F the time t is taken so large that the same condition obtains. For $t=0$ the coefficients $C_{\mathbf{k}}$ can be found by Fourier-analyzing the preassigned $\Psi_{t=0}$ and first constructing the packet out of plane waves $e^{i\mathbf{kr}}$ rather than out of the $\psi_{\mathbf{k}}$. The coefficients so found may be identified with the $C_{\mathbf{k}}$ as may be seen by explicit construction in special cases or from the following argument. Before the packet hits the field there can be no scattered wave. Therefore the presence of the second term in Eq. (1) cannot matter in this epoch. Hence

$$\int C_{\mathbf{k}}e^{i[kr-Et/\hbar]}d\mathbf{k}=0, \quad (t<0). \tag{2'}$$

Since for $t<0$, $z<0$ one has

$$kr=-kz, \quad (t<0), \tag{2''}$$

which shows that as one goes along z through the packet the phases change in opposite ways for the two terms of Eq. (1). This circumstance accounts for the difference in the behavior of the two terms, one of which reproduces the plane-wave packet while the other gives zero by destructive interference. On the other hand for $t>t_{\mathrm{coll}}$ the signs of r and z are the same so that constructive interference occurs in the same locations for both terms of Eq. (1). If one used e^{-ikr} in the second term the relative phase relations would reverse themselves and the process described would not be the intended one.

It is now possible to visualize the actual problem. It is supposed that one deals with solutions of

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial t}+H_0\right)\Psi_0=0, \tag{3}$$

where H_0 is the Hamiltonian of a scattering problem. In Eq. (3) there may be variables additional to the space variables such as the spin coordinate. The function Ψ_0 is taken to be a wave packet such as that of Eq. (2). The problem is now modified through the introduction of a perturbing Hamiltonian H'. The latter may be caused by the incidence of a plane light wave, coupling to radiation oscillators, emission of a nuclear γ ray, etc. The wave equation is

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial t}+H_0+H'\right)\Psi=0. \tag{4}$$

It is convenient to introduce time-independent solu-

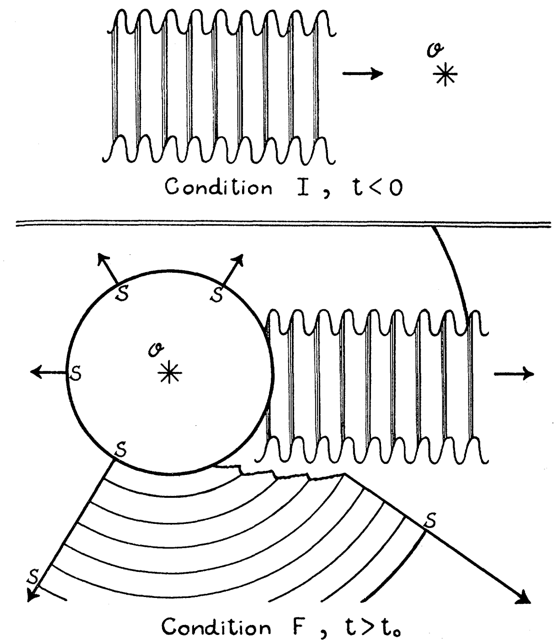

FIG. 1. Illustration of scattering conditions. In condition I the incident train of waves (wave packet) has not yet reached scattering center 0. In condition II the train has passed over 0 and has produced scattered waves S.

tions of Eq. (3) by means of

$$(H_0-E_j)u_{j,s}=0, \tag{5}$$

where the subscript s enumerates the continuum of possibilities for solutions at a fixed energy E_j. It covers the possibilities of waves having different propagation vectors and spin specifications, and in more complicated problems it could be made to enumerate additional variables such as internal excitation quantum numbers. The normalization of the functions $u_{j,s}$ may be chosen so that

$$\int\int u_{j,s}^*(\mathbf{r},\mu)u_{j,s}(\mathbf{r}',\mu')djds=\delta_{\mu\mu'}\delta(\mathbf{r}-\mathbf{r}'), \tag{6}$$

where μ is the spin coordinate. In the region of discrete levels the integrals in the above formula are understood to be replaced by summations. By means of Eq. (6) one can solve Eq. (4) to first order in H' obtaining

$$\Psi(\mathbf{r},\mu,t)=\Psi_0(\mathbf{r},\mu,t)$$
$$-\frac{i}{\hbar}\int_0^t dt\int\int djds(u_{j,s}(\mathbf{r}',\mu'),$$
$$H'\Psi_0(\mathbf{r}',\mu',t))\exp[iE_j(t'-t)/\hbar]u_{j,s}(\mathbf{r},\mu), \tag{7}$$

where the ( ) indicate the scalar product involving integration over $\mathbf{r}'$ and summation over μ'. In the specification of the $u_{j,s}$ the boundary conditions at large r were left unrestricted, the only essential requirement being that of completeness as expressed by Eq. (6). One can satisfy this relation by having the $u_{j,s}$ be asymptotic at large r to either (a) plane+outgoing wave or (b) plane+ingoing wave. Either modification gives a complete set of functions.

