Theory of Bremsstrahlung and Pair Production. II. Integral Cross Section for Pair Production

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(Received October 29, 1953)

The differential cross section for bremsstrahlung and pair production at high energies obtained in the preceding paper by Bethe and Maximon has been integrated over all angles, and formulas are given for the integral cross section for all Z. For small Z, the correction to the Born approximation is proportional to Z^2, and the constant of proportionality is given. The correction for heavier elements is somewhat less than the Z^2 law would indicate. It is shown that the correction is associated only with large recoil momenta of the nucleus whereas screening is important only for small recoil momenta; and, therefore, the same correction is valid in the case of complete, incomplete, or no screening. Agreement of these new predictions with observations of pair-production cross section at 88 and 280 Mev is excellent and not unreasonable at 17.6 Mev.

I. INTRODUCTION

It is well known that the predictions of pair-production cross sections made by the Born approximation are not verified for the heavy elements, the discrepancy in the case of Pb being of the order of 10 percent at high energies. The error arises because of the fact that in the Born approximation, the Coulomb interaction is treated as a small perturbation. In the paper immediately preceding this, Bethe and Maximon[1] have made a calculation of the bremsstrahlung process in which the Coulomb interaction is included in the *unperturbed* Hamiltonian and which is valid for all Z as long as the energies of all of the particles involved are large compared with the rest energy of the electron.

The purpose of this paper is to carry out the integration of this differential cross section over all angles, and this is done by a method similar to that used by one of us[2] in determining the influence of screening on bremsstrahlung and pair production. The integration is facilitated by the fact that the energies involved are high and, therefore, the angles between the photon and the electrons are small for significant values of the differential cross section. A brief statement of the results has already been published.[3]

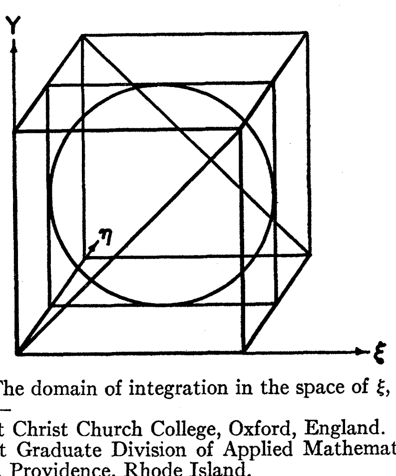

FIG. 1. The domain of integration in the space of ξ, η, and y.

The notation employed is that of Bethe and Maximon,[1] and a natural system of units is used in which $m=c=\hbar=1$.

II. THE INTEGRATION

The most convenient starting point is Eq. (7.14) of Bethe and Maximon which gives for the differential cross section

$$d\sigma=8\left(\frac{\pi a}{\sinh\pi a}\right)^2\frac{a^2}{2\pi}\frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2\frac{\epsilon_1{}^2\epsilon_2{}^2}{k^3}d\epsilon_1$$
$$\times\theta_1 d\theta_1\theta_2 d\theta_2 d\phi\{q^{-4}V^2(x)[k^2(u^2+v^2)\xi\eta$$
$$-2\epsilon_1\epsilon_2(u^2\xi^2+v^2\eta^2)+2(\epsilon_1{}^2+\epsilon_2{}^2)uv\xi\eta\cos\phi]$$
$$+a^2W^2(x)\xi^2\eta^2[k^2(1-(u^2+v^2)\xi\eta)$$
$$-2\epsilon_1\epsilon_2(u^2\xi^2+v^2\eta^2)-2(\epsilon_1{}^2+\epsilon_2{}^2)uv\xi\eta\cos\phi]\}. \quad (1)$$

The variables of integration are θ_1, θ_2, and ϕ. The other quantities can be expressed in terms of these (see Sec. VII of A):

$$u=\epsilon_1\theta_1, \qquad v=\epsilon_2\theta_2, \quad (2)$$

$$\xi=(1+u^2)^{-1}, \quad \eta=(1+v^2)^{-1}, \quad (3)$$

$$q^2=u^2+v^2+2uv\cos\phi, \quad (4)$$

$$x=1-y, \qquad y=q^2\xi\eta. \quad (5)$$

In all of these expressions, it has been assumed that the angles θ_1, θ_2 are small, of order $1/\epsilon$. In Eq. (4), it has been assumed, furthermore, that q is of order 1 rather than of order $1/\epsilon$. There is an important range of angles, *viz.*, $\epsilon_1\theta_1\approx\epsilon_2\theta_2$, $\phi\approx 0$, for which $q^2=O(1/\epsilon^2)$; and then Eq. (4) is useless, not being of sufficient accuracy. We have found it necessary to consider the two domains (I) $q^2=O(1)$ and (II) $q^2=O(1/\epsilon^2)$ separately. Domain I corresponds to $y=O(1)$ and domain II

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[1] H. A. Bethe and L. C. Maximon, *Differential Cross Section for Bremsstrahlung and Pair Production*, quoted as A in the following.
[2] H. A. Bethe, Cambridge Phil. Soc. **30**, 524 (1934).
[3] Handel Davies and H. A. Bethe, Phys. Rev. **87**, 156 (1952).

to $y=O(\epsilon^{-2})$. Classically these two domains correspond to small and large impact parameters, respectively.

