Theory of Bremsstrahlung and Pair Production. I. Differential Cross Section

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The differential cross sections for bremsstrahlung and pair production are calculated without the use of the Born approximation, assuming the energy of the electron to be large compared to mc^2 both in initial and final state. The wave functions in initial and final state are essentially those previously proposed by Furry (Sec. II). It is proved in Sec. III that our wave functions agree with the exact ones of Darwin except for terms of relative order a^2/l^2, where $a=Ze^2/\hbar c$ and l the angular momentum, and that this agreement holds for any energy of the electron. An independent proof is given in Sec. IX, showing that the Furry wave functions give the matrix element correctly except for terms of relative order $1/\epsilon$.

In the matrix element for bremsstrahlung, the initial state of the electron must be represented by a plane wave plus an outgoing spherical wave, whereas the final state has an ingoing spherical wave (Sec. IV). In pair production, both electrons contain ingoing spherical waves (Sec. V). This causes essential differences between the cross sections for the two processes.

The cross section for pair production is calculated in Sec. VI; the result consists of the Bethe-Heitler formula multiplied by a relatively simple factor, plus another term of similar structure. A simplified derivation is given, which is valid for the important case of small angles between electrons and quantum (Sec. VII); it provides a useful check of the cross section of Sec. VI. In Sec. VIII, the bremsstrahlung cross section is calculated and found to be the Bethe-Heitler result multiplied by a factor. This factor is different from that encountered in pair production and becomes important only for very small momentum transfer q. In the limit of complete screening, these small q do not contribute and the cross section goes over into that of the Born approximation.

The error in the cross sections calculated in this paper is estimated (Sec. X) to be of order $1/\epsilon$, where ϵ is the energy of the final electron in bremsstrahlung, or that of the less energetic electron in pair production, in units of mc^2. The total cross section for pair production by a quantum of energy k may be in error by $\log k/k$.

I. INTRODUCTION AND GENERAL DESCRIPTION OF METHOD

THE exact calculation of bremsstrahlung and pair production is a problem of long standing. For nonrelativistic energies, the bremsstrahlung problem was solved exactly by Sommerfeld.[1] For relativistic energies, a solution has been obtained only in Born approximation, by Bethe and Heitler.[2] The total cross section for pair production by high-energy x-rays has been repeatedly tested by experiment[3–9] and the Bethe-Heitler cross section was found to be correct for light elements, but too high (by about 10 percent) for heavy elements. This discrepancy is less than might have been expected since the error in the Born approximation should be of the order of magnitude $(Z/137)^2$ which is 36 percent for lead. Still it is reasonable to assume that the discrepancy is due to failure of the Born approximation, and it was attributed to this cause in the experimental papers.

Sommerfeld's success in the nonrelativistic case was due to the fact that the Schrödinger equation for an electron in the Coulomb field can be separated in parabolic coordinates. This provides a wave function for an electron traveling in a definite direction so that, for instance, the problem of Rutherford scattering can be solved directly and in closed form.[10] This is not possible in relativistic theory, whether Dirac or Klein-Gordon; in this case only a separation in polar coordinates is possible. This makes even the simple problem of electron scattering very cumbersome.[11,12]

If we tried to use wave functions separated in polar coordinates for the bremsstrahlung problem, the calculation would become incredibly cumbersome. We should have to calculate the transition matrix element from any initial angular momentum l_1 of the electron to any final l_2. Now the important values of l extend somewhat beyond $l_0=b/\lambda\!\!\!^{-}$, where b is the atomic radius and $\lambda\!\!\!^{-}$ the de Broglie wavelength of the electron divided by 2π; $b/\lambda\!\!\!^{-}$ is about 137ϵ, where ϵ is the electron energy in units of mc^2, so that for a 100-Mev electron l_0 is nearly 30 000 and the number of matrix elements required about 10^9. Worse than that, in each of these matrix elements the retardation factor for the light quantum, $e^{i\mathbf{k}\cdot\mathbf{r}}$, has to be expanded in spherical harmonics, and l_2 terms in this expansion will contribute to the matrix element if $l_2<l_1$. Presumably, these various contributions will interfere partly destructively. The whole calculation would have to be carried out for any pair of initial and final energies, ϵ_1 and ϵ_2. Even if many shortcuts should prove feasible, it is clear that this calculation would be an essentially impossible task.[13]

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[13] It is true that Harvey Hall, Phys. Rev. **38**, 622 (1934), and Revs. Modern Phys. **8**, 358 (1936), was able to solve the related problem of the photoelectric effect in the K shell by considering each angular momentum l_1 of the final electron state separately. However, in this case the angular momentum of the initial state

It is therefore essential to find at least an appropriate solution in parabolic coordinates, sufficiently accurate for the calculation of the radiative matrix element. We shall show that a slight modification of Furry's wave function[14] satisfies this condition. Indeed, it will be shown in Sec. III that this wave function is obtained from the exact wave function (i.e., from the expansion in terms of spherical harmonics times radial functions) if in the latter we neglect[15] $a^2=(Z/137)^2$ as compared with l^2. Thus the large values of l which were disturbing in the solution by expansion in spherical harmonics, turn out to be helpful in this approach. Small values of l are known to be unimportant: both from the Born approximation solution,[2] and from the Weizsäcker-Williams[16] method, it can be shown that impact parameters less than the Compton wavelength, and thus values of l less than ϵ, give relatively little contribution, the contribution in this region being proportional to ldl. The error due to neglect of a^2 would therefore appear to be of order a^2/ϵ^2. Actually, a more detailed consideration of the error in the wave function, along the lines of Sec. III of this paper, shows that the error is of order $a^2/\epsilon\log\epsilon)$. A similar result, a^2/ϵ, will be found from a discussion of the matrix element in Sec. IX. Assuming the latter behavior, the total cross section, integrated over all energies of the final state, will then have an error of order $a^2\epsilon_1^{-1}\log\epsilon_1$ [see Eq. (10.6)]. Thus our theory will be satisfactory for energies of 50 Mev or more (error less than 2 percent) but will give appreciable errors below 20 Mev.

We propose, then, to use the Furry wave function, which was indeed originally developed for the bremsstrahlung problem. The same wave function, for the same purpose, was developed by Sommerfeld and Maue,[17] but neither they nor Furry used it to calculate the bremsstrahlung itself. The first serious attempt at an exact calculation of bremsstrahlung was made by Bess,[18] who used a wave function of similar structure as Furry's but containing one additional term which Bess believed necessary for the purpose. Unfortunately, as Nordsieck pointed out to us,[19] this additional term is incorrect (see Sec. II) and so is, therefore, Bess' result for the cross section. However, Bess' work is still valuable for the integration of the matrix element and also because he wrote the Furry wave function in a much more convenient form than Furry had done, namely,

$$\psi=Ne^{i\mathbf{p}\cdot\mathbf{r}}[1-(i/2\epsilon)\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}]F(ia\epsilon/p;1;ipr-i\mathbf{p}\cdot\mathbf{r}), \quad (1.1)$$

where $\mathbf{p}$ is the momentum of the electron in units of mc, $\epsilon=(p^2+1)^{\frac{1}{2}}$ its energy in units of mc^2, $a=Ze^2/\hbar c$, $\boldsymbol{\alpha}$ the Dirac operator, $\mathbf{r}$ the electron's coordinate in units of $\hbar/mc$, and N a normalization factor. The derivation of (1.1) will be given in Sec. II (see also Secs. III and IX), and we shall use the wave function in this form for both initial and final state.

The matrix element to be calculated is

$$M=\int\psi_2^*\alpha_\lambda\psi_1 e^{-i\mathbf{k}\cdot\mathbf{r}}d\tau, \quad (1.2)$$

where λ is the direction of polarization of the light wave, α_λ the Dirac α matrix in this direction, and the subscripts 1 and 2 refer to initial and final state of the electron. Using (1.1) and neglecting certain terms discussed in Sec. IX, the integral may then be written in the form (6.2), (6.3). (See Sec. VI.) Now the integrals in (6.3) can be evaluated exactly, which is done in Sec. VI, and all other contributions to M can be neglected as is shown in Sec. IX.

It is somewhat surprising that the three terms in (6.2) are all of the same order of magnitude, whereas it might be expected that the term 1 in the wave function (1.1) would give a much greater contribution than the term $(i/2\epsilon)\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}$ which at least *seems* to be of relative order $1/\epsilon$. In other words, I_1 of (6.3) may be expected to be much larger than I_2 and I_3. This is indeed the case, as is shown in Sec. VI and VII; I_1 is of order ϵ, I_2 and I_3 of order 1. Nevertheless, the three integrals give contributions of the *same* order to the matrix element M. This is due to the matrix factors in (6.2): the matrix $(u_2^*\alpha_\lambda u_1)$ is of order $1/\epsilon$ whereas the matrix vector $(u_2^*\alpha_\lambda\boldsymbol{\alpha}u_1)$ has a component of order unity. This makes the contributions of I_1, I_2 and I_3 all of order unity.

The statement about the order of magnitude of the matrix factors can be seen as follows: In the first term of (6.2), the matrix factor $(u_2^*\alpha_\lambda u_1)$ represents essentially the velocity of the electron (in units of c) perpendicular to the direction of propagation of the quantum, $\mathbf{k}$. But the differential cross section is known to be large only if the angles θ_1, θ_2 between the electron directions $\mathbf{p}_1$, $\mathbf{p}_2$ and the quantum $\mathbf{k}$ are small, of order $1/\epsilon$. Then the electron velocities perpendicular to $\mathbf{k}$ will also be small of this order, and so will the matrix factor $(u_2^*\alpha_\lambda u_1)$. On the other hand, in the second and third term in (6.2), we may, for instance, choose the component of the vector $\boldsymbol{\alpha}$ in the direction λ. This makes the matrix factor equal to $(u_2^*u_1)$ and this is nearly equal to unity because the electron momenta in states 1 and 2 are nearly parallel. This proves the statement above. This fact that I_2 and I_3 give contributions of the same order as I_1, although they look at first sight much smaller, has been a major cause for obscuring this entire problem in the past.

The contributions to the matrix element M which are *not* contained in (6.2) might be expected to fall into three classes: (a) terms which contain an integral of

is $l_2=0$ so that only one (actually a few) terms of the expansion of $e^{i\mathbf{k}\cdot\mathbf{r}}$ in spherical harmonics contribute. Therefore, at least the individual matrix elements are rather simple; only the summation of the cross section over l_1 is laborious.

[14] W. H. Furry, Phys. Rev. **46**, 391 (1934).

[15] This proof goes beyond that of Furry, who had to neglect also terms of order $1/\epsilon^2$.

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[17] A. Sommerfeld and A. W. Maue, Ann. Physik **22**, 629 (1935).

[18] L. Bess, Phys. Rev. **77**, 550 (1950).

[19] A. Nordsieck (private communication).

order 1, and a matrix factor of order $1/\epsilon$, (b) terms containing an integral of order $1/\epsilon$ and a matrix factor of order 1, and (c) terms of order $\epsilon^{-2}\log\epsilon$ or smaller. The term which Bess tried to take into account[20] is of type (a), as is shown in Sec. IX-c. Examination shows that there is no term of type (b). The term $\nabla F_2{}^*\nabla F_1$ which naturally must occur is of type (c) and is shown to be small of order $\epsilon^{-2}\log\epsilon$ in Sec. IX-d; this term could not easily be evaluated by analytical methods. Generally, the proof that (6.2) determines the matrix element with sufficient accuracy, is given in Sec. IX, and we regard this proof as a central part of our calculation. The most important consequence of this proof is that it is unnecessary to calculate the wave function to a better accuracy than the Furry wave function.

