Reduction of an Integral in the Theory of Bremsstrahlung

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A matrix element integral for bremsstrahlung or pair production without Born approximation is reduced to an ordinary hypergeometric function by contour integration methods. The result is stated in Eqs. (1), (2) for bremsstrahlung and in (1′), (2′) for pair production.

In the theoretical calculation of bremsstrahlung and pair production cross sections without Born approximation,[1–4] i.e., without the assumption that the deflecting forces exerted by the nucleus on the electrons are small, a certain fundamental matrix element integral occurs which can be reduced to an ordinary hypergeometric function. The reduction is carried out in this note by contour integral methods. For the bremsstrahlung case the correct matrix element integral[5] and its value in terms of a hypergeometric function are

$$I\equiv\int d\mathbf{r}e^{-\lambda r}\frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{r}F(ia_1,1,ip_1r-i\mathbf{p}_1\cdot\mathbf{r})\times F(ia_2,1,ip_2r+i\mathbf{p}_2\cdot\mathbf{r})$$
$$=\frac{2\pi}{\alpha}e^{-\pi a_1}\left(\frac{\alpha}{\gamma}\right)^{ia_1}\left(\frac{\gamma+\delta}{\gamma}\right)^{-ia_2}\times F\left(1-ia_1,ia_2,1,\frac{\alpha\delta-\beta\gamma}{\alpha(\gamma+\delta)}\right). \quad (1)$$

In the original expression for I, $\mathbf{q}=\mathbf{p}_1-\mathbf{p}_2-\mathbf{k}$, $\mathbf{p}_1$ is the initial momentum of the electron, $\mathbf{p}_2$ is its final momentum, and $\mathbf{k}$ is the momentum of the radiated quantum. $a_1=Ze^2/\hbar v_1$ and $a_2=Ze^2/\hbar v_2$, where v_1 and v_2 are the initial and final electron speeds. λ is a real positive parameter introduced for the purpose of allowing other integrals to be evaluated by differentiation with regard to it. Only infinitesimal values of λ are pertinent in applications of the result. In the final expression for I,

$$\alpha=\tfrac{1}{2}(q^2+\lambda^2),\qquad \beta=\mathbf{p}_2\cdot\mathbf{q}-i\lambda p_2,$$
$$\gamma=\mathbf{p}_1\cdot\mathbf{q}+i\lambda p_1-\alpha,\quad \delta=p_1p_2+\mathbf{p}_1\cdot\mathbf{p}_2-\beta. \quad (2)$$

The pair-production integral will be discussed separately below.

Although a result of this type has been derived previously by Bess[3] in a manner patterned after Sommerfeld[1] and employing transformations to parabolic coordinates and several theorems on Bessel functions, it is hoped that a more direct evaluation of I by the powerful methods of contour integration may suggest ways of evaluating other similar integrals. The factor $e^{-\pi a_1}$ in (1) is unambiguously determined by the present method, whereas it is not by the method of the Bessel function theorems.

The confluent hypergeometric functions involved in I are special in that the second parameter is unity (Laguerre functions) and a convenient representation for them is

$$F(ia,1,z)=\frac{1}{2\pi i}\oint^{(0+,1+)}dt\,t^{ia-1}(t-1)^{-ia}e^{zt}. \quad (3)$$

The integration contour is closed and encircles each of the two points 0 and 1 once anticlockwise. At the point where the contour crosses the real axis to the right of 1 arg$t$ and arg$(t-1)$ are both zero. Let integral representations of this form be inserted into the expression for I and consider carrying out the space integration first:

$$I=-\frac{1}{4\pi^2}\int dt_1\int dt_2t_1^{ia_1-1}(t_1-1)^{-ia_1}\times t_2^{ia_2-1}(t_2-1)^{-ia_2}V(t_1,t_2),$$
$$V=\int d\mathbf{r}\frac{1}{r}\exp\{-\lambda r+i\mathbf{q}\cdot\mathbf{r}+it_1(p_1r-\mathbf{p}_1\cdot\mathbf{r})+it_2(p_2r+\mathbf{p}_2\cdot\mathbf{r})\}. \quad (4)$$

The interchange of order of integrations is justified if the space integration converges uniformly in t_1 and t_2 on the contours, and this will be the case provided the contours are such that

$$-\lambda-2p_1\mathfrak{s}(t_1)-2p_2\mathfrak{s}(t_2)<0 \quad (5)$$

everywhere on them. This in turn is possible for any positive λ, and we so restrict the contours for the time being.