The completeness of the set can be verified as follows. The distorted plane waves for no spin may be represented in the form

$$\varphi_{\mathbf{k}}=\Sigma_L i^L(2L+1)P_L((\mathbf{k}\cdot\mathbf{r})/kr)\mathfrak{F}_L(kr)/(kr), \qquad (8)$$

where outside the region of action of H_0

$$\mathfrak{F}_L=(F_L\cos\delta_L+G_L\sin\delta_L)\exp(\pm i\delta_L), \qquad (9)$$

the + sign corresponding to outgoing and the − sign to ingoing waves, and a standard notation of regular and irregular functions F and G being used. Inside the region of action of H_0 the function $\mathfrak{F}_L$ is the continuation of the free-space functions by means of the radial equation. Applying the summation theorem for spherical harmonics to P_L in Eq. (8), one sees that polar coordinate solutions

$$Y_L{}^m(\mathbf{r}/r)\mathfrak{F}_L(kr)/(kr) \qquad (10)$$

are connected with $\varphi_{\mathbf{k}}$ by a unitary transformation, the coefficients of which are to within a constant factor

$$Y_L{}^{m*}(\mathbf{k}/k)d\Omega_{\mathbf{k}}.$$

Here the $Y_L{}^m$ are normalized spherical harmonics of order L and magnetic quantum number m and $d\Omega_{\mathbf{k}}$ is the element of solid angle in the direction of $\mathbf{k}$. The two arguments of the spherical harmonics are indicated by specification of the unit vector defined by the polar angles. If the functions of Eq. (10) form a complete set, then the $\varphi_{\mathbf{k}}$ do also as a consequence of the unitary character of the transformation. It is immediately obvious that in field-free space the functions of Eq. (10) form a complete set. Establishing H_0 adiabatically the set remains complete as long as for every L the radial equations have no discrete spectrum. In order to keep it complete when there is a discrete spectrum, all the discrete-level functions must be included among the $u_{j,s}$ and the integration must be supplemented by a summation. The extension of the above proof of completeness to the Dirac equation is readily made by observing, in analogy to a similar contact transformation used by Dirac[3] in his introduction of the two Dirac-Darwin radial functions, that the transformation to the two radial functions from the four-component spinor is a contact transformation. The $Y_L{}^m$ are now replaced by column matrices containing the $Y_L{}^m$ and the single radial function $\mathfrak{F}_L$ is replaced by a column matrix having for elements two radial functions. In order to keep the set complete negative-energy states have to be included but, on account of the fact that the physical problem with Dirac's equation is concerned with an infinite number of electrons, the negative-energy solutions are removed from the first-order answer by the exclusion principle.

In the above sense one may use Eq. (7) with the $u_{j,s}$ being either ingoing or outgoing wave modifications of plane waves, the choice of sign in Eq. (9) being immaterial for the unitary nature of the transformation.

If the outgoing wave modification is employed, then in the region of large positive $\mathbf{k}\cdot\mathbf{r}$ for $\mathbf{r}$ along $\mathbf{k}$ the $u_{j,s}$ contain terms in e^{ikr} coming from $e^{i\mathbf{k}\cdot\mathbf{r}}$ and also terms in e^{ikr} coming from the outgoing wave as in the second term of Eq. (1). In (7) the second term represents the first-order effect of H' in the form of a superposition of the $u_{j,s}$. For large t standard considerations show that there is a selection by interference arising in $\int dt$ of narrow-energy regions corresponding to conservation of energy for inelastic processes such as photon absorption in inverse bremsstrahlung. The inelastically-scattered electron wave in each energy region is represented as a superposition of the $u_{j,s}$. Since in the direction of $\mathbf{k}_s$ each $u_{j,s}$ contains two terms with the same phase, they both contribute to the wave packet of inelastically-scattered particles. It might be thought at first sight that $e^{i\mathbf{k}\cdot\mathbf{r}}$ and e^{ikr}/kr contribute amounts of different order at large r. This is not the case, however, as may be seen from the fact that $e^{i\mathbf{k}\mathbf{r}}$ when analyzed in Legendre functions of angle consists mainly of terms in e^{ikr}/kr and e^{-ikr}/kr. In fact it is interference of the outgoing wave parts of Eq. (1) with the unscattered wave packet that accounts for the reduction of intensity of the primary beam due to the existence of particles scattered out of the beam in connection with Eq. (2). One cannot neglect, therefore, the outgoing wave part of $u_{j,s}$ in the calculation of the inelastically-scattered wave packet. The calculation by means of Eq. (7) in terms of distorted plane waves with outgoing wave modifications is thus possible but not directly interpretable in terms of a differential cross section.

On the other hand, if the $u_{j,s}$ are taken to be distorted plane waves with *ingoing* wave modifications, then in the direction of each $\mathbf{k}_s$ the phases of the ingoing waves are just opposite to those of the plane wave parts. For the ingoing wave part one has, accordingly, destructive interference as in Eq. (2′). The second part of Eq. (7) gives, therefore, a representation of the inelastically-scattered wave packet in terms of *undistorted* plane waves and is consequently immediately interpretable in terms of the number of particles. The coefficients, given by the scalar products in Eq. (7) must of course be calculated by means of distorted wave functions since in the matrix element the wave function is needed at small values of r', within the interaction zone, so that the mere knowledge of the asymptotic behavior does not suffice. The transition to undistorted plane waves occurs on account of the vanishing of contributions from the ingoing part of the $u_{j,s}$ *outside* the matrix element. It is clear that Eq. (7) gives the standard formula for the transition probability in terms of squares of matrix elements and it follows, therefore, that for the usually desired application to the calculation of the flux of particles in a given direction, one should use the ingoing wave modification.

[3] P. A. M. Dirac, Proc. Roy. Soc. (London) A118, 351 (1928).