The minimum value of q is

$$q_{\min}=\delta=k/2\epsilon_1\epsilon_2. \tag{6}$$

The upper limit of q may be taken to be infinite. From Eqs. (4) and (5), the upper limit of y is 1 (which is clear since x is positive); and a more accurate expansion of q^2 shows that the minimum value of y occurs when $q=\delta$ and then $y=\delta^2$.

Domain I: $y=O(1)$

In this domain, Eq. (4) is valid.

Since the hypergeometric functions $V(x)$ and $W(x)$ are complicated functions [see Eq. (6.24) of Bethe and Maximon], the integration over x should be left to the end. We shall therefore transform the integration variables. As a first step, we transform to the variables u, v, and q.

Using Eqs. (2) and (4), we have

$$\frac{\partial(u,v,q^2)}{\partial(\theta_1,\theta_2,\phi)}=2\epsilon_1{}^2\epsilon_2{}^2\theta_1\theta_2\sin\phi, \tag{7}$$

so that

$$\theta_1 d\theta_1\theta_2 d\theta_2 d\phi=\frac{uvdudvdq^2}{\epsilon_1{}^2\epsilon_2{}^2\{4u^2v^2-(u^2+v^2-q^2)^2\}^{\frac{1}{2}}}. \tag{8}$$

The last term in each of the square brackets of Eq. (1) becomes

$$(\epsilon_1{}^2+\epsilon_2{}^2)\xi\eta(q^2-u^2-v^2), \tag{9}$$

and Eq. (1) goes over into

$$\begin{aligned}d\sigma=8Audu vdvdq^2[4u^2v^2-(u^2+v^2-q^2)^2]^{-\frac{1}{2}}\\ \times\{q^{-4}V^2(x)[(\epsilon_1{}^2+\epsilon_2{}^2)q^2\xi\eta-2\epsilon_1\epsilon_2(u^2\xi-v^2\eta)(\xi-\eta)]\\ +a^2W^2(x)\xi^2\eta^2[k^2-(\epsilon_1{}^2+\epsilon_2{}^2)q^2\xi\eta\\ -2\epsilon_1\epsilon_2(u^2\xi+v^2\eta)(\xi+\eta)]\},\end{aligned} \tag{10}$$

with

$$A=\left(\frac{\pi a}{\sinh\pi a}\right)^2\frac{a^2}{2\pi}\frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2k^{-3}d\epsilon_1. \tag{11}$$

Next we change to the variables ξ, η, and y and get

$$\begin{aligned}d\sigma=2Ad\xi d\eta dy[2y(\xi+\eta-2\xi\eta)-(\xi-\eta)^2-y^2]^{-\frac{1}{2}}\\ \times\{y^{-2}V^2(x)[(\epsilon_1{}^2+\epsilon_2{}^2)y+2\epsilon_1\epsilon_2(\xi-\eta)^2]\\ +a^2W^2(x)[k^2-(\epsilon_1{}^2+\epsilon_2{}^2)y\\ -2\epsilon_1\epsilon_2(\xi+\eta)(2-\xi-\eta)]\}.\end{aligned} \tag{12}$$

The limits of integration are 0 and $2p_1$ for u and 0 and $2p_2$ for v; the upper limits can be taken as infinite since only $u, v=0(1)$, and less give significant contributions to the cross section. From Eq. (3), therefore, the limits of ξ, η are 0 and 1. According to Eq. (5),

$$y=q^2\xi\eta=\xi+\eta-2\xi\eta+2\cos\phi[\xi\eta(1-\xi)(1-\eta)]^{\frac{1}{2}}, \tag{13}$$

and the limits of y are obtained by letting ϕ go from 0 to π and doubling the resulting integral. The volume of integration can be conveniently represented in the $\xi\eta y$ space as in Fig. 1.

The volume of integration is contained between two surfaces in the $\xi\eta y$ space, both of which intersect $\eta=0$ in the straight line $y=\xi$, and the plane $\eta=1$ in the straight line $y=1-\xi$, one of which cuts the plane $\eta=\frac{1}{2}$ in the upper half circle and the other in the lower half circle (Fig. 1). When $\xi=\eta$,

$$y=2\xi(1-\xi)(1-\cos\phi), \tag{14}$$

and near $\phi=0$, y is very small. This is precisely the region in which Eq. (4) for q^2 is invalid and must therefore be avoided in our present integration over domain I. It is necessary to insert then a lower cutoff, $y=y_1$, for the present such that

$$\delta^2\ll y_1\ll 1. \tag{15}$$

The integration can now be further simplified by introducing the symmetrical variables:[4]

$$z=\xi+\eta-1,\quad w=\xi-\eta. \tag{16}$$

Then, after some algebra,

$$\begin{aligned}d\sigma=Adzdwdy[y(1-y)-yz^2-(1-y)w^2]^{-\frac{1}{2}}\\ \times\{y^{-2}V^2[(\epsilon_1{}^2+\epsilon_2{}^2)y+2\epsilon_1\epsilon_2w^2]\\ +a^2W^2[(\epsilon_1{}^2+\epsilon_2{}^2)(1-y)+2\epsilon_1\epsilon_2z^2]\}.\end{aligned} \tag{17}$$

If we now change the order of integration, integrating first over w and z and finally over y, then the limits of integration over w and z will both be given by the vanishing of the square root in the denominator of (17) which arises from the Jacobian (7) of the original transformation. These limits on z and w suggest the introduction of auxiliary polar coordinates χ, ψ by setting

$$z=(1-y)^{\frac{1}{2}}\sin\chi\cos\psi,\quad w=y^{\frac{1}{2}}\sin\chi\sin\psi. \tag{18}$$