[1] A. Sommerfeld, Ann. Physik **11**, 257 (1931).
[2] H. Wergeland, Phys. Rev. **76**, 184 (1949).
[3] L. Bess, Phys. Rev. **77**, 550 (1950).
[4] L. Maximon and H. Bethe, Phys. Rev. **87**, 156 (1952); H. Davies and H. Bethe, Phys. Rev. **87**, 156 (1952) and the paper accompanying the present paper.
[5] As pointed out by Bethe, Low, and Maximon, Phys. Rev. **91**, 417 (1953), the wave function for the final state must be taken to behave asymptotically like a plane wave plus an *ingoing* spherical wave, whereas the initial state wave function is asymptotically a plane wave plus an outgoing spherical wave. The function with the ingoing spherical wave may be derived from the one with the outgoing spherical wave by taking the complex conjugate and reversing the vector momentum. Then since the complex conjugate of the final state wave function appears in the matrix element, the two functions appearing in the matrix element are alike in all respects except for the sense of the momentum vector.

The space integral V is a standard integral for real t_1 and t_2, having the value

$$V=4\pi[|\mathbf{q}-t_1\mathbf{p}_1+t_2\mathbf{p}_2|^2-(t_1p_1+t_2p_2+i\lambda)^2]^{-1}$$
$$=-2\pi[(\beta+\delta)t_1t_2+(\alpha+\gamma)t_1-\beta t_2-\alpha]^{-1}. \quad (6)$$

Since both members of Eq. (6) are analytic functions of t_1 and t_2 wherever (5) holds, this result is valid for all complex t_1 and t_2 obeying (5) as well.

Next we carry out the t_1 integration with t_2 fixed at some value not in violation of (5). The quantity (6) considered as a function of t_1 has one singularity, a simple pole, at

$$t_1=t^*\equiv\frac{\alpha+\beta t_2}{\alpha+\gamma+(\beta+\delta)t_2}. \quad (7)$$

Is t^* inside or outside the t_1 contour? Assume that $\mathscr{I}(t_2)=0$, a value compatible with (5), and that λ is so small that its square may be neglected. Then calculate the left member of (5) for $t_1=t^*$:

$$-\lambda-2p_1\mathscr{I}(t^*)=\frac{\lambda(At_2^2+2Bt_2+C)}{[\alpha+\gamma+(\beta+\delta)t_2]^2}, \quad (8)$$

where A, B, C are best expressed in terms of the angles θ between $\mathbf{p}_1$ and $\mathbf{p}_2$, ψ_1 between $\mathbf{p}_1$ and $\mathbf{q}$, ψ_2 between $\mathbf{p}_2$ and $\mathbf{q}$ and φ between the planes containing $\mathbf{p}_1$, $\mathbf{q}$, respectively $\mathbf{p}_2$, $\mathbf{q}$: $A=p_1^2p_2^2\sin^2\theta$; $B=p_1^2p_2q(\cos\psi_2-\cos\theta\cos\psi_1)$; $C=p_1^2q\sin^2\psi_1$. The quadratic in the numerator of (8) is positive for all real t_2, for $A\geqslant 0$ and

$$AC-B^2=p_1^4p_2^2q^2\sin^2\psi_1\sin^2\psi_2\sin^2\varphi\geqslant 0. \quad (9)$$

Thus under the assumptions $\mathscr{I}(t_2)=0$ and λ infinitesimal, we find that t^* violates condition (5) and must therefore lie outside the t_1 contour. But the conclusion that t^* lies outside the t_1 contour cannot depend on any special assumptions made in order to deduce the conclusion because all pairs of contours in keeping with (5) yield the same result for I and because I is an analytic function of λ. Hence t^* lies outside the t_1 contour in any case.