Once the Furry wave function is established as sufficient, the evaluation of the matrix elements is possible using the method of Bess or a similar one of Nordsieck. The resulting differential cross section differs from that of Bess only by the correction of minor algebraic mistakes and the omission of terms arising from the spurious term in Bess' wave function. It is therefore simpler than that of Bess and more similar to the Bethe-Heitler cross section.

In the accompanying paper by Davies and ourselves, the differential cross section is integrated over angles and furthermore it is shown that screening can be taken into account easily.

II. WAVE FUNCTION

We start with the first-order Dirac equation for an electron in a Coulomb field,

$$(E+Ze^2/\rho)\psi=\beta mc^2\psi-i\hbar c\boldsymbol{\alpha}\cdot\nabla\psi. \tag{2.1}$$

Introducing $\hbar/mc$ as the unit of length, mc^2 as that of energy, $a=Ze^2/\hbar c$ and the other notations as described below Eq. (1.1), we get

$$(\epsilon+a/r-\beta+i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_r)\psi=0, \tag{2.2}$$

where $\boldsymbol{\nabla}_r$ denotes the gradient with respect to the coordinate $\mathbf{r}$, no longer with respect to $\boldsymbol{\varrho}=(\hbar/mc)\mathbf{r}$ as in (2.1). In (2.2), ψ is the usual four-row unicolumnar matrix, and $\boldsymbol{\alpha}$ and β the four-row four-column matrices of the Dirac equation.

Since we wish to derive wave functions which are the relativistic generalization of the exact solution of the Schrödinger equation with a Coulomb potential, we transform (2.2) into a second-order equation which reduces to the Schrödinger equation when spin and relativity effects are neglected, i.e., for $a\ll1$ and $v\ll c$. This may be accomplished by applying the operator $[(\epsilon+a/r)+\beta-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_r]$ to (2.2), which results in

$$[\nabla_r{}^2+p^2+2\epsilon a/r]\psi=[i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_r(a/r)-a^2/r^2]\psi, \tag{2.3}$$

after making use of the commutation relations for $\boldsymbol{\alpha}$ and β. Here $\mathbf{p}$ is the momentum of the electron in units of mc and $\epsilon^2=p^2+1$. The terms on the right-hand side of (2.3) are negligible for $a\ll1$ and $v\ll c$, in which case we are left with the Schrödinger equation

$$[\nabla_r{}^2+p^2+2\epsilon a/r]\psi=0. \tag{2.4}$$

As is well known,[1,10] the exact solution of this is

$$\psi=Ne^{i\mathbf{p}\cdot\mathbf{r}}uF, \tag{2.5}$$

where N is a normalization constant, F is the confluent hypergeometric function

$$F=F(ia\epsilon/p;\,1;\,ipr-i\mathbf{p}\cdot\mathbf{r}), \tag{2.6}$$

and $u=u(\mathbf{p})$, the normalized Dirac matrix coefficient for a free electron of momentum $\mathbf{p}$. This factor, which is independent of $\mathbf{r}$, is the only one by which (2.5) differs from the purely nonrelativistic solution; it insures that (2.5) satisfies also the first order Eq. (2.2) asymptotically for large r. The asymptotic behavior of (2.5), (2.6) is a plane wave plus outgoing spherical waves.

Following Bess[18] and starting with the wave function $Ne^{i\mathbf{p}\cdot\mathbf{r}}uF$, we try to find a solution of the form

$$\psi=Nei^{\mathbf{p}\cdot\mathbf{r}}(1+\Omega)uF, \tag{2.7}$$

where Ω is an operator to be determined. Then, remembering that (2.5) is an exact solution of (2.4), substitution of (2.7) in (2.3) gives

$$(\nabla_r{}^2+2i\mathbf{p}\cdot\boldsymbol{\nabla}_r+2\epsilon a/r)\Omega uF =[ia\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_r(1/r)-a^2/r^2](1+\Omega)uF. \tag{2.8}$$

If we now assume that Ω commutes with the operator $\nabla_r{}^2+2i\mathbf{p}\cdot\boldsymbol{\nabla}_r$, then (2.8) becomes

$$(2\epsilon a/r)\Omega uF-\Omega(2\epsilon auF/r) =[ia\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_r(1/r)-a^2/r^2](1+\Omega)uF. \tag{2.9}$$

Since the simplest operators that commute with $\boldsymbol{\nabla}_r{}^2+2i\mathbf{p}\cdot\boldsymbol{\nabla}_r$ are those which are made up of any number of differentiations with respect to x,y,z and of Dirac matrices, we assume Ω to be of the form

$$\Omega=\textstyle\sum_n\sigma_n,$$

where

$$\sigma_0=a_{01},$$

$$\sigma_1=a_{11}\partial/\partial x+a_{12}\partial/\partial y+a_{13}\partial/\partial z,$$

$$\sigma_2=a_{21}\partial^2/\partial x^2+a_{22}\partial^2/\partial y^2+\cdots+a_{24}\partial^2/\partial x\partial y+\cdots, \tag{2.10}$$

and the a_{nm} are constants or Dirac matrices. In particular, σ_0 would merely affect the asymptotic behavior of the wave function. However, since $u(\mathbf{p})$ is already the correct asymptotic Dirac amplitude, σ_0 must not contain any Dirac operators, but must merely be a constant, and as such can be absorbed in the normalizing factor N. Therefore we may set $\sigma_0=0$ and find that Ω contains at least one differentiation with respect to coordinates.

[20] With the incorrect wave function used by Bess, this term appeared to be of order 1 instead of $1/\epsilon$.

Substituting (2.10) in (2.9) we may attempt to satisfy (2.9) to terms of $O(1/r^2)$, and hence obtain

$$-2\epsilon a[\sigma_1(1/r)]uF = [ia\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_r(1/r)-a^2/r^2]uF+O(1/r^3). \quad (2.11)$$

Mere inspection shows that if we set

$$\sigma_1=-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_r/2\epsilon, \quad (2.12)$$

then the first of the terms on the right is canceled. Furthermore, it may be noted that there is no σ_1 of the form assumed in (2.10) such that $\sigma_1(1/r)\sim 1/r^2$, and thus we cannot satisfy (2.3) to $O(1/r^2)$ with Ω of the form assumed in (2.10).[21] Actually, as will be shown in Sec. IX, it is unnecessary to solve (2.3) with the term involving a^2/r^2 on the right; the solution so far obtained, i.e., with $\Omega=\sigma_1$ as given in (2.12), is sufficient.

We consider now the wave function $Ne^{i\mathbf{p}\cdot\mathbf{r}}(1+\Omega)uF$, where $\Omega=-(i/2\epsilon)\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_r$, and see to what extent it satisfies (2.3), forgetting that we were led to Ω by consideration of the terms in (2.9) which contain the lowest power of $1/r$. Denoting the wave function that satisfies (2.3) *exactly* by ψ, we may write

$$\psi=\psi_a+\psi_b+\psi_c, \quad (2.13)$$

where

$$\psi_a=Ne^{i\mathbf{p}\cdot\mathbf{r}}uF, \quad (2.14)$$

$$\psi_b=-(i/2\epsilon)Ne^{i\mathbf{p}\cdot\mathbf{r}}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}Fu, \quad (2.15)$$

and ψ_c remains undetermined. Thus we wish to see how satisfactory ψ_b is as a correction to ψ_a, i.e., to what extent we can neglect ψ_c. Substituting (2.13), (2.14), (2.15) in (2.3) and noting that

$$[\nabla_r^2+p^2+2a\epsilon/r]\psi_a=0, \quad (2.16)$$

we have

$$[\nabla_r^2+p^2+2\epsilon a/r](\psi_b+\psi_c) = [ia\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_r(1/r)-a^2/r^2](\psi_a+\psi_b+\psi_c). \quad (2.17)$$

Substituting (2.15) and using (2.16), we find

$$[\nabla_r^2+p^2+2\epsilon a/r]\psi_b=ia\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_r(1/r)\psi_a, \quad (2.18)$$

so that

$$[\nabla_r^2+p^2+2\epsilon a/r]\psi_c=ia\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_r(1/r)(\psi_b+\psi_c) - (a^2/r^2)(\psi_a+\psi_b+\psi_c). \quad (2.19)$$

We shall return to (2.19) in Sec. IXc when showing that the contribution to the differential cross section coming from ψ_c may be neglected when compared with that arising from ψ_a and ψ_b.

III. COMPARISON OF THE WAVE FUNCTION WITH THE SOLUTION IN RADIAL WAVES

In this section, we intend to show that our solution to the wave equations (2.14, 2.15) agrees with the exact solution within terms of the order a^2/l^2 for *any* energy. Since the matrix element of bremsstrahlung comes mostly from impact parameters equal to $\hbar/mc$ or larger, the values of l contributing most are of order ϵ or greater. Therefore, the calculations in this section give an added proof that the error in our cross section becomes negligible for large ϵ.

The exact solution of the Dirac equation in a Coulomb field has been given by Darwin[22] in the form of an infinite series. Starting from the Darwin solution, Furry[14] has derived wave functions in closed form by neglecting, in each of the terms of the summation, terms of $O(1/\epsilon^2)$ and of $O(a^2/l^2)$ compared to those kept, where l is the angular momentum. We shall show that the wave function which we have been using here, namely $\psi_a+\psi_b$, Eq. (2.14, 2.15), may be obtained directly from the Darwin solution by neglecting only terms of $O(a^2/l^2)$ in the infinite series, not those of $O(1/\epsilon^2)$. Thus the wave function $\psi_a+\psi_b$ is actually valid for *all* energies, and, moreover, satisfies [up to terms of $O(a^2/l^2)$] the more restrictive first-order Dirac equation as well as the second-order equation.

Since the Darwin solution is given as an expansion in spherical harmonics, we shall first expand $\psi_a+\psi_b$ in spherical harmonics and then show that, apart from a different normalization factor, this is identical with Darwin's series solution when one neglects terms of $O(a^2/l^2)$ in the latter. We shall indicate the steps for the initial state wave function; the procedure for the final state is identical.

From the work of Gordon[23] we have, directly,

$$\psi_a\equiv N_1e^{i\mathbf{p}\cdot\mathbf{r}}uF=u\sum_l(2l+1)i^lL_l(r)P_l(\cos\theta), \quad (3.1)$$

where

$$L_l(r)=e^{\frac{1}{2}\pi a_1}[\Gamma(l+1-ia_1)/(2l+1)!](2pr)^l \times e^{ipr}F(l+1-ia_1;\,2l+2;\,-2ipr), \quad (3.2)$$

and

$$a_1=a\epsilon/p=Ze^2/\hbar v. \quad (3.3)$$

Solving (3.1) for uF and substituting in ψ_b, Eq. (2.15), we obtain

$$\psi_b=-\frac{i\alpha_z u}{2\epsilon}\sum i^le^{\frac{1}{2}\pi a_1}\frac{(2l+1)\Gamma(l+1-ia_1)}{(2l+1)!}(2pr)^le^{ipr} \times\{\cos\theta P_l[ipF(l+1-ia_1;\,2l+2;\,-2ipr) + (l/r)F(l+1-ia_1;\,2l+2;\,-2ipr) - 2ip[(l+1-ia_1)/(2l+2)] \times F(l+2-ia_1;\,2l+3;\,-2ipr)] + (\sin\theta/r)P_l^{(1)}F(l+1-ia_1;\,2l+2;\,-2ipr) - ipP_lF(l+1-ia_1;\,2l+2;\,-2ipr)\}$$

(*continued on next page*)

[21] It should be noted that Bess, in the paper referred to, assumes that (2.9) is satisfied to $O(1/r^2)$ by choosing $\sigma_1=-(i/2\epsilon)\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_r+(a/2\epsilon)\partial/\partial r$. As pointed out by Dr. Nordsieck this is incorrect, for although $(a/2\epsilon)\partial/\partial r(2\epsilon a/r)=-a^2/r^2$, the operator $\partial/\partial r$ does not commute with $\nabla_r^2+2i\mathbf{p}\cdot\boldsymbol{\nabla}_r$.