Then

$$dzdw=y^{\frac{1}{2}}(1-y)^{\frac{1}{2}}\sin\chi\cos\chi d\chi d\psi,$$

$$[y(1-y)-yz^2-(1-y)w^2]^{\frac{1}{2}}=y^{\frac{1}{2}}(1-y)^{\frac{1}{2}}\cos\chi, \tag{19}$$

and the integral reduces simply to one over the solid angle $\sin\chi d\chi d\psi$. However, it should be noted that the entire domain of w, from $-y^{\frac{1}{2}}$ to $+y^{\frac{1}{2}}$, and the entire domain of z are covered by letting χ go from 0 to $\pi/2$, so that the integral over solid angle gives 2π rather than 4π for the terms independent of χ and ψ. On the other hand, the integral has to be doubled according to the remark after Eq. (13). The averages over solid angle become obviously

$$\langle z^2\rangle=\tfrac{1}{3}(1-y),\quad \langle w^2\rangle=\tfrac{1}{3}y \tag{20}$$

[4] The introduction of $\xi+\eta-1$, rather than $\xi+\eta$ itself, is suggested by the symmetry of the integration volume, Fig. 1, about $\xi=\frac{1}{2}$ and $\eta=\frac{1}{2}$, as well as by the form of the last term in Eq. (12). As is shown by Eq. (17), this procedure leads to the elimination of all linear terms in w and z.

and, after integration,

$$d\sigma=4\pi A dy(\epsilon_1^2+\epsilon_2^2+\tfrac{2}{3}\epsilon_1\epsilon_2)[V^2y^{-1}+a^2(1-y)W^2]. \quad (21)$$

This exceedingly simple formula is valid for all y down to $y=y_1$, as defined by Eq. (15). It can be shown that all neglected terms are of order $1/\epsilon^2$.

Domain II:[5] $y=O(\delta^2)$

It will now be shown that to the order of approximation in which we are working, the contribution of domain II to the integral cross section is the same as in the Born approximation. The proof depends on the behavior of the functions $V(x)$ and $W(x)$ near $x=1$. It is shown in the Appendix that $V(x)$ is convergent up to and including $x=1$ and that the derivative of $V(x)$ which is related to $W(x)$ by

$$V'(x)=a^2W(x) \quad (22)$$

has a logarithmic singularity at $x=1$, with an expansion of $W(x)$:

$$W(x)=-\frac{\sinh\pi a}{\pi a}[\log y+c_0+O(y\log y)]. \quad (23)$$

When $y=O(\delta^2)$, we can therefore replace $V(x)$ by $V(1)$ with an error of $O(\delta^2\log\delta^2)$ which we will neglect in comparison with unity. Further,[6]

$$V(1)=F(ia,-ia,1,1)=\frac{1}{\Gamma(1-ia)\Gamma(1+ia)}=\frac{\sinh\pi a}{\pi a}, \quad (24)$$

which cancels the first factor in Eq. (1) and subsequent formulas for $d\sigma$.

Again the term in Eq. (1) containing $W(x)$ can be neglected altogether in domain II. This is because W^2, unlike V^2, does not have the denominator q^4 in Eq. (1) or the denominator y in Eq. (21). We therefore need simply the integral over W^2, and, the range of integration in domain II being only δ^2, the contribution to the integral from this domain is $O(\delta^2\log\delta^2)$, which we neglect in comparison with unity.

Therefore, the differential cross section reduces in domain II to that of the Born approximation. Now the integration in the Born approximation has been carried out and yields[2] for the *total* cross section (domains I+II),

$$\sigma_B=4\pi A V^2(1)(\epsilon_1^2+\epsilon_2^2+\tfrac{2}{3}\epsilon_1\epsilon_2)(-\log\delta^2-1), \quad (25)$$

where $V^2(1)$ must be inserted into the cross section to remove that factor $(\pi a/\sinh\pi a)^2$ in the definition of A [Eq. (11)]. The contribution from domain I in the Born limit is obtained by putting $a=0$ in (21), which yields $V(x)=1$, and integrating over y from y_1 to 1; this procedure gives

$$\sigma_{BI}=4\pi A V^2(1)(\epsilon_1^2+\epsilon_2^2+\tfrac{2}{3}\epsilon_1\epsilon_2)(-\log y_1). \quad (26)$$

Therefore, the contribution which comes from domain II in the Born approximation and also in our theory is

$$\sigma_{BII}=4\pi A V^2(1)(\epsilon_1^2+\epsilon_2^2+\tfrac{2}{3}\epsilon_1\epsilon_2)(\log y_1-\log\delta^2-1). \quad (27)$$

III. CROSS SECTION

We now add σ_{BII} of Eq. (27) to the contribution of domain I obtained from Eq. (21) by integrating over y from y_1 to 1; this procedure gives

$$\sigma=4\pi A V^2(1)(\epsilon_1^2+\epsilon_2^2+\tfrac{2}{3}\epsilon_1\epsilon_2)\Big\{\log y_1-\log\delta^2-1+[V(1)]^{-2}\Big[\int_{y_1}^1 V^2(x)y^{-1}dy+a^2\int_{y_1}^1(1-y)W^2(x)dy\Big]\Big\}. \quad (28)$$