The t_1 integrand in (4) is single-valued and $O(1/t_1^2)$ as $t_1\rightarrow\infty$, so that we may expand the complete t_1 contour to infinity, and upon doing so we find that the only contribution to the t_1 integral is from the residue at the pole t^*. We may of course expand the contour thus at this stage, in violation of (5), because the integrand is analytic. The result of the t_1 integration is

$$-2\pi i[\text{residue at } t^*]=\frac{2\pi(t^*)^{ia_1-1}(t^*-1)^{-ia_1}}{\alpha+\gamma+(\beta+\delta)t_2}. \quad (10)$$

The values of $\arg(t^*)$ and $\arg(t^*-1)$ require to be determined. Now for $\lambda=0$ and t_2 on the real interval (0,1) t^* is likewise on the real interval (0,1) because

$$\mathbf{p}_1\cdot\mathbf{q}-\tfrac{1}{2}q^2>0,$$
$$\mathbf{p}_1\cdot\mathbf{p}_2+p_1p_2+\mathbf{p}_1\cdot\mathbf{q}>\mathbf{p}_2\cdot\mathbf{q}+\tfrac{1}{2}q^2>0. \quad (11)$$

These inequalities follow from the conservation laws and the relation between the energy and the momentum of an electron. Furthermore it follows from the arguments above relative to Eq. (8) that for λ positive but small $\mathscr{I}(t^*)$ is negative. Thus t^* lies below the t_1 contour opposite the interval (0,1) and having regard to the complex arguments stipulated for (3), we find $\arg(t^*)-\arg(t^*-1)=\pi+O(\lambda)$ for $0<t_2<1$. Consequently it is convenient to recast (10) into the form

$$2\pi e^{-\pi a_1}(\alpha+\beta t_2)^{ia_1-1}(\gamma+\delta t_2)^{-ia_1}. \quad (12)$$

In virtue of (11) both linear factors in (12) are real and positive on the interval $0<t_2<1$ for vanishing λ; they are both assigned arguments $0+O(\lambda)$ on that interval.

The t_2 integrand has now acquired two new singularities, $t_2=-\alpha/\beta$ and $t_2=-\gamma/\delta$, both of which have negative imaginary parts (as follows from (2) and $p_1>p_2$) and are therefore excluded from the t_2 contour. In general the exclusion of the new singularities from the contours in both contour integrations is the only essential consequence of the requirement for uniform convergence of the space integration, and once the contours are laid so as to exclude the new singularities any stronger restrictions on t_1 and t_2, such as (5), may be disregarded.

I has now been reduced to

$$I=-ie^{-\pi a_1}\int dt_2t_2^{ia_2-1}(t_2-1)^{-ia_2}\times(\alpha+\beta t_2)^{ia_1-1}(\gamma+\delta t_2)^{-ia_1}. \quad (13)$$

Again the integrand is single valued and $O(1/t_2^2)$ as $t_2\rightarrow\infty$, and this makes the integral reducible to one with only three linear factors, i.e., to an ordinary hypergeometric function. We set $t=1/t_2$ and find:

$$I=-ie^{-\pi a_1}\oint^{(-\beta/\alpha+,-\delta/\gamma+)}dt(1-t)^{-ia_2}\times(\alpha t+\beta)^{ia_1-1}(\gamma t+\delta)^{-ia_1}, \quad (14)$$

with the point $t=1$ excluded from the contour. Equation (14) is now to be put into a standard form for the hypergeometric function, and as such a form we take

$$F(a,b,c,x)=\frac{\Gamma(c)\Gamma(b-c+1)}{2\pi i\Gamma(b)}\int_{-\infty}^{(0+,x+)}d\tau\tau^{a-c}\times(1-\tau)^{c-b-1}(\tau-x)^{-a}, \quad (15)$$

where for x real and <1, $\arg\tau=\arg(1-\tau)=\arg(\tau-x)=0$ on the real axis to the right of 0 and x and to the left of 1, and a cut exists in the τ plane from 1 to $+\infty$. The appropriate change of variable for reducing I to the standard form is a linear fractional transformation which takes $t=1$ into $\tau=1$ and either $t=-\beta/\alpha$ or $t=-\delta/\gamma$ into $\tau=0$. Of the two alternatives we choose the latter because it makes x lie on the interval (0,1).

Thus $\tau=(\gamma t+\delta)/(\gamma+\delta)$ and (14) becomes

$$I=\frac{2\pi}{\alpha}e^{-\pi a_1}\left(\frac{\alpha}{\gamma}\right)^{ia_1}\left(\frac{\gamma+\delta}{\gamma}\right)^{-ia_2}\frac{1}{2\pi i}\int^{(0+,x+)}d\tau\tau^{-ia_1}$$
$$\times(1-\tau)^{-ia_2}(\tau-x)^{ia_1-1},\quad(16)$$

where $x=(\alpha\delta-\beta\gamma)/\alpha(\gamma+\delta)$. x ix real and positive but less than 1 for vanishing λ, see (2), (11), and one further inequality, namely,

$$1+\cos\theta-2\cos\psi_1\cos\psi_2\geqslant 0,\quad(17)$$

which follows from the fact that θ, ψ_1 and ψ_2 are sides of a spherical triangle. Since x is real and <1 the arguments of the linear factors in (16) are as required in (15). Identifying the parameters a, b, c, we have the final result stated in (1), (2).