[22] C. G. Darwin, Proc. Roy. Soc. (London) **A118**, 654 (1928).

[23] W. Gordon, Z. Physik **48**, 180 (1928).

$$
\begin{aligned}
&-i(\alpha_x\cos\varphi+\alpha_y\sin\varphi)(u/2\epsilon)\sum i^l e^{\frac{1}{2}\pi a_1}(2l+1)\\
&\times[\Gamma(l+1-ia_1)/(2l+1)!](2pr)^l e^{ipr}\\
&\times\{\sin\theta P_l{}^{(1)}[ipF(l+1-ia_1;\,2l+2;\,-2ipr)\\
&+(l/r)F(l+1-ia_1;\,2l+2;\,-2ipr)\\
&-2ip[(l+1-ia_1)/(2l+2)]\\
&\times F(l+2-ia_1;\,2l+3;\,-2ipr)]-(\cos\theta/r)P_l{}^{(1)}\\
&\qquad\times F(l+1-ia_1;\,2l+2;\,-2ipr)\},
\end{aligned}
\tag{3.4}
$$

where we have defined $P_l{}^{(1)}=\sin\theta(dP_l/d\cos\theta)$, and note that $\boldsymbol{\alpha}\cdot\mathbf{r}/r=\alpha_x\sin\theta\cos\varphi+\alpha_y\sin\theta\sin\varphi+\alpha_z\cos\theta$ and $\boldsymbol{\alpha}\cdot\mathbf{p}=\alpha_z p$, since the z axis has been chosen in the direction of $\mathbf{p}$.

In order to obtain an expansion of ψ_b in spherical harmonics, we make use of the following recurrence relations involving the Legendre polynomials P_l and the associated Legendre functions $P_l{}^{(1)}$:

$$
\begin{aligned}
\sin\theta P_l{}^{(1)}&=l[P_{l-1}-\cos\theta P_l],\\
(2l+1)\sin\theta P_l&=P_{l+1}{}^{(1)}-P_{l-1}{}^{(1)},\\
(2l+1)\cos\theta P_l&=(l+1)P_{l+1}+lP_{l-1},\\
(2l+1)\cos\theta P_l{}^{(1)}&=lP_{l+1}{}^{(1)}+(l+1)P_{l-1}{}^{(1)},
\end{aligned}
\tag{3.5}
$$

in which we define $P_0{}^{(1)}=0$ and $P_{-1}{}^{(1)}=0$ in order that these recurrence relations will also be valid for $l=0$. We also make use of the following recurrence relations for the confluent hypergeometric function:

$$
\begin{aligned}
&xF(a+1;b+1;x)=b[F(a+1;b;x)-F(a;b;x)],\\
&xaF(a+1;b+1;x)=b(b-1)[F(a;b-1;x)\\
&\qquad\qquad-F(a;b;x)],\\
&aF(a+1;b+1;x)=(a-b)F(a;b+1;x)\\
&\qquad\qquad+bF(a;b;x).
\end{aligned}
\tag{3.6}
$$

The last of these three relations can be derived directly from the first two.

By straightforward but tedious application of (3.5) and (3.6), we may obtain

$$
\begin{aligned}
\psi_b=&(a_1/2\epsilon)\alpha_z u\sum e^{\frac{1}{2}\pi a_1}(2l+1)i^l\\
&\times[\Gamma(l+1-ia_1)/(2l+1)!](2pr)^l(e^{ipr}/r)P_l\\
&\times[F(l+1-ia_1;\,2l+2;\,-2ipr)\\
&+ia_1(l-ia_1)^{-1}F(l-ia_1;\,2l+2;\,-2ipr)]\\
&+(a_1/2\epsilon)(\alpha_x\cos\varphi+\alpha_y\sin\varphi)u\sum e^{\frac{1}{2}\pi a_1}\\
&\times(2l+1)i^l[\Gamma(l+1-ia_1)/(2l+1)!]\\
&\times(2pr)^l(e^{ipr}/r)P_l{}^{(1)}(l-ia_1)^{-1}\\
&\qquad\times F(l-ia_1;\,2l+2;\,-2ipr).
\end{aligned}
\tag{3.7}
$$

Having obtained the expansions of ψ_a and ψ_b in spherical harmonics, we turn now to the Darwin solution as given in Mott and Massey,[24] where the following two components of the wave function are given:

$$
\begin{aligned}
\psi_3&=\sum[(l+1)e^{i\eta_l}G_l+le^{i\eta_{-l-1}}G_{-l-1})i^lP_l,\\
\psi_4&=\sum[-e^{i\eta_l}G_l+e^{i\eta_{-l-1}}G_{-l-1})i^lP_l{}^{(1)}e^{i\varphi},
\end{aligned}
\tag{3.8}
$$

where

$$
\begin{aligned}
e^{i\eta_{-l-1}}G_{-l-1}=&\frac{\Gamma(\rho_l+1-ia_1)ie^{\frac{1}{2}\pi a_1}e^{\frac{1}{2}i\pi(l-\rho_l)}}{2\Gamma(2\rho_l+1)(\rho_l-ia_1)}(2pr)^{\rho_l}\frac{e^{ipr}}{pr}\\
&\times\{-(l-ia_1/\epsilon)F(\rho_l-ia_1;\,2\rho_l+1;\,-2ipr)\\
&+(\rho_l-ia_1)F(\rho_l+1-ia_1;\,2\rho_l+1;\,-2ipr)\},
\end{aligned}
\tag{3.9}
$$

and $\rho_l=(l^2-a^2)^{\frac{1}{2}}$ (positive square root implied). $e^{i\eta_l}G_l$ may be obtained directly from (3.9) by replacing l by $-l-1$ everywhere except in the factor $e^{\frac{1}{2}i\pi(l-\rho_l)}$ which is replaced by $e^{\frac{1}{2}i\pi(l-\rho-l-1)}$. In obtaining (3.9) from Mott and Massey, we have replaced n, $\alpha=Ze^2/\hbar c$, $\gamma=Ze^2/\hbar v$, and $\gamma'=\gamma(1-v^2/c^2)^{\frac{1}{2}}$, by their corresponding symbols in the notation that we have been using, namely l, a, a_1, and a_1/ϵ, respectively. We have also divided G_{-l-1} and G_l (as given in Mott and Massey) by p in order that they have the same asymptotic behavior as $L_l(r)$, given in (3.1). Further, the relation

$$F(a;b;x)=e^xF(b-a;b;-x)$$

has been substituted in the Darwin solution in order that the arguments of the confluent hypergeometric functions be the same as those in (3.1) and (3.2) for ψ_a and ψ_b.

If we assume $a^2/l^2\ll1$ in each term of the sum, then ρ_l and ρ_{-l-1} may be replaced by l and $l+1$, respectively, so that we have then, from (3.9),

$$
\begin{aligned}
e^{-i\eta_{-l-1}}G_{-l-1}=&-\frac{i\,\Gamma(l+1-ia_1)e^{\frac{1}{2}\pi a_1}}{2\;\Gamma(2l+1)(l-ia_1)}(2pr)^l\frac{e^{ipr}}{pr}\\
&\times\{-(l-ia_1/\epsilon)F(l-ia_1;\,2l+1;\,-2ipr)\\
&+(l-ia_1)F(l+1-ia_1;\,2l+1;\,-2ipr)\},
\end{aligned}
\tag{3.10}
$$

and

$$
\begin{aligned}
e^{i\eta_l}G_l=&\frac{\Gamma(l+2-ia_1)e^{\frac{1}{2}\pi a_1}}{2\Gamma(2l+3)(l+1-ia_1)}(2pr)^l\frac{e^{ipr}}{pr}\\
&\times\{(l+1+ia_1/\epsilon)F(l+1-ia_1;\,2l+3;\,-2ipr)\\
&+(l+1-ia_1)F(l+2-ia_1;\,2l+3;\,-2ipr)\}.
\end{aligned}
\tag{3.11}
$$

By substituting (3.10) and (3.11) in (3.8) we may obtain, after numerous applications of the relations given in (3.6),

$$
\begin{aligned}
\psi_3=&\sum(2l+1)(2l+1)!^{-1}i^le^{\frac{1}{2}\pi a_1}\Gamma(l+1-ia_1)(2pr)^l\\
&+e^{ipr}F(l+1-ia_1;\,2l+2;\,-2ipr)P_l\\
&+(a_1p/2\epsilon)(\epsilon+1)^{-1}\sum(2l+1)(2l+1)!^{-1}i^le^{\frac{1}{2}\pi a_1}\\
&\times\Gamma(l+1-ia_1)(2pr)^l(e^{ipr}/r)P_l\\
&\times\{F(l+1-ia_1;\,2l+2;\,-2ipr)\\
&+ia_1(l-ia_1)^{-1}F(l-ia_1;\,2l+2;\,-2ipr)\},
\end{aligned}
\tag{3.12}
$$

[24] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Clarendon Press, Oxford, 1949), second edition, p. 76, Eq. (25); p. 79, Eqs. (35), (37), and (38).

and

$$\psi_4=(a_1p/2\epsilon)(\epsilon+1)^{-1}e^{i\varphi}\sum(2l+1)(2l+1)!^{-1}i^le^{\frac{1}{2}\pi a_1}\times\Gamma(l+1-ia_1)(2pr)^l(e^{ipr}/r)P_l^{(1)}\times(l-ia_1)^{-1}F(l-ia_1;2l+2;-2ipr). \quad (3.13)$$

We note first that, apart from the matrix factors, the terms in the summation for ψ_3 [Eq. (3.12)] are identical to those in (3.1) for ψ_a and in (3.7) for the part of ψ_b with the factor α_z. Further, apart from the matrix factors, the terms in the summation for ψ_4 (Eq. (3.13)) are identical to those with the factors α_x and α_y in the summation for ψ_b in (3.7).

Now, if u represents the state with positive energy and "spin up," then, since we have chosen the z axis in the direction of $\mathbf{p}$, we have $p_x=p_y=0$, $p_z=p$, and

$$u=[1+p^2/(\epsilon+1)^2]^{-\frac{1}{2}}\begin{pmatrix}1\\0\\p/(\epsilon+1)\\0\end{pmatrix}. \quad (3.14)$$

Thus with

$$\alpha_x=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\quad \alpha_y=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix},$$
$$\alpha_z=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}, \quad (3.15)$$

and numbering the Dirac components of our wave function (3.1) by ψ_{a1}, ψ_{a2}, ψ_{a3}, ψ_{a4}, we have, from (3.1), (3.7), (3.12), and (3.13),

$$\psi_{a1}+\psi_{b1}=[1+p^2/(\epsilon+1)^2]^{-\frac{1}{2}}\psi_3,$$
$$\psi_{a2}+\psi_{b2}=[1+p^2/(\epsilon+1)^2]^{-\frac{1}{2}}\psi_4, \quad (3.16)$$

where ψ_3 and ψ_4 refer to the function of Mott and Massey. The wave function used here, $\psi_a+\psi_b$, is, therefore, apart from a difference of normalization factor,[25] identical with that of Darwin after neglecting terms of $O(a^2/l^2)$ in the latter.*

It may be noted that since we have assumed only $a^2/l^2\ll 1$ rather than $a^2\ll 1$ in each of the terms, further corrections to $\psi_a+\psi_b$ may be obtained fairly easily. We need only write

$$\psi=\psi^{(1)}+\Sigma_l[f_l(r,\theta,\phi;a^2)-f_l(r,\theta,\phi;0)], \quad (3.17)$$

where $\psi^{(1)}$ is our old solution of Sec. II, $f_l(r,\theta,\phi;a^2)$ is the component of the exact Darwin solution for a given l, and $f_l(r,\theta,\phi;0)$ is the same component with a^2/l^2 replaced by zero. In practice, the sum is extended over a *finite* number of terms according to the accuracy desired.

IV. THE MATRIX ELEMENT IN BREMSSTRAHLUNG

It is well known[26] that the matrix element for bremsstrahlung is

$$H_{12}'=C\int\psi_2^*\alpha_\lambda e^{-i\mathbf{k}\cdot\mathbf{r}}\psi_1 d\tau, \quad (4.1)$$

where

$$C=-e\hbar c(2\pi/k)^{\frac{1}{2}}, \quad (4.2)$$

and α_λ is the component of the Dirac matrix operator $\boldsymbol{\alpha}$ in the direction of polarization, λ, which is perpendicular to the propagation vector $\mathbf{k}$. The wave functions ψ_1 and ψ_2 of the initial and final state of the electron are taken in the Coulomb potential and are given by the expressions in Secs. II and III.

The wave function of a continuum state, however, is not fully defined until its asymptotic behavior is given. In the case of ψ_1, it is clear that it has to be represented by a plane wave propagating in the direction of the initial momentum $\mathbf{p}_1$, plus outgoing spherical waves. In the past,[1,18,27] ψ_2 has often been chosen in the same manner. That this is incorrect was recognized first by Mott and Massey[28] in a more general context and then by Sommerfeld[29] in relation to our particular problem. A physical argument was given by us in an earlier paper.[30] The correct theory was used for internal conversion by Rose *et al.*[31] who give references to several previous papers in which the incorrect assumption was made, as well as to Rarita and Schwinger[32] who were familiar with the correct treatment.

A mathematical argument which leads to that of Mott and Massey is as follows. The Hamiltonian of our system is

$$H=H_c+H_r+H', \quad (4.3)$$

where H_r is the Hamiltonian of the pure radiation field, H_c that of the electron including its Coulomb interaction with the nucleus, and H' the interaction between electron and radiation. The latter is to be regarded as a small perturbation, and to be treated only in first approximation. The Schrödinger equation is

$$H\Psi=E_1\Psi, \quad (4.4)$$

where E_1 is the energy of the incident electron. In zero order, we have no radiation present, and the wave function therefore satisfies

$$H_c\Psi^{(0)}=E_1\Psi^{(0)}. \quad (4.5)$$

[25] Darwin, Mott, and Massey, take the particle density to be $|\psi_3|^2+|\psi_4|^2$ and consider ψ_1 and ψ_2 as expressed in terms of ψ_3, ψ_4. We take the density to be $|\psi_1|^2+|\psi_2|^2+|\psi_3|^2+|\psi_4|^2$; this explains the difference in normalization.

* *Note added in proof:*—Closer examination shows that our wave function (3.2) differs from Darwin's true wave function by terms of order a^2/l rather than a^2/l^2. This will be shown in a forthcoming paper by one of us (L. C. M.). It has the consequence of making the error in our matrix element of order $a^2/(\epsilon\log\epsilon)$, as mentioned in Sec. I.

[26] See Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1944), second edition, p. 96.

[27] L. Maximon and H. A. Bethe, Phys. Rev. **87**, 156 (1952).

[28] Reference 24, pp. 111–13.

[29] A. Sommerfeld, *Atombau und Spektrallinien* (F. Vieweg and Son, Braunschweig, 1939), Vol. 2, pp. 457 and 502.

[30] Bethe, Maximon, and Low, Phys. Rev. **91**, 417 (1953).

[31] Rose, Biedenharn, and Arfken, Phys. Rev. **85**, 5 (1952).

[32] W. Rarita and J. Schwinger, Phys. Rev. **59**, 436 (1941).

The solution of this equation is the initial electron wave function, ψ_1.

In first order, we have to solve

$$(H_c+H_r-E_1)\Psi^{(1)}=-H'\Psi^{(0)}. \tag{4.6}$$

Now if we take the part of $\Psi^{(1)}$ in which there is a quantum k present, $H_r\Psi^{(1)}=k\Psi^{(1)}$ and

$$(H_c-E_2)\Psi^{(1)}=-H'\Psi^{(0)}=-H'\psi_1, \tag{4.7}$$

since $E_2=E_1-k$. Now if we take the factor of $\Psi^{(1)}$ which describes the electron and denote it by ϕ, then

$$(H_c-E_2)\phi=C'\alpha_\lambda\psi_1 e^{-i\mathbf{k}\cdot\mathbf{r}}\equiv S, \tag{4.8}$$

where C' is a constant closely related to C, Eq. (4.2). The right-hand side of (4.8) will be called "the source" and denoted by S. Our task is now to solve (4.8) by a wave function ϕ which consists exclusively of outgoing spherical waves, without any plane wave, because there is no incident electron having the final momentum $\mathbf{p}_2$.

Writing the left-hand side of (4.8) more explicitly, we seek a solution of

$$[\nabla^2+p_2^2+U(r)]\phi=S, \tag{4.9}$$

where $U(r)$ is the "potential operator" occurring in (2.3), *viz.*,

$$U(r)=2\epsilon a/r-ia\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}(1/r)+a^2/r^2, \tag{4.10}$$

and the solution ϕ is supposed to contain only outgoing spherical waves. Exactly the same problem is solved by Mott and Massey, with somewhat different notation and with the restrictive assumption that

$$rU(r)\to 0, \text{ as } r\to\infty.$$

This restriction is not satisfied by our potential (4.10) but can be satisfied by introducing a screening factor $e^{-\alpha r}$ in (4.10) and letting α go to zero in the end.

Mott and Massey show that ϕ behaves asymptotically, as

$$\phi(\mathbf{r})=-(4\pi r)^{-1}e^{ip_2r}\int d\tau' S(\mathbf{r}')\mathfrak{F}(r',\pi-\Theta), \tag{4.11}$$

where

$$\mathfrak{F}(r',\theta)=\sum(2l+1)i^l e^{i\eta_l}L_l(r')P_l(\theta) \tag{4.12}$$

is the well-known solution of the *homogeneous* wave equation in the given potential, i.e., of (4.9) with zero on the right-hand side. The asymptotic behavior of $\mathfrak{F}$ is the usual one, *viz.*, plane wave plus outgoing spherical waves. L_l is the regular (and real) solution of the radial wave equation and η_l the phase shift for angular momentum l in the given potential. According to (4.11), $\mathfrak{F}$ must be taken in the integrand at the angle $\pi-\Theta$, where Θ is defined by Mott and Massey as the angle between the vectors $\mathbf{r}$ and $\mathbf{r}'$. Now we are interested in the asymptotic behavior of $\phi(\mathbf{r})$ in the direction $\mathbf{p}_2$; therefore we must choose $\mathbf{r}$ in (4.11) in the direction of $\mathbf{p}_2$, and $\pi-\Theta=\chi$ is the angle between $-\mathbf{p}_2$ and $\mathbf{r}'$. Therefore $\mathfrak{F}(r',\pi-\Theta)$ is a function which behaves asymptotically like a plane wave propagating in the direction $-\mathbf{p}_2$, plus the outgoing spherical waves associated with it by the potential $U(r)$; thus,

$$\mathfrak{F}(r,\pi-\Theta)\to e^{-i\mathbf{p}_2\cdot\mathbf{r}}+r^{-1}e^{ip_2r}f(\chi), \tag{4.13}$$

where $\chi=\pi-\Theta$ is the angle between $-\mathbf{p}_2$ and $\mathbf{r}$, and $f(\chi)$ is the amplitude of the wave scattered through an angle χ from the direction of the plane wave, $-\mathbf{p}_2$.

It should be noted that in (4.11) $\mathfrak{F}$ occurs, not $\mathfrak{F}^*$. Therefore $\mathfrak{F}$ must be identified with ψ_2^* in the matrix element (4.1). Therefore ψ_2 itself[33] will be

$$\psi_2=\mathfrak{F}^*(r,\pi-\Theta)=e^{+i\mathbf{p}_2\cdot\mathbf{r}}+r^{-1}e^{-ip_2r}f^*(\chi). \tag{4.14}$$

In other words, ψ_2 is a plane wave propagating as usual in the direction $+\mathbf{p}_2$, plus a spherical wave which is *ingoing* rather than outgoing. The amplitude of the ingoing wave is large near $\chi=0$, or $\Theta=\pi$, i.e. on the side *from which the plane wave is coming*, as is reasonable for an ingoing spherical wave associated with the plane wave $e^{i\mathbf{p}_2\cdot\mathbf{r}}$.[34]

We can prove more explicitly that ψ_2 is just the solution of the homogeneous equation which satisfies the boundary condition of having no *outgoing* spherical waves added to the plane wave $e^{i\mathbf{p}_2\cdot\mathbf{r}}$. Inserting $\theta=\pi-\Theta$ in (4.12), we note that $P_l(\pi-\Theta)=(-1)^lP_l(\Theta)$ and $i^l(-1)^l=i^{-l}$; in fact, this is the way in which $\mathfrak{F}(r',\pi-\Theta)$ is first introduced in the derivation by Mott and Massey. Therefore

$$\psi_2(\mathbf{r}')=\mathfrak{F}^*(r',\pi-\Theta)$$
$$=\sum(2l+1)i^l e^{-i\eta_l}L_l(r')P_l(\Theta). \tag{4.15}$$

This differs from the usual expression for the wave function, (4.12), only by having $e^{-i\eta_l}$ instead of $e^{i\eta_l}$. By considering the asymptotic behavior of $L_l(r')$, it is easily seen[35] that

$$\psi_2=e^{i\mathbf{p}_2\cdot\mathbf{r}}+r^{-1}e^{-ip_2r}g(\theta), \tag{4.16}$$

with

$$g(\theta)=(-2ik)^{-1}\sum(2l+1)(e^{-2i\eta_l}-1)(-1)^lP_l(\theta)$$
$$=f^*(\pi-\theta)=f^*(\chi). \tag{4.17}$$

For the actual evaluation of the matrix element, it is more convenient to use $\mathfrak{F}$ than ψ_2. As is seen from (4.13), $\mathfrak{F}$ is the usual solution, plane wave plus outgoing spherical waves, only with $-\mathbf{p}_2$ instead of $+\mathbf{p}_2$. Therefore, if we consider only the part of the matrix element (4.1) arising from the "main" part ψ_a, defined in (2.13),

[33] There is of course no need to go to ψ_2; the actual calculation could be done equally well directly with $\mathfrak{F}$.

[34] It is good to note that in the Born approximation one doesn't need to pay attention to the incoming wave because one doesn't consider any spherical waves at all in the wave function. This was recognized by Mott and Massey, p. 356, reference 10.

[35] As in Mott and Massey, pp. 22–24, reference 10.

of the wave functions ψ_1 and ψ_2, we get

$$H_{12}'=CN_1N_2^*(u_2^*\alpha_\lambda u_1)\int d\tau e^{-i\mathbf{k}\cdot\mathbf{r}}e^{i(\mathbf{p}_1-\mathbf{p}_2)\cdot\mathbf{r}}$$
$$\times F(ia_1;1;ip_1r-i\mathbf{p}_1\cdot\mathbf{r})F(ia_2;1;ip_2r+i\mathbf{p}_2\cdot\mathbf{r}) \quad (4.18)$$

with

$$a_1=a\epsilon_1/p_1, \quad a_2=a\epsilon_2/p_2. \quad (4.19)$$

The usual choice of ψ_2, with outgoing spherical waves, would replace the last factor in (4.18) by

$$F(-ia_2;1;-ip_2r+i\mathbf{p}_2\cdot\mathbf{r}). \quad (4.20)$$

The evaluation of the matrix element will be given in Sec. VIII.

V. WAVE FUNCTION AND MATRIX ELEMENT IN PAIR PRODUCTION

In the case of pair production, *both* the electron and the positron are produced, and therefore both should be represented by plane waves with convergent spherical waves. Thus, in contrast to the Born approximation theory, there is a significant difference between the matrix element for bremsstrahlung and pair production. The latter is

$$H_{12}'=C\int\psi_-^*\alpha_\lambda\psi_+^*e^{i\mathbf{k}\cdot\mathbf{r}}d\tau, \quad (5.1)$$

where ψ_+ is the wave function of the positron.

The spinor factor u_+^* in the wave function ψ_+^* of a positron of momentum $\mathbf{p}_+$ is the same as the spinor factor u_1 (not conjugate) in the wave function of an electron of momentum $-\mathbf{p}_+$ and negative energy $-\epsilon_+$. The matrix element (5.1) thus contains the matrix factor

$$b=(u_-^*\alpha_\lambda u_1). \quad (5.2)$$

The spatial part of ψ_+ differs from that of an electron by having the sign of the Coulomb interaction reversed. This can be accomplished by changing the sign of $a\epsilon/p$ in the first argument of the hypergeometric function (2.6), thus obtaining

$$\psi_{+a}'=N_+e^{i\mathbf{p}_+\cdot\mathbf{r}}u_+F(-ia\epsilon_+/p_+;1;ip_+r-i\mathbf{p}_+\cdot\mathbf{r}) \quad (5.3)$$

for a positron going in the direction $\mathbf{p}_+$ plus *outgoing* spherical waves. For ψ_+ in (5.1) we must again take ingoing spherical waves, so that ψ_+^* is represented, in analogy with (4.18), by

$$\psi_{+a}^*=N_+^*e^{-i\mathbf{p}_+\cdot\mathbf{r}}u_1F(-ia_1;1;ip_+r+i\mathbf{p}_+\cdot\mathbf{r}), \quad (5.4)$$

with

$$a_1=a\epsilon_+/p_+. \quad (5.5)$$

Similarly, the negaton is described by a wave function ψ_-^* exactly like the last factor of (4.18), and the "main" part of the matrix element (5.1) becomes

$$H_{12}'=CN_+^*N_-^*(u_-^*\alpha_\lambda u_1)\int d\tau e^{i(\mathbf{k}-\mathbf{p}_1-\mathbf{p}_2)\cdot\mathbf{r}}$$
$$\times F(-ia_1;1;ip_+r+i\mathbf{p}_+\cdot\mathbf{r})$$
$$\times F(ia_-;1;ip_-r+i\mathbf{p}_-\cdot\mathbf{r}). \quad (5.6)$$

Asymptotically for large r, the hypergeometric function gives a large scattered wave where its argument is small. The scattered waves of *both* electron and positron in (5.6) are therefore large when $\mathbf{r}$ is in the direction $-\mathbf{p}$ (convergent waves), whereas in (4.18) one scattered wave (for the incident electron) is large in the direction $+\mathbf{p}$, the other (outgoing electron) in the direction $-\mathbf{p}$. This will make a considerable difference in the matrix element.

As is well known, ψ_+^* can also be considered as the wave function (not conjugate) of a negative energy, negatively charged electron. The usual prescription for obtaining this is to take the electron wave function

$$\psi_1=N_1e^{i\mathbf{p}_1\cdot\mathbf{r}}F(ia\epsilon_1/p_1;1;ip_1r-i\mathbf{p}_1\cdot\mathbf{r})u(\mathbf{p}_1,\epsilon_1), \quad (5.7)$$

and substitute $\epsilon_1=-\epsilon_+$, $\mathbf{p}_1=-\mathbf{p}_+$, while leaving $p_1=+p_+$. It is easily seen that this leads to the correct function

$$\psi_1=N_1e^{-i\mathbf{p}_+\cdot\mathbf{r}}$$
$$\times F(-ia\epsilon_+/p_+;1;ip_+r+i\mathbf{p}_+\cdot\mathbf{r})u(-\mathbf{p}_+,-\epsilon_+), \quad (5.8)$$

which is exactly the function occurring in (5.6).

VI. EVALUATION OF THE PAIR MATRIX ELEMENT AND CROSS SECTION

We proceed now to the evaluation of the matrix element (5.1). For ease of writing, we shall denote the quantities referring to the positron by the subscript 1 rather than +, and those referring to the electron by 2. Thus $\mathbf{p}_1=\mathbf{p}_+$ in the next two sections, F_1 is the hypergeometric function relating to the positron and F_2^* that for the electron.

We substitute $d^3\rho=(\hbar/mc)^3d^3r$ and $\psi_1=\psi_{1a}+\psi_{1b}+\psi_{1c}$, as in (2.13), and similarly for the negaton. Of the resulting integrals we will show that three may be evaluated in closed form, and those remaining will be shown in Sec. IX to give contributions to the differential cross-section which may be neglected when the energies of the positive and negative electron are much larger than mc^2.

We proceed then to those integrals that may be evaluated in closed form and to the differential cross-section determined from them alone. These integrals are

$$C\int\psi_{2a}^*\alpha_\lambda e^{i\mathbf{k}\cdot\mathbf{r}}\psi_{1a}d^3\rho+C\int\psi_{2a}^*\alpha_\lambda e^{i\mathbf{k}\cdot\mathbf{r}}\psi_{1b}d^3\rho$$
$$+C\int\psi_{2b}^*\alpha_\lambda e^{i\mathbf{k}\cdot\mathbf{r}}\psi_{1a}d^3\rho. \quad (6.1)$$

Using (2.14), (2.15), the matrix element which we will use may be written in the form

$$H_{12}'=C'[(u_2^*\alpha_\lambda u_1)I_1+(u_2^*\alpha_\lambda\boldsymbol{\alpha}\cdot\mathbf{I}_2u_1)+(u_2^*\boldsymbol{\alpha}\cdot\mathbf{I}_3\alpha_\lambda u_1)], \quad (6.2)$$

where

$$I_1=\int e^{i\mathbf{q}\cdot\mathbf{r}}F_2^*F_1d^3r, \quad (6.3)$$

$$\mathbf{I}_2=-(i/2\epsilon_1)\int e^{i\mathbf{q}\cdot\mathbf{r}}F_2^*\nabla F_1d^3r, \quad (6.3a)$$

$$\mathbf{I}_3=(i/2\epsilon_2)\int e^{i\mathbf{q}\cdot\mathbf{r}}(\nabla F_2^*)F_1d^3r, \quad (6.3b)$$

and

$$C'=C(\hbar/mc)^3N_2^*N_1, \quad (6.4)$$

$$\mathbf{q}=\mathbf{k}-\mathbf{p}_1-\mathbf{p}_2. \quad (6.5)$$

N_1 and N_2 will be determined so that ψ_1 and ψ_2 are normalized to unit amplitude asymptotically. Following Bess again, the integrals in (6.3) may be obtained from the integral

$$I_0=\int e^{-\lambda r}e^{i\mathbf{q}\cdot\mathbf{r}}F_2^*F_1d^3r/r \quad (6.6)$$

by differentiation with respect to parameters. Thus

$$I_1=-\partial I_0/\partial\lambda|_{\lambda=0}. \quad (6.7)$$

Further, according to (5.6),

$$F_1=F(-ia_1;1;ip_1r+i\mathbf{p}_1\cdot\mathbf{r}),$$

and differentiation of the argument gives

$$\nabla_r(p_1r+\mathbf{p}_1\cdot\mathbf{r})=(p_1/r)\nabla_{p_1}(p_1r+\mathbf{p}_1\cdot\mathbf{r}), \quad (6.8)$$

where

$$\nabla_{p_1}\equiv(\partial/\partial p_{1x},\partial/\partial p_{1y},\partial/\partial p_{1z}), \quad (6.9)$$

and F_2 is independent of $\mathbf{p}_1$. Therefore we have

$$\mathbf{I}_2=-(ip_1/2\epsilon_1)\nabla_{p_1}\int e^{i\mathbf{q}\cdot\mathbf{r}}F_2^*F_1d^3r/r$$

$$=-(ia/2a_1)\nabla_{p_1}I_0|_{\lambda=0}, \quad (6.10)$$

in which both a_1 and $\mathbf{q}$ are to be considered independent of $\mathbf{p}_1$ when operating with ∇_{p_1}. Similarly, since $F_2^*=F(ia_2;1;ip_2r+i\mathbf{p}_2\cdot\mathbf{r})$,

$$\mathbf{I}_3=(ia/2a_2)\nabla_{p_2}I_0|_{\lambda=0}. \quad (6.11)$$

The method used in the evaluation of I_0 (involving contour integration) is due to Dr. A. T. Nordsieck.[36] The authors are indebted to Dr. Nordsieck for his generous communication of the essential steps of this evaluation. By a calculation entirely analogous to that in the accompanying paper,[36] we obtain

$$I_0=\frac{2\pi}{\alpha}\left(\frac{\gamma}{\alpha}\right)^{ia_1}\left(\frac{\alpha}{\alpha+\beta}\right)^{ia_2}F(-ia_1,ia_2;1;x), \quad (6.12)$$

where

$$\alpha=\tfrac{1}{2}(q^2+\lambda^2),\qquad \beta=\mathbf{p}_2\cdot\mathbf{q}-i\lambda p_2,$$
$$\gamma=\mathbf{p}_1\cdot\mathbf{q}-i\lambda p_1+\alpha,\quad \delta=\mathbf{p}_1\cdot\mathbf{p}_2-p_1p_2+\beta, \quad (6.13)$$

$$x=(\beta\gamma-\alpha\delta)/[\gamma(\alpha+\beta)]. \quad (6.14)$$

For small λ, the three quantities α, $\alpha+\beta$, and γ are all real; in fact we have

$$D_1\equiv 2(\alpha+\beta)_{\lambda=0}=2\mathbf{q}\cdot\mathbf{p}_2+q^2=(\mathbf{k}-\mathbf{p}_1)^2-p_2^2=2k(\epsilon_1-p_1\cos\theta_1);$$
$$D_2\equiv 2(\gamma)_{\lambda=0}=2\mathbf{q}\cdot\mathbf{p}_1+q^2=(\mathbf{k}-\mathbf{p}_2)^2-p_1^2=2k(\epsilon_2-p_2\cos\theta_2). \quad (6.15)$$

Therefore the two powers with imaginary exponent in (6.12) will be of absolute magnitude unity. The argument of the hypergeometric function, x, can be shown to be between 0 and 1. For most calculations, it is preferable to consider

$$y\equiv 1-x=\left.\frac{\alpha(\gamma+\delta)}{\gamma(\alpha+\beta)}\right|_{\lambda=0}. \quad (6.16)$$

Some algebra yields

$$y=q^2\mu/D_1D_2, \quad (6.17)$$

where

$$\mu=2(\gamma+\delta)_{\lambda=0}=k^2-(p_1+p_2)^2 \quad (6.18)$$

is a positive quantity (since $k=\epsilon_1+\epsilon_2$), independent of angles. It is convenient to introduce the abbreviations:

$$K=4\pi a(D_2/q^2)^{ia_1}(D_1/q^2)^{-ia_2},\quad |K|=4\pi a, \quad (6.19)$$

$$V_{12}(x)=F(-ia_1,ia_2;1;x), \quad (6.20)$$

$$W_{12}(x)=(a_1a_2)^{-1}dV_{12}/dx=F(1-ia_1,1+ia_2;2;x). \quad (6.21)$$

We now differentiate (6.12) as required in (6.7), (6.10), and (6.11). For this purpose, $\mathbf{q}$ must be kept constant, i.e., (6.15) and (6.18) must not be used until *after* the differentiation. We note that, for $\lambda=0$,

$$-p_1\nabla_{p_1}\delta=p_2\mathbf{p}_1-p_1\mathbf{p}_2\equiv\mathbf{P},\quad \nabla_{p_2}\delta=\mathbf{P}/p_2+\mathbf{q}. \quad (6.22)$$

We obtain

$$I_1=2K\left\{\frac{V_{12}}{q^2}\left(\frac{\epsilon_2}{D_1}-\frac{\epsilon_1}{D_2}\right)+\frac{iW_{12}}{D_1D_2}\left[a_1\epsilon_2\left(\frac{\mu}{D_1}-1\right)+a_2\epsilon_1\left(\frac{\mu}{D_2}-1\right)\right]\right\},$$
$$\mathbf{I}_2=K\left\{-\frac{V_{12}}{D_2}\frac{\mathbf{q}}{q^2}+ia_2\frac{W_{12}}{D_1D_2}\left[\mathbf{q}\left(\frac{\mu}{D_2}-1\right)+\frac{\mathbf{P}}{p_1}\right]\right\}, \quad (6.23)$$
$$\mathbf{I}_3=K\left\{\frac{V_{12}}{D_1}\frac{\mathbf{q}}{q^2}+ia_1\frac{W_{12}}{D_1D_2}\left[\mathbf{q}\left(\frac{\mu}{D_1}-1\right)-\frac{\mathbf{P}}{p_2}\right]\right\}.$$

[36] A. T. Nordsieck, following paper in this issue. The integration was already carried out by Bess, but it is somewhat more difficult to follow the complex arguments of all the quantities involved in Bess' paper.

In the limit of high energies, $a_1=a_2=a$ with an error of the relative order $1/\epsilon^2$. We may therefore replace V_{12} and W_{12}, respectively, by

$$V(x)=F(-ia, ia; 1; x)$$
$$=1+\frac{a^2}{1!^2}x+\frac{a^2(1+a^2)}{2!^2}x^2+\frac{a^2(1+a^2)(2^2+a^2)}{3!^2}x^3+\cdots, \quad (6.24)$$

$$W(x)=a^{-2}dV/dx$$
$$=1+\frac{1+a^2}{1!2!}x+\frac{(1+a^2)(2^2+a^2)}{2!3!}x^2+\cdots. \quad (6.25)$$

Both of these functions are real so that the expressions for I_1 to $\mathbf{I}_3$, (6.23), fall into a real part proportional to V and an imaginary part proportional to W. The resulting cross section therefore contains only V^2 and W^2, not VW.

In the cross section we require the absolute square of the matrix element H_{12}', Eq. (5.1), summed over the spin directions of electron and positrons, and averaged over the directions of polarization of the incident quantum. Since the spin sums of $|H_{12}'|^2$ involve the free-electron Dirac matrix coefficients u_1 and u_2 we may use the spur and closure theorems. From (6.2) we have then

$$\Sigma|H_{12}'|^2=|C'|^2(\epsilon_1\epsilon_2)^{-1}$$
$$\times\sum_{\nu=1}^{2}\{I_{1\nu}^2(\epsilon_1\epsilon_2+1-p_1p_2\cos\theta_1\cos\theta_2)$$
$$+(\mathbf{I}_{2\nu}^2+\mathbf{I}_{3\nu}^2)(\epsilon_1\epsilon_2-1+p_1p_2\cos\theta_1\cos\theta_2)$$
$$-2I_{2\nu z}I_{3\nu z}(\epsilon_1\epsilon_2-1+\mathbf{p}_1\cdot\mathbf{p}_2)$$
$$+2(\mathbf{I}_{2\nu}-\mathbf{I}_{3\nu})\cdot(\mathbf{p}_2I_{3\nu z}p_{1z}-\mathbf{p}_1I_{2\nu z}p_{2z})$$
$$+2\mathbf{I}_{3\nu}\cdot\mathbf{p}_2\mathbf{I}_{2\nu}\cdot\mathbf{p}_1-2\mathbf{I}_{2\nu}\cdot\mathbf{p}_2\mathbf{I}_{3\nu}\cdot\mathbf{p}_1$$
$$+2I_{1\nu}\epsilon_2(\mathbf{I}_{2\nu}\cdot\mathbf{p}_1-I_{3\nu z}p_{1z})$$
$$+2I_{1\nu}\epsilon_1(\mathbf{I}_{3\nu}\cdot\mathbf{p}_2-I_{2\nu z}p_{2z})$$
$$+2(\mathbf{p}_1\cdot\mathbf{p}_2-p_{1z}p_{2z})\mathbf{I}_{2\nu}\cdot\mathbf{I}_{3\nu}\}, \quad (6.26)$$

where $I_{k\nu}$, with $\nu=1$ and 2, denote the real and imaginary part, respectively, of the integral I_k, $k=1$ to 3 (see remarks following Eq. (6.25)). The subscript z used in the expression above denotes the z component of the vector to which it applies, i.e., the component in the direction of $\mathbf{k}$.

The results (6.23), (6.26) do not differ greatly from Bess' results (37) and (36). In (6.26), the only difference is that Bess omitted the last line which is in fact not important for small angles θ_1, θ_2. In (6.23), $\mathbf{I}_2$ and $\mathbf{I}_3$ differ from Bess' expression only by the denominators p_1 and p_2 in the last terms which are needed for dimensional reasons. I_1 is much simpler than the corresponding expression of Bess which contains a number of additional terms arising from the incorrect part of his wave function.[21] It is reasonable that the results should differ only in I_1 because Bess' additional term in ψ, $(a/2\epsilon)\partial F/\partial r$, involves no matrix operator and should have the same matrix factor as the contribution of ψ_a. A minor change is in the coefficient K for which Bess has $4\pi a(D_1/D_2)^{ia}$; this does not affect the result.

The transition probability per unit time is

$$w=(2\pi/\hbar)\rho_f\Sigma|H_{12}'|^2, \quad (6.27)$$

where $\Sigma|H_{12}'|^2$ is given by (6.26) and ρ_f is the density of final states for the two emitted electrons, i.e.,

$$\rho_f=(mc^2)^4p_1\epsilon_1p_2\epsilon_2d\Omega_1d\Omega_2(2\pi\hbar c)^{-6}, \quad (6.28)$$

where $d\Omega_1$, and $d\Omega_2$ are the solid angles.

The differential cross section, $d\sigma$, is equal to the transition probability normalized to unit current of the incident particle. Thus we must divide w by the velocity of the incident quantum, c, and obtain

$$d\sigma=\frac{(2\pi)^2}{\hbar c}\frac{(mc^2)^4}{(2\pi\hbar c)^6}p_1p_2\epsilon_1\epsilon_2\Sigma|H_{12}'|^2d\epsilon_1$$
$$\times\sin\theta_1d\theta_1\sin\theta_2d\theta_2d\phi, \quad (6.29)$$

where $\phi=\phi_1-\phi_2$ and the integration over ϕ_2 has been carried out, yielding a factor 2π.

The factor $|C'|^2$ in $|H_{12}'|^2$ is given by (6.4) and (4.2) in terms of the normalization factors N_1 and N_2. The wave function of the negative electron is asymptotically $\psi_2=N_2e^{i\mathbf{p}_2\cdot\mathbf{r}}u_2F_2$. Since the exponential and u_2 are normalized to unity we must examine the asymptotic behavior of F_2 which is

$$F_2\sim[\Gamma(1+ia_2)]^{-1}e^{-\pi a_2/2}e^{ia_2\log(p_2r+\mathbf{p}_2\cdot\mathbf{r})}. \quad (6.30)$$

Thus, normalizing ψ_2 to unit amplitude, we have

$$|N_2|^2=|\Gamma(1+ia_2)|^2e^{\pi a_2}=\pi a_2e^{\pi a_2}/\sinh\pi a_2, \quad (6.31)$$

whereas for the positive electron the sign of a is reversed, yielding

$$|N_1|^2=\pi a_1e^{-\pi a_1}/\sinh\pi a_1, \quad (6.32)$$

with positive a_1.

Substituting (4.2), (6.4), (6.19), (6.31), and (6.32), we have

$$|C'K|^2=(4\pi a)^2\frac{2\pi e^2\hbar^2c^2}{k}\left(\frac{\hbar}{mc}\right)^6\frac{\pi^2a_1a_2e^{\pi(a_2-a_1)}}{\sinh\pi a_1\sinh\pi a_2}. \quad (6.33)$$

Upon making the approximations $a_1\approx a$, $a_2\approx a$ [involving errors of $O(1/\epsilon^2)$], this becomes

$$|C'K|^2=\frac{(2\pi a)^4}{(\sinh\pi a)^2}\frac{2\pi e^2\hbar^2c^2}{k}\left(\frac{\hbar}{mc}\right)^6. \quad (6.34)$$

Substituting (6.23), (6.26), and (6.34) in (6.29), we

have the differential cross-section for pair production:

$$d\sigma=-\left(\frac{\pi a}{\sinh\pi a}\right)^2\frac{a^2}{2\pi}\left(\frac{\hbar}{mc}\right)^2\frac{e^2}{\hbar c}\frac{p_-p_+}{k^3}d\epsilon_+$$
$$\times\sin\theta_-\sin\theta_+d\theta_-d\theta_+d\phi$$
$$\times\left\{\frac{V^2(x)}{q^4}\left[\frac{p_-^2\sin^2\theta_-(4\epsilon_+^2-q^2)}{(\epsilon_--p_-\cos\theta_-)^2}\right.\right.$$
$$+\frac{p_+^2\sin^2\theta_+(4\epsilon_-^2-q^2)}{(\epsilon_+-p_+\cos\theta_+)^2}$$
$$+\frac{(4\epsilon_-\epsilon_++q^2-2k^2)2p_-p_+\sin\theta_-\sin\theta_+\cos\phi}{(\epsilon_--p_-\cos\theta_-)(\epsilon_+-p_+\cos\theta_+)}$$
$$\left.-\frac{2k^2(p_-^2\sin^2\theta_-+p_+^2\sin^2\theta_+)}{(\epsilon_--p_-\cos\theta_-)(\epsilon_+-p_+\cos\theta_+)}\right]$$
$$+\frac{a^2[k^2-(p_-+p_+)^2]^2W^2(x)}{[4k^2(\epsilon_--p_-\cos\theta_-)(\epsilon_+-p_+\cos\theta_+)]^2}$$
$$\times\left[\frac{p_-^2\sin^2\theta_-(4\epsilon_+^2-q^2)}{(\epsilon_--p_-\cos\theta_-)^2}+\frac{p_+^2\sin^2\theta_+(4\epsilon_-^2-q^2)}{(\epsilon_+-p_+\cos\theta_+)^2}\right.$$
$$-\frac{(4\epsilon_-\epsilon_++q^2-2k^2)2p_-p_+\sin\theta_-\sin\theta_+\cos\phi}{(\epsilon_--p_-\cos\theta_-)(\epsilon_+-p_+\cos\theta_+)}$$
$$+\frac{2k^2(p_-^2\sin^2\theta_-+p_+^2\sin^2\theta_+)}{(\epsilon_--p_-\cos\theta_-)(\epsilon_+-p_+\cos\theta_+)}$$
$$\left.\left.-4k^2(\epsilon_-\epsilon_++p_-p_+\cos\theta_-\cos\theta_+)\right]\right\}. \quad (6.35)$$

It will be recognized that the first group of terms, those proportional to $V^2(x)$, is identical with the Bethe-Heitler cross section except for the factor

$$[V(x)\pi a/\sinh\pi a]^2. \quad (6.36)$$

It can be shown that[37]

$$V(x=1)=\sinh\pi a/\pi a, \quad (6.37)$$

so that the cross section is simply multiplied by

$$[V(x)/V(1)]^2. \quad (6.38)$$

In addition, there is another group of terms proportional to $W^2(X)$. These terms have a form very similar to that of the Bethe-Heitler terms, but the deviations from the latter are real, as will be further demonstrated in the next section.

The pair-production cross section (6.35) may be compared with Bess's result (38) for bremsstrahlung. When the translation from one phenomenon to the other is made in the usual way, the main difference is that (6.35) does not contain those terms which arise from Bess' spurious terms in I_1, i.e., all his terms containing a^2V^2, a^2VW or a^4W^2. In addition, Bess omitted the q^2 terms in the first square bracket, which is not serious. The second bracket has been completely rewritten to make its similarity with the first bracket evident; apart from this, it differs from Bess' appreciably in content.

[37] Davies, Bethe, and Maximon, Eq. (24), this issue [Phys. Rev. **93**, 788 (1954)].

VII. SMALL-ANGLE APPROXIMATION

It is well known that the main contribution to the *integral* cross section for either pair production or bremsstrahlung comes from *small* angles, θ_1 and θ_2 of order $1/\epsilon$. Moreover, our whole approximation, i.e., the representation of the electron wave functions by the contributions a and b, alone, Eqs. (2.14), (2.15), is only justified for small angles. It is therefore interesting to investigate the behavior of our matrix elements and cross sections in the limit of small angles. This will permit a considerable simplification, and thereby better insight into the orders of magnitude and the structure of the expression. Furthermore, this simplification is an essential first step in the integration of the cross section over angles.[38]

We introduce the vectors $\mathbf{u}$ and $\mathbf{v}$ to denote the components of $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively, perpendicular to $\mathbf{k}$. Their magnitudes are

$$u=p_1\theta_1, \quad v=p_2\theta_2, \quad (7.1)$$

if we make the approximation $\sin\theta_1=\theta_1$, $1-\cos\theta_1=\frac{1}{2}\theta_1^2$ which we shall use generally. It is convenient to split $\mathbf{q}$ into the part in the z direction and the part perpendicular to $\mathbf{k}$ which are, respectively,

$$q_z=(1+u^2)/2\epsilon_1+(1+v^2)/2\epsilon_2, \quad (7.2)$$

$$\mathbf{q}_\perp=-(\mathbf{u}+\mathbf{v}),$$
$$q_\perp^2=u^2+v^2+2uv\cos\phi. \quad (7.3)$$

Here, and in the following, we neglect consistently terms of relative order $1/\epsilon^2$. The minimum value of q, for $u=v=0$, is

$$q_{\min}=q_{z\,\min}\equiv\delta=k-p_1-p_2\approx k/2\epsilon_1\epsilon_2. \quad (7.4)$$

If u and v are of order 1, q is of the same order; but q_z is generally of order $1/\epsilon$ and q *can* be of that order if the vectors $\mathbf{u}$ and $\mathbf{v}$ are nearly equal and opposite.

The quantities introduced in (6.15) to (6.18) are, in our approximation and notation,

$$D_1=(k/\epsilon_1)(1+u^2), \quad D_2=(k/\epsilon_2)(1+v^2), \quad (7.5)$$

$$\mu=2k\delta=k^2/\epsilon_1\epsilon_2, \quad (7.6)$$

$$y=q^2/[(1+u^2)(1+v^2)]. \quad (7.7)$$

It will be convenient in the following to use the further

[38] We are indebted to Dr. Handel Davies who carried out the angular integration, for some of the calculations reported in this section.

abbreviations:

$$\xi=1/(1+u^2),\quad \eta=1/(1+v^2), \tag{7.8}$$

in terms of which (7.5) and (7.7) can easily be rewritten. If k and ϵ_1, ϵ_2 are regarded as of the same order of magnitude (order ϵ), and are large compared with 1, then D_1, D_2, μ, y, u, v, ξ, and η are all of order unity.

The vector $\mathbf{P}$ introduced in (6.22) has the components

$$\mathbf{P}_\perp=\epsilon_2\mathbf{u}-\epsilon_1\mathbf{v},\quad P_z=(\epsilon_1/2\epsilon_2)v^2-(\epsilon_2/2\epsilon_1)u^2. \tag{7.9}$$

Since p occurs only with the denominators p_1, p_2 in (6.23), the component P_z (which is of order 1) gives a negligible contribution compared with either $P_\perp$ (order ϵ) or the remaining terms in (6.23).

Inserting into (6.23), we obtain

$$\begin{aligned}
&I_1=2K(\epsilon_1\epsilon_2/k)[q^{-2}(\eta-\xi)V+ia(\xi+\eta-1)W],\\
&\mathbf{I}_2=K(\epsilon_2/k)[-\mathbf{q}q^{-2}\eta V+ia(\mathbf{u}+\eta\mathbf{q})W],\\
&\mathbf{I}_3=K(\epsilon_1/k)[\mathbf{q}q^{-2}\xi V+ia(\mathbf{v}+\xi\mathbf{q})W].
\end{aligned} \tag{7.10}$$

These expressions are very much simpler than (6.23). They show clearly that I_1 is of order ϵ, while I_2 and I_3 are of order 1. This is to be expected from the definitions (6.3) in which $\mathbf{I}_2$, $\mathbf{I}_3$ have extra denominators ϵ_1, ϵ_2. It can also be shown by the methods of Sec. IX.

The spin sum (6.26) contains many terms which are negligible in the limit of high energies and small angles. Clearly, the factor of I_2^2 inside the braces of (6.26) is of order ϵ^2, therefore the complete expression in the braces must be expected to be (and turns out to be) of this order. This means that we should retain coefficients of order 1 multiplying I_1^2, of order ϵ multiplying I_1I_2 or I_1I_3, and of order ϵ^2 multiplying I_2^2, I_2I_3, or I_3^2; all lower powers of ϵ can be neglected. Doing this, (6.26) becomes after some algebra,

$$\begin{aligned}
\Sigma|H_{12}'|^2=|C'|^2(\epsilon_1\epsilon_2)^{-1}\textstyle\sum_\nu\{&(2\epsilon_1\epsilon_2)^{-1}\\
&\times(k^2+\epsilon_2^2u^2+\epsilon_1^2v^2)I_{1\nu}^2\\
&+2I_{1\nu}(\epsilon_2\mathbf{u}\cdot\mathbf{I}_{2\nu\perp}+\epsilon_1\mathbf{v}\cdot\mathbf{I}_{3\nu\perp})\\
&+2\epsilon_1\epsilon_2(I_{2\nu\perp}^2+I_{3\nu\perp}^2)\}.
\end{aligned} \tag{7.11}$$

This can also be written

$$\begin{aligned}
\Sigma|H_{12}'|^2=|C'|^2\textstyle\sum_\nu\{&(k^2/2\epsilon_1\epsilon_2)I_{1\nu}^2\\
&+2(I_{1\nu}\mathbf{u}/2\epsilon_1+\mathbf{I}_{2\nu\perp})^2+2(I_{1\nu}\mathbf{v}/2\epsilon_2+\mathbf{I}_{3\nu\perp})^2\}.
\end{aligned} \tag{7.12}$$

The simplification achieved in (7.11), (7.12) is quite remarkable. The result can also be understood directly by evaluating approximately the matrix elements occurring in (6.2), *viz.*,

$$b=(u_2^*\alpha_\lambda u_1),\quad \mathbf{c}=(u_2^*\alpha_\lambda\boldsymbol{\alpha}u_1), \tag{7.13}$$

between an initial state of energy $-\epsilon_1$, momentum $-\mathbf{p}_1$, and a final state ϵ_2, $\mathbf{p}_2$. Taking the λ component of $\mathbf{c}$, we obtain the operator $\alpha_\lambda^2=1$ whose matrix element is very nearly 1 if the spins of electron and positron are parallel. There are two possibilities of achieving this, *viz.*, both spins up or both down; hence the factor of I_2^2 and I_3^2 should be 2, as indeed it is in (7.12). Only the components of $\mathbf{I}_2$ and $\mathbf{I}_3$ perpendicular to $\mathbf{k}$ can come in because λ is perpendicular to $\mathbf{k}$. If the electron and positron have opposite spin, the matrix element of c_λ is of order $1/\epsilon$.

The matrix element b, for parallel spin of electron and positron, represents the sum of the velocities of the two particles in the direction of λ (or, after summing over polarization, the velocity component perpendicular to $\mathbf{k}$). The velocity of the positron perpendicular to $\mathbf{k}$ (in units of c) is $\mathbf{u}/\epsilon_1$, that of the electron $\mathbf{v}/\epsilon_2$. It is at least plausible that the positron velocity $\mathbf{u}/\epsilon_1$ will interfere only with $\mathbf{I}_2$ which is due to the correction to the positron wave function, and *vice versa*. The first term in (7.12) arises from the matrix elements of b which correspond to a reversal of spin.

From (7.12) and from the last two paragraphs, we can see clearly what has already been said in the introduction: $\mathbf{I}_2$ and $\mathbf{I}_3$ have matrix factors of order unity, thanks to the extra operator $\boldsymbol{\alpha}$ occurring in $\mathbf{c}$, while I_1 has a matrix factor of order $1/\epsilon$. Therefore, even though I_1 itself is much larger than I_2, *viz.*, of order ϵ rather than 1, its contribution to the complete matrix element is only of the same order as that of I_2 or I_3.

With the simplified expressions (7.10), (7.11), it is now easy to calculate the differential cross section which becomes

$$\begin{aligned}
d\sigma=8\left(\frac{\pi a}{\sinh\pi a}\right)^2\frac{a^2}{2\pi}\left(\frac{\hbar}{mc}\right)^2\frac{e^2}{\hbar c}\frac{\epsilon_1^2\epsilon_2^2}{k^3}&d\epsilon_1\theta_1d\theta_1\theta_2d\theta_2d\phi\\
\times\{q^{-4}V^2(x)[&k^2(u^2+v^2)\xi\eta-2\epsilon_1\epsilon_2(u^2\xi^2+v^2\eta^2)\\
&+2(\epsilon_1^2+\epsilon_2^2)uv\xi\eta\cos\phi]+a^2W^2(x)\xi^2\eta^2\\
\times[&k^2(1-(u^2+v^2)\xi\eta)-2\epsilon_1\epsilon_2(u^2\xi^2+v^2\eta^2)\\
&-2(\epsilon_1^2+\epsilon_2^2)uv\xi\eta\cos\phi]\}.
\end{aligned} \tag{7.14}$$

This result can also be obtained by making the high-energy, small-angle approximations in (6.35) and taking out a factor $8\epsilon_1\epsilon_2$ from the braces. However, the fact that (7.14) can also be deduced from the simple theory of this section gives confidence that no algebraic error has been made on its derivation, and that the difference in structure between the factors of V^2 and W^2 is real.

The result (7.14) is also the most suitable starting point for the integration over angles.

VIII. THE CROSS SECTION FOR BREMSSTRAHLUNG

The matrix element for bremsstrahlung is given in (4.18). Its evaluation is quite similar to that for pair production, with the result, analogous to (6.12),

$$\begin{aligned}
I_0=\frac{2\pi}{\alpha}e^{-\pi a_1}\left(\frac{\alpha}{\gamma+\delta}\right)^{ia_1}&\left(\frac{\gamma+\delta}{\gamma}\right)^{i(a_1-a_2)}\\
&\times F(1-ia_1,ia_2;1;x),
\end{aligned} \tag{8.1}$$

where the meaning of α and β is the same as in (6.13)

but the other parameters are changed to

$$\gamma=\mathbf{p}_1\cdot\mathbf{q}+i\lambda p_1-\alpha,\quad \delta=p_1p_2+\mathbf{p}_1\cdot\mathbf{p}_2-\beta. \tag{8.2}$$

Also,

$$\mathbf{q}=\mathbf{p}_1-\mathbf{p}_2-\mathbf{k}, \tag{8.3}$$

and the argument of F is

$$x=(\alpha\delta-\beta\gamma)/[\alpha(\gamma+\delta)]. \tag{8.4}$$

It is again more convenient to use

$$y=1-x=\gamma(\alpha+\beta)/(\gamma+\delta)\alpha. \tag{8.5}$$

The reason for the appearance of the factor $e^{-\pi a_1}$ is explained in the paper by Nordsieck following this.

The most important change is that δ, Eq. (8.2), now contains the sum of p_1p_2 and $\mathbf{p}_1\cdot\mathbf{p}_2$ rather than their difference, as in (6.13). This change can be traced directly to the form of the matrix element (4.18) for bremsstrahlung which contains $\mathbf{p}_1\cdot\mathbf{r}$ and $\mathbf{p}_2\cdot\mathbf{r}$ with *opposite* signs while in the pair matrix elements (5.6) they occur with the same sign. The effect of the changed expression for δ is to make this quantity very large, of order ϵ^2, if $\mathbf{p}_1$ and $\mathbf{p}_2$ have nearly the same direction, as is usual. The three other parameters, α, β, γ in bremsstrahlung, and all four parameters in pair production, are of order 1. Since δ is now so large, the definition (6.14) of x is no longer convenient because it would make x larger than 1. The changed argument x in turn causes a change of the first parameter of the hypergeometric function, from $-ia_1$ to $1-ia_1$. The new definition (8.4) of x keeps $x<1$; in fact it is easy to show that $y=1-x$ is in general very small (see below). Since for small angles between $\mathbf{p}_1$ and $\mathbf{p}_2$ we may replace δ by $2\epsilon_1\epsilon_2$, we get

$$y=D_1D_2/4\epsilon_1\epsilon_2q^2, \tag{8.6}$$

where D_1 is the same as in (6.15), and

$$D_2=2(\gamma)_{\lambda=0}=2\mathbf{p}_1\cdot\mathbf{q}-q^2=p_1^2-(\mathbf{k}+\mathbf{p}_2)^2 =2k(\epsilon_2-p_2\cos\theta_2) \tag{8.7}$$

is given by the same final expression as in (6.15) even though intermediate steps are different. Since D_1 and D_2 are of order 1, y will in general be of order $1/\epsilon^2$ (i.e., where q is of order 1) but will be larger for small q. In particular, if both $\mathbf{p}_1$ and $\mathbf{p}_2$ are in the direction $\mathbf{k}$, then q attains its minimum value (7.4); u and v, Eq. (7.1), are zero and this makes $y=1$ and $x=0$. Thus the argument of F in (8.1) is extremely close to one except for very small q: this is the essential difference between the results for bremsstrahlung and pair production.

We must now differentiate (8.1) with respect to λ, $\mathbf{p}_1$, and $\mathbf{p}_2$ as required by (6.7), (6.10), and (6.11). Consider first the differentiation of the factors in front of F. Of these, α does not depend on λ in first order, nor on the $\mathbf{p}$'s; γ has the exponent $i(a_1-a_2)$ which is of order $1/\epsilon^2$, and δ is so large that $\partial\log(\gamma+\delta)/\partial\lambda$, and the other derivatives, are small of order $1/\epsilon$ or less. Thus one can show that the differentiation of the factors outside of F in (8.1) contributes only a term of relative order $1/\epsilon$ which may be neglected.

It is then only necessary to differentiate with respect to the argument x of the hypergeometric function, and we obtain

$$I_1=\frac{2K}{a}\frac{1}{q^2}\frac{dF}{dx}yi\left(\frac{p_1}{D_2}-\frac{p_2}{D_1}\right), \tag{8.8}$$

where

$$K=4\pi ae^{-\pi a_1}(q^2/4\epsilon_1\epsilon_2)^{ia_1}(4\epsilon_1\epsilon_2/D_2)^{i(a_1-a_2)}. \tag{8.9}$$

We now transform the hypergeometric function. Setting $a_1=a_2=a$, it can easily be shown that

$$F\equiv F(1-ia,\,ia;\,1;\,x)=F(-ia,\,ia;\,1;\,x) +iaxF(1-ia,\,1+ia;\,2;\,x)=V(x)+iaxW(x), \tag{8.10}$$

with V and W defined in (6.24), (6.25). One of the terms occurring in (8.8) is then

$$y\frac{d}{dx}(xW)=a^{-2}(1-x)\frac{d}{dx}\left(x\frac{dV}{dx}\right). \tag{8.11}$$

Now the hypergeometric function $F(a,b;c;x)$ satisfies the differential equation[39]

$$x(1-x)F''+[c-(a+b+1)x]F'-abF=0. \tag{8.12}$$

For $V=F(-ia,\,ia;\,1;\,x)$, this becomes

$$x(1-x)V''+(1-x)V'-a^2V=0, \tag{8.13}$$

and, therefore,

$$(1-x)\frac{d}{dx}\left(x\frac{dV}{dx}\right)=a^2V. \tag{8.14}$$

Inserting in (8.8), (8.10), (8.11) gives

$$I_1=2Kq^{-2}(p_2/D_1-p_1/D_2)[V-iayW]. \tag{8.15}$$

This is just the first part I_1 in (6.23), with $V(x)$ replaced by $V-iayW$. Similarly, $\mathbf{I}_2$ and $\mathbf{I}_3$ also become equal to the first term in the respective formulas (6.23), with the same replacement.[40] This result is very similar to the Born approximation which corresponds to taking the limit $a=0$. Doing this in Eq. (6.23) removes the second term which is proportional to aW, and replaces V by 1. Thus our result agrees with the Born approximation except for a factor.

To determine this factor, we must consider the normalization factors of the initial and final electron wave function, N_1 and N_2. N_2 is given by (6.31) and, in our case, N_1 will be given by a similar formula because the initial state is now also an electron of positive energy, whereas the normalization factor (6.32) applied to a negative-energy electron. The two normalization factors $|N_1N_2|^2$ together yield a factor $e^{\pi(a_1+a_2)}$, but our ex-

[39] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (The Macmillan Company, New York, 1946), p. 283.

[40] There are also some changes of sign, due to the difference between pair production and bremsstrahlung; these are the same as in Born approximation.

pression (8.9) shows that $|C'K|^2$ now contains a factor $e^{\pi(a_2-a_1)}$, just as in (6.33), and for high energies this factor goes to one. This leaves the same factor as compared with the Born approximation as in (6.34), *viz.*,

$$\mathfrak{N}=(\pi a/\sinh\pi a)^2. \tag{8.16}$$

This expression is closely related to the hypergeometric function $V(x)$ for $x=1$. In fact, we have[41] for argument 1:

$$F(a,b,c,1)=\frac{\Gamma(c)\Gamma(1-a-b)}{\Gamma(1-a)\Gamma(1-b)}, \tag{8.17}$$

and therefore from the definition (6.24)

$$V(1)=\frac{\Gamma^2(1)}{\Gamma(1-ia)\Gamma(1+ia)}=\frac{\sinh\pi a}{\pi a}. \tag{8.18}$$

Therefore, when the normalization factor is included, the matrix element for bremsstrahlung differs from that in Born approximation by the factor

$$[V(x)-iayW(x)]/V(1), \tag{8.19}$$

and the cross section by the factor

$$R=[V^2(x)+a^2y^2W^2(x)]/V^2(1). \tag{8.20}$$

As we have already mentioned after (8.7), the argument x of the hypergeometric function is almost always very close to one. Whenever this is so, we may use the result of Davies, Bethe, and Maximon, *viz.*, that for $y=1-x$ near zero,

$$W(x)=-V(1)\log y. \tag{8.21}$$

Then $yW(x)\sim y\log y$ is very small, and the factor (8.20) becomes equal to unity. Thus, for most of the possible angles, θ_1, θ_2, and ϕ, the differential cross section for bremsstrahlung *is given exactly by the Born approximation.* This is in contrast to the cross section for pair production, and is probably due to the fact that the scattered waves for the initial and final electron do not overlap appreciably in bremsstrahlung (see Sec. IV) while they do for pair production (Sec. V).

An exception is the case of q very close to its minimum value, δ. Then, as shown below Eq. (8.7), the argument x is substantially below 1, and then R, Eq. (8.20), will differ from unity. In fact, it can easily be shown to be always less than 1. For this purpose, we take the derivative of the numerator of (8.20) with respect to x (note $y=1-x$):

$$dR'/dx=VdV/dx-a^2yW^2+a^2y^2WdW/dx. \tag{8.22}$$

Using (8.14), we find

$$dR'/dx=a^2yWdW/dx. \tag{8.23}$$

But from the power series expansion (6.25) it is clear that both W and dW/dx are positive for all x between 0 and 1. Therefore (8.20) increases monotonically with x, and since it reaches 1 at $x=1$, it is less than 1 for any other value of x.

Thus we have shown that the bremsstrahlung cross section is always less than the Born approximation value. Concerning the integration over angle, there is one limiting case for which the result is obvious, namely that of complete screening: in this limit, the differential cross section for small q is essentially eliminated by the atom form factor $F(q)$. Then q cannot come close to its lower limit δ, which was the only region in which we obtained a deviation from the Born approximation. Therefore, in the limit of complete screening, the Born approximation is valid for bremsstrahlung.

The integration in the absence of screening has been carried out and will appear shortly. For this limit the correction to the Born approximation turns out to be the same as for pair production [see DBM Eq. (35)]. Experimental cross sections, in which screening is incomplete should therefore (for high energies) be less than the Born approximation value. However, the reduction will be less than that for pair production, where the influence of screening is only in the region of small momentum transfer, for which the Coulomb correction is unimportant.

This has consequences for the theory of cascade showers in heavy elements: the cross section for pair production is reduced by about 10 percent (for lead) while that for bremsstrahlung is unchanged. Therefore the ratio of high-energy photons to electrons in the shower is increased by about 10 percent compared with the conventional theory, and the radiation length is increased by about 5 percent.[42]

IX. THE NEGLIGIBLE MATRIX ELEMENTS

(a) General

It remains for us to prove that the approximate wave function derived in Secs. II, III is really sufficient for the calculation of the matrix element. In particular, we must consider the correction to the wave function which arises from the term a^2/r^2 in the wave equation, i.e., the function ψ_e defined by

$$(\nabla^2+p^2+2\epsilon a/r)\psi_e=-a^2r^{-2}\psi_a, \tag{9.1}$$

and we must show that its contributions to the matrix element, such as (in the case of bremsstrahlung)

$$I_4=\int\psi_{2a}{}^*e^{-i\mathbf{k}\cdot\mathbf{r}}\psi_{1e}d\tau, \tag{9.2}$$

are actually negligible. It was just this contribution I_4 which Bess tried