The integral in (28) may be written in the form

$$\int_0^{1-y_1}\Big(\frac{V^2}{1-x}+a^2xW^2\Big)dx. \quad (29)$$

Using the differential equation for the hypergeometric function V which is given in A [Eq. (8.4)], *viz.*,

$$(1-x)\frac{d}{dx}\Big(x\frac{dV}{dx}\Big)=a^2V, \quad (30)$$

we have, since $dV/dx=a^2W$ [A, Eq. (6.25)],

$$\frac{V}{1-x}=\frac{d}{dx}(xW) \quad (31)$$

and

$$\frac{V^2}{1-x}+a^2xW^2=V\frac{d}{dx}(xW)+xW\frac{dV}{dx}=\frac{d}{dx}(xVW). \quad (32)$$

Thus the integral in Eq. (29) is

$$xVW]_0^{1-y_1}=(1-y_1)V(1-y_1)W(1-y_1)=V(1)W(1-y_1)+O(y_1\log y_1) \quad (33)$$

[5] In a paper on the integral cross section for bremsstrahlung to appear shortly, the integration over domain II will be done without using the variables of reference 2. Making the approximations appropriate in domain II and performing the integrations over w and z, we find that the remaining differential cross section is very similar to Eq. (21) in form and equal to it in the region where domains I and II overlap. The differential cross section (in which one has only to integrate over y) valid in both domains may then be written and the integral over the entire range $\delta^2\leq y\leq 1$ performed at one time.

[6] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, 1950), p. 282.

since from Eq. (15) $y_1 \ll 1$. Further, it is shown in the Appendix, Eqs. (A12), (A13), that for y_1 small

$$W(1-y_1) = -V(1)\left[\log y_1 + 2a^2 \sum_{\nu=1}^{\infty} \frac{1}{\nu(\nu^2+a^2)}\right] + O(y_1 \log y_1). \quad (34)$$

Substituting Eqs. (11), (33), and (34) in Eq. (28) gives

$$\sigma = 2a^2 \frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2 \frac{d\epsilon_1}{k^3}(\epsilon_1^2+\epsilon_2^2+\tfrac{2}{3}\epsilon_1\epsilon_2) \times\left[2\log\frac{2\epsilon_1\epsilon_2}{k} - 1 - 2f(Z)\right], \quad (35)$$

where

$$f(Z) = a^2 \sum_{1}^{\infty} \frac{1}{\nu(\nu^2+a^2)}. \quad (36)$$

A convenient way to evaluate the sum is to write

$$\sum = (1+a^2)^{-1} + \sum_{n=1}^{\infty} (-a^2)^{n-1}[\zeta(2n+1)-1], \quad (37)$$

or, numerically,[7]

$$\sum = (1+a^2)^{-1} + 0.20206 - 0.0369a^2 + 0.0083a^4 - 0.002a^6. \quad (38)$$

This expression is sufficient to evaluate $\sum$ to 4 decimals up to $a=\frac{2}{3}$, which corresponds to uranium. A curve, going up to $a=0.4$, is given by Jackson and Blatt.[8]

For *small a*, we have

$$\sum = 1.2021, \quad f(Z) = 1.2021a^2. \quad (39)$$

This conclusion means that the correction is proportional to a^2 and hence to Z^2. This has been verified in some[9–16] of the many experimental papers on the absorption of gamma rays by pair production. The coefficient is also in good agreement with experiment (see Sec. V).

For larger a, $\sum$ decreases. For Pb, for instance, $\sum = 0.9250$, which is only about $\frac{3}{4}$ of the low-Z value. Therefore, the correction $f(Z)$ to the Born approximation, as defined by Eq. (36), increases with Z somewhat less rapidly than Z^2.

IV. THE EFFECT OF SCREENING

The differential and integral cross sections have been derived under the assumption that the field is a pure Coulomb field. The screening effect of the atomic electrons has been completely neglected. In the Born approximation, the influence of screening on the bremsstrahlung and pair-production processes has been investigated by Bethe,[2] who showed that the screening effect is important only for recoil momenta,

$$q < Z^{\frac{1}{3}}/137. \quad (40)$$

For Pb, therefore, the region of $q > 0.03$ is not seriously affected by the screening.

Because of (8), $y < q^2$; and therefore screening will seriously affect only the region $y < 0.001$. Already in Sec. II (integration over domain II) we have shown that the Born approximation is valid for such small y. Indeed, from the explicit Eq. (28) and from Eq. (23), we find that the contribution of $y < y_1$ to the Coulomb correction $2f(Z)$ is approximately

$$-a^2 \int_0^{y_1} \log^2 y\, dy = -a^2 y_1[\log^2 y_1 - 2\log y_1 + 2]. \quad (41)$$

For $y_1 = 0.001$, this is about $-0.06a^2$, or 2 percent of the total Coulomb correction. The Coulomb correction is therefore not seriously affected by screening.

We can now make a similar statement about the *differential cross section:* the corrections resulting from screening and resulting from the Coulomb effect are independent; wherever one of them is important, the other is not. This can be seen most easily from Eq. (21) for the differential cross section. As we have already seen, screening is important only for small y, let us say, $y < y_1 = 0.001$. V^2 is multiplied by a factor $1/y$, and V^2 itself is of order 1, Eq. (24). On the other hand, W^2 is multiplied by a factor of order a^2 and is itself of order $(\log y)^2$. Therefore, for small y, the term with W^2 is of relative order

$$y(\log y)^2 a^2,$$

which is small compared with 1. In the same approximation, $V(x)$ may be replaced by $V(1)$, and this factor cancels the factor $(\pi a/\sinh \pi a)^2$ in front of the entire cross section. Therefore, *in the limit of small momentum transfer and in the entire region in which screening can be important, the differential cross sections of Eqs. (21) and (17) reduce to the Born approximation result.*

This is physically reasonable: screening, and small momentum transfer generally, imply large impact parameters of the electron. Under these conditions, the electron wave function should be well represented by a plane wave, and, therefore, the Born approximation should be valid. Only for close collisions with q (and therefore y) of the order of 1, and with impact parameters of the order $\hbar/mc$, will the Coulomb correction be important. The soundness of this physical argument is, however, somewhat doubtful because for

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[7] For accurate numerical values of the Riemann zeta function, see J. P. Gram, Kgl. Danske Videnskab. Selskabs Skrifter, Naturvidenskab. math. Afdel, **10**, 313 (1925).
[8] J. D. Jackson and J. M. Blatt, Revs. Modern Phys. **22**, 77 (1950).
[9] G. D. Adams, Phys. Rev. **74**, 1707 (1948).
[10] R. L. Walker, Phys. Rev. **76**, 527 (1949).
[11] R. L. Walker, Phys. Rev. **76**, 1440 (1949).
[12] J. L. Lawson, Phys. Rev. **75**, 433 (1949).
[13] DeWire, Ashkin, and Beach, Phys. Rev. **83**, 505 (1951).
[14] C. R. Emigh, Phys. Rev. **86**, 1028 (1952).
[15] Rosenblum, Schrader, and Warner, Phys. Rev. **88**, 612 (1952).
[16] A. I. Berman, Phys. Rev. **90**, 210 (1953).

bremsstrahlung[17] the Born approximation breaks down just for small q.

Returning to the integral cross section, we may say that in domain I, the only correction to the Born approximation is the Coulomb correction calculated by Bethe and Maximon, and screening is insignificant, whereas in domain II the Coulomb correction is insignificant and results are sufficiently accurate if the screening effect in the Born approximation is treated as by Bethe.[2] Hence the correction calculated in this paper is valid in the case of screening, partial or complete; and in the latter case, using Bethe's result for a Thomas-Fermi model of the atom, we get, in place of Eq. (35),

$$\sigma=4a^2\frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2\frac{d\epsilon_1}{k^3}\{(\epsilon_1{}^2+\epsilon_2{}^2+\tfrac{2}{3}\epsilon_1\epsilon_2)\times[\log(183Z^{-\frac{1}{3}})-f(Z)]+\tfrac{1}{9}\epsilon_1\epsilon_2\},\quad(42)$$

where $f(Z)$ is defined in Eq. (36) and evaluated in Eq. (38).

For partial screening the correction is the same, and Eq. (31) of Bethe and Heitler[18] may be written

$$\sigma=a^2\frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2\frac{d\epsilon_1}{k^3}\times\{(\epsilon_1{}^2+\epsilon_2{}^2)[\phi_1(\gamma)-(4/3)\log Z-4f(Z)]+\tfrac{2}{3}\epsilon_1\epsilon_2[\phi_2(\gamma)-(4/3)\log Z-4f(Z)]\},\quad(43)$$

where $\phi_{1,2}(\gamma)$ are the functions given graphically in that paper.

Integration over ϵ_- gives for the total cross section for pair production

$$\sigma=\frac{28}{9}\frac{Z^2r_0{}^2}{137}\left[\log 2k-\frac{109}{42}-f(Z)\right]\quad(44)$$

without screening. When screening is complete,

$$\sigma=\frac{28}{9}\frac{Z^2r_0{}^2}{137}\left[\log(183Z^{-\frac{1}{3}})-\frac{1}{42}-f(Z)\right].\quad(45)$$

At energies at which measurements have been made, the screening effect is incomplete; but the screening calculation in the Born approximation can be done numerically[18] and the Coulomb correction of this paper subtracted. The term to be subtracted is always

$$\Delta\sigma=\frac{28}{9}\frac{Z^2r_0{}^2}{137}f(Z),\quad(46)$$

no matter whether screening is absent, partial, or complete.

[17] A, Sec. VIII.

[18] H. A. Bethe and W. Heitler, Proc. Roy. Soc. (London) **146A**, 83 (1934).

V. COMPARISON WITH EXPERIMENT

The correction calculated in this paper amounts to a *decrease* of the cross section for pair production, as is in accord with experiment. The magnitude of the theoretical correction is relatively small: although it is of order a^2, as might be expected for a correction to the Born approximation, it has to be compared with a main term which is the logarithm of a large number and is usually of the order 3 to 5: therefore, the Coulomb correction for lead is not $a^2=36$ percent but only about 10 percent. The smallness of the correction as deduced from experiment has often given rise to comment.

One of the remarkable features of the result, Eqs. (35), (36), (42), (43), is that the energy distribution of the pair electrons is essentially unchanged: the major energy dependence is contained in the factor $\epsilon_+{}^2+\epsilon_-{}^2+\frac{2}{3}\epsilon_+\epsilon_-$ in Eq. (35), and this is unaltered; only the slowly variable logarithm, $\log(2\epsilon_1\epsilon_2/k)$, is modified by the subtraction of a constant term, $f(Z)$, which is only 10 percent of the logarithm for Pb. Now if the energy of the positron changes from $0.1k$ to $0.5k$, the log changes only by about 1 unit, or about $\frac{1}{4}$ of its value: therefore the relative cross section at these two values of ϵ_1 is changed by the Coulomb correction by only $2\frac{1}{2}$ percent (in the case of no screening). For complete screening, the energy dependence is not changed at all, as shown by Eq. (42); and for partial screening, the change is less than in the absence of screening. This is in agreement with the experimental results of DeWire and Beach[19] who measured the energy distribution of the positrons and electrons produced by 270-Mev photons in a $\frac{1}{2}$-mil gold foil. In this case, screening is *close* to complete, and the experiments give agreement with the Bethe-Heitler distribution within experimental error, as expected from our theory.

Turning now to the total cross section for pair production, the measurements fall into two classes, *viz.*, those of the pair production itself and those of the total cross section for gamma-ray absorption. The former measurements are all relative,[11,12,14] comparing pair production in different targets; for their interpretation, it is necessary that the pair production for one standard element be known. If this standard is chosen to be a very light element, a considerable fraction of the pairs is formed in the field of the atomic electrons, and the cross section for this process is still only approximately known. (This difficulty is absent for energies below $4mc^2$ where the pair formation in the field of an electron is impossible.)[20] For the purpose of minimizing the combined error from this cause and from the Coulomb correction, the standard should be chosen at intermediate Z, e.g., Al or Fe. In any case, measurements of the relative pair cross section are only of moderate accuracy, of the order of 2 percent.

[19] J. W. DeWire and L. A. Beach, Phys. Rev. **83**, 476 (1951).

[20] Experiments in this region have been done by I. E. Dayton, Phys. Rev. **89**, 544 (1953).

The measurement of total absorption cross section[9,10,12,13,15] has the advantage of higher precision which can be as good as ½ percent. It also has the advantage of being absolute, thus avoiding the reference to a standard which itself may not be accurately known. On the other hand, the total cross section must be corrected for other absorption processes. Fortunately, the most important of these, the Compton effect, is very accurately known from theory. The photoelectric effect in the atom is quite unimportant at high energies where our theory of pair production may be tested; its behavior is reasonably well known theoretically[21] and has been studied experimentally at a few Mev by Colgate.[22]

The most troublesome correction to the absorption coefficient arises from the nuclear photoeffect because neither a quantitative theory nor comprehensive experiments on this effect exist. However, it is known that the photoeffect is strong only for photon energies around 20 Mev; in the neighborhood of its maximum, the cross section is about ½ to 1 barn for lead and about proportional to the atomic weight A for other elements. For Pb, this is equal to 2.5–5 percent of the total gamma-ray absorption cross section; for lighter elements, it may be even more. However, at 100 Mev and over, the nuclear photoeffect is probably negligible.

The most careful correction for nuclear photoeffect, by use of both experimental and theoretical information, was applied by Rosenblum, Schrader, and Warner.[15] They find the correction to be 1–1.5 percent of the total cross section at 10.3 Mev and 2–4 percent at 17.6 Mev.

The experimental results for energies above 10 Mev are listed in Table I and compared with theory. We give the percentage reduction of the pair-production cross section for lead relative to that of the Bethe-Heitler theory. The experimental data of references 9 to 12 are also shown in Fig. 2, in which the solid curve represents the theory of this paper and the dashed curve the Born approximation. Both from the table and the figure, it is seen that the agreement is excellent with the absolute cross section measurements of Lawson at 88 Mev and of DeWire *et al.* at 280 Mev. It is equally good with Emigh's experiments in which the continuous gamma rays from a 300-Mev betatron were used to produce pairs in various targets. The pairs were observed in a cloud chamber, and all pairs below 50 Mev combined energy were rejected; Al was used as the standard target. The theoretical value in this case was calculated on the assumption that the photon spectrum behaves like dk/k between 50 and 300 Mev.

TABLE I. Percentage reduction of pair-production cross section for lead compared with Born approximation.

Energy Mev	This theory	Experiment	Author	Method[a]	Reference
10.3	28	6.8±1.0	Rosenblum *et al.*	Absol.	c
11.0	26	8.8	Adams	Absol.	d
13.7	23	10.0	Adams	Absol.	d
17.6	20	10.4±0.6	Walker[j]	Absol.	e
17.6	20	12.9±1	Walker[b]	Absol.	e
17.6	20	15.5±2	Walker	Rel.	f
17.6	20	13.5±1.0	Rosenblum *et al.*	Absol.	c
19.0	19	13.5	Adams	Absol.	d
88	11.8	10.7±1.6	Lawson	Absol.	g
280	10.0	9.7±1.2	DeWire *et al.*	Absol.	h
50–300	11.2	11.7±1.7	Emigh	Rel.	i

[a] Absolute measurement of total absorption cross section, or relative measurement of pair production in Pb as compared with a light element.
[b] Corrected for nuclear photoeffect as given in reference 15.
[c] See reference 15.
[d] See reference 9.
[e] See reference 10.
[f] See reference 11.
[g] See reference 12.
[h] See reference 13.
[i] See reference 14.
[j] Uncorrected.

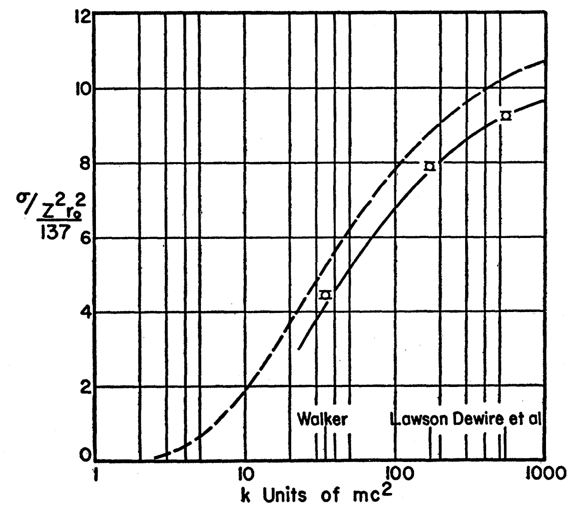

FIG. 2. Total cross section for pair production in Pb. Dotted curve, Born approximation. Solid curve, theory of this paper. Measured points are indicated.

Agreement with Walker's measurements at 17.6 Mev is not good, the discrepancy being greater with his absolute absorption coefficient than with his relative measurement of pair production. As we have discussed above, it is probable that the absolute cross section at this energy contains a substantial contribution from nuclear photoeffect. Indeed, if one applies the correction for nuclear photoeffect given by Rosenblum *et al.*[15] for Pb at 17.6 Mev, then Walker's absolute measurement implies a reduction of the cross section compared with Born approximation of 12.9±1 percent, the error including the uncertainty of the correction for the photonuclear effect. This corrected result is not inconsistent with his relative cross section and is in good agreement with the more recent measurement of Rosenblum *et al.* at the same energy. Agreement with Adams'[9] older measurements is not so good since they were not corrected for photonuclear effect.

Figure 3 gives the deviation from Born approximation

[21] Gladys White, National Bureau of Standards Report NBS-1003 (unpublished).
[22] S. A. Colgate, Phys. Rev. 87, 592 (1952).

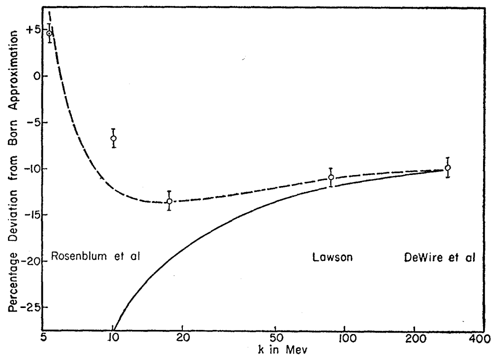

FIG. 3. Percentage deviation from Born approximation of pair-production cross section in Pb. Dashed curve, theory of this paper with empirical second correction [Eq. (48)]. Solid curve, theory of this paper. Measured points are indicated.

as a function of energy for Pb. The solid curve represents the theory of this paper, the points the experiments with probable errors. The good agreement at high energies has already been noted. The discrepancy between theory and experiment below 20 Mev is not surprising. The error in our theory, according to Sec. X of the paper by Bethe and Maximon, should be of the order

$$\delta = a^2 \epsilon^{-1} \log \epsilon. \tag{47}$$

At 17.6 Mev we have $\epsilon = 35$ giving 3.5 percent from Eq. (47). The actual error is about twice this figure, which is quite understandable considering the roughness of the error estimate. That the experimental points should lie *above* our theoretical curve is consistent with the prediction of the theory of Jaeger and Hulme,[23] who made an exact calculation of the pair-production cross section at 1.53 and 2.55 Mev and found results substantially *higher* than the Born approximation, by 100 and 25 percent, respectively. Their predictions are well confirmed by Colgate's[22] measurement of the absolute total cross section at 2.62 Mev and especially by Dayton's[20] measurement of the relative pair production by various elements at 1.33 and 2.62 Mev. Therefore, one must expect that the correction to the Born approximation crosses zero at some energy. This has been realized by many authors, including Rosenblum *et al.*,[15] who put the crossing point at about 6 Mev. All experiments on Pb above 5 Mev are well represented by the empirical formula

$$\sigma_{\text{pair}} = \sigma_{BH} - 4.0 + 46/\epsilon, \tag{48}$$

where σ_{BH} is the Bethe-Heitler cross section, 4.0 the correction derived in this paper, and the last term an empirical second correction. All cross sections are given in barns. Equation (48) is represented by the dashed line in Fig. 3. The equation is no longer good at energies below 5 Mev.

In summary, we can say that the agreement of our theory with experiment is very satisfactory at high energy (88 Mev and up) and that below 20 Mev deviations exist as should be expected.

ACKNOWLEDGMENT

We are indebted to F. J. Dyson for the prevention of a mistake, to R. R. Wilson for helpful discussion, and one of us (H.D.) to the Commonwealth Fund of New York for financial support.

APPENDIX

The arguments involved in the above calculation require a knowledge of the behavior of the hypergeometric functions $V(x)$ and $W(x)$ between $x=0$ and $x=1$.

$V(x) = F(ia, -ia; 1; x)$ is convergent at all points in this range, but $W(x)$ is divergent for $x=1$, and the series converges very slowly near $x=1$.

There is a connection[24] between the hypergeometric functions of z and $1-z$,

$$\begin{aligned}\Gamma(c-a)\Gamma(c-b)\Gamma(a)\Gamma(b)F(a,b;c;z) \\ = \Gamma(a)\Gamma(b)\Gamma(c)\Gamma(c-a-b)F(a, b; a+b-c+1; 1-z) \\ + \Gamma(c)\Gamma(c-a)\Gamma(c-b)\Gamma(a+b-c)(1-z)^{c-a-b} \\ \times F(c-a, c-b; c-a-b+1; 1-z),\end{aligned} \tag{A1}$$

which is valid if $c-a-b$ is not an integer.

We wish to investigate $W(x) = F(1+ia, 1-ia; 2; x)$, for which the above equation is not valid. We proceed in the same way as Whittaker and Watson:[24]

$$\begin{aligned}\frac{\Gamma(1+ia)\Gamma(1-ia)}{\Gamma(2)} F(1+ia, 1-ia; 2; z) \\ = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(1+ia+s)\Gamma(1-ia+s)\Gamma(-s)}{\Gamma(2+s)}(-z)^s ds.\end{aligned} \tag{A2}$$

Now, by Barnes' lemma,[24]

$$\begin{aligned}\Gamma(1+ia+s)\Gamma(1-ia+s)/\Gamma(2+s) \\ = \frac{1}{\Gamma(1+ia)\Gamma(1-ia)} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(1+ia+t) \\ \times \Gamma(1-ia+t)\Gamma(s-t)\Gamma(-t)dt,\end{aligned} \tag{A3}$$

where the path of integration keeps the poles of $\Gamma(s-t)\Gamma(-t)$ on the right and the poles of $\Gamma(1+ia+t)\Gamma(1-ia+t)$ on the left.

Hence,

$$\begin{aligned}\Gamma^2(1+ia)\Gamma^2(1-ia)F(1+ia, 1-ia; 2; z) \\ = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left\{\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(1+ia+t)\Gamma(1-ia+t) \right. \\ \left. \times \Gamma(s-t)\Gamma(-t)dt\right\}\Gamma(-s)(-z)^s ds.\end{aligned} \tag{A4}$$

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[23] J. C. Jaeger and H. R. Hulme, Proc. Roy. Soc. (London) **A153**, 443 (1936); J. C. Jaeger, Nature **148**, 86 (1941).

[24] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, 1950), fourth edition, p. 290.

If the order of integration is interchanged,

$$\Gamma^2(1+ia)\Gamma^2(1-ia)F(1+ia,1-ia;2;z)$$
$$=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(1+ia+t)\Gamma(1-ia+t)\Gamma(-t)\times\left\{\frac{1}{2\pi i}\int_{i\infty}^{i\infty}\Gamma(s-t)\Gamma(-s)(-z)^s ds\right\}dt \tag{A5}$$
$$=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(1+ia+t)\Gamma(1-ia+t)\times\Gamma^2(-t)(1-z)^t dt. \tag{A6}$$

This integral is evaluated by completing the contour with a semicircle to the right. The only poles within the contour are double poles at $t=0, 1, 2\cdots$. The contribution of the semicircle to the integral goes to zero in the limit of infinite radius. We must find the residue of the integrand at a pole $t=n$. We write $t=n+\zeta$, and, expanding,

$$\Gamma(1+ia+t)\Gamma(1-ia+t)\Gamma^2(-t)(1-z)^t$$
$$=[\Gamma(1+ia+n)+\zeta\Gamma'(1+ia+n)][\Gamma(1-ia+n)+\zeta\Gamma'(1-ia+n)]\frac{\pi^2}{\sin^2\pi\zeta}[\Gamma(1+n)+\zeta\Gamma'(1+n)]^{-2}\times[(1-z)^n+\zeta\log(1-z)(1-z)^n]. \tag{A7}$$

The coefficient of $1/\zeta$ is

$$\frac{\Gamma(1+ia+n)\Gamma(1-ia+n)}{\Gamma^2(1+n)}(1-z)^n\left[\frac{\Gamma'(1+ia+n)}{\Gamma(1+ia+n)}+\frac{\Gamma'(1-ia+n)}{\Gamma(1-ia+n)}-2\frac{\Gamma'(1+n)}{\Gamma(1+n)}+\log(1-z)\right]. \tag{A8}$$

Hence,

$$\Gamma^2(1+ia)\Gamma^2(1-ia)F(1+ia,1-ia;2;z)$$
$$=-\sum_{n=0}^{\infty}\frac{\Gamma(1+ia+n)\Gamma(1-ia+n)}{\Gamma^2(1+n)}(1-z)^n\times\left\{\frac{\Gamma'(1+ia+n)}{\Gamma(1+ia+n)}+\frac{\Gamma'(1-ia+n)}{\Gamma(1-ia+n)}-2\frac{\Gamma'(1+n)}{\Gamma(1+n)}+\log(1-z)\right\}. \tag{A9}$$

Remembering that

$$\Gamma(1+ia)\Gamma(1-ia)=\frac{\pi a}{\sinh\pi a}=[V(1)]^{-1} \tag{A10}$$

and neglecting terms of order $1-x=y$ or higher, we get

$$W(x)=-V(1)[\log(1-x)+\Psi(1+ia)+\Psi(1-ia)-2\Psi(1)+O(y\log y)], \tag{A11}$$

where $\Psi(n)=\Gamma'(n)/\Gamma(n)$. This may be written in the form

$$W(x)=-V(1)[\log(1-x)+c_0]+O(y\log y). \tag{A12}$$

If we use the expression for the Ψ-function, the constant c_0 is found to be

$$c_0=2a^2\sum_{\nu=1}^{\infty}\frac{1}{\nu(\nu^2+a^2)}. \tag{A13}$$

The Eq. (A12) is used in Eq. (23), and Eq. (A13) in Eq. (34). A practical formula for the sum in Eq. (A13) is given in Eq. (38).