In the case of pair production the corresponding integral to be evaluated and the result are as follows:

$$I'=\int d\mathbf{r}e^{-\lambda r}\frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{r}F(-ia_1,1,ip_1r-i\mathbf{p}_1\cdot\mathbf{r})$$
$$\times F(ia_2,1,ip_2r+i\mathbf{p}_2\cdot\mathbf{r})$$
$$=\frac{2\pi}{\alpha}\left(\frac{\gamma'}{\alpha}\right)^{ia_1}\left(\frac{\alpha}{\alpha+\beta}\right)^{ia_2}$$
$$\times F\left(-ia_1,ia_2,1,\frac{\beta\gamma'-\alpha\delta'}{\gamma'(\alpha+\beta)}\right).\quad(1')$$

Since the derivation of this result is so similar to the derivation for bremsstrahlung, we shall indicate only the differences for the sake of brevity. $\mathbf{q}=\mathbf{k}-\mathbf{p}_1-\mathbf{p}_2$ is the nuclear recoil momentum, $\mathbf{p}_1$ is the positron momentum and $\mathbf{p}_2$ is the electron momentum. The wave functions for both electron and positron are asymptotic to plane waves plus ingoing waves. In the final expression for I', α and β have the same meaning as in (2) and γ' and δ' are given by

$$\gamma'=\mathbf{p}_1\cdot\mathbf{q}-i\lambda p_1+\alpha;\quad \delta'=\mathbf{p}_1\cdot\mathbf{p}_2-p_1p_2+\beta.\quad(2')$$

The space integral now has the form

$$V'=-2\pi[(\beta-\delta')t_1t_2+(\alpha-\gamma')t_1-\beta t_2-\alpha]^{-1}.\quad(6')$$

The pole of the t_1 integrand appears at

$$t_1=t^*\equiv\frac{\alpha+\beta t_2}{\alpha-\gamma'+(\beta-\delta')t_2},\quad(7')$$

and this is again outside the t_1 contour by the same arguments as above. The result of the t_1 integration is now

$$-2\pi i[\text{residue at }t^*]=\frac{2\pi(t^*)^{-ia_1-1}(t^*-1)^{ia_1}}{\alpha-\gamma'+(\beta-\delta')t_2}.\quad(10')$$

For $\lambda=0$ and t_2 on the real interval (0,1), t^* is in this case on the real axis *outside* the interval (0,1) because

$$\mathbf{p}_1\cdot\mathbf{q}+\tfrac{1}{2}q^2>0;\quad \mathbf{p}_2\cdot\mathbf{q}+\tfrac{1}{2}q^2>0;$$
$$\mathbf{p}_2\cdot\mathbf{q}+\tfrac{1}{2}q^2>p_1p_2-\mathbf{p}_1\cdot\mathbf{p}_2-\mathbf{p}_1\cdot\mathbf{q}.\quad(11')$$

Hence $\arg(t^*)-\arg(t^*-1)=0+O(\lambda)$. The convenient form of (10′) is then

$$2\pi(\alpha+\beta t_2)^{-ia_1-1}(\gamma'+\delta' t_2)^{ia_1}.\quad(12')$$

Both linear factors in (12′) are real and positive on the interval $0<t_2<1$ for vanishing λ on account of (11′); they are assigned arguments $0+O(\lambda)$ on that interval.

The analog of Eq. (14) is

$$I'=-i\oint^{(-\beta/\alpha+,-\delta'/\gamma'+)}dt(1-t)^{-ia_2}$$
$$\times(\alpha t+\beta)^{-ia_1-1}(\gamma' t+\delta')^{ia_1}.\quad(14')$$

In order to put this into the standard form (15) we may again make $t=1$ go into $\tau=1$ and make either $t=-\beta/\alpha$ or $t=-\delta'/\gamma'$ go into $\tau=0$. In this case we choose the first alternative because it makes x lie on the interval (0,1). Thus $\tau=(\alpha t+\beta)/(\alpha+\beta)$. Identifying the parameters a, b, c we have the result stated in (1′) and (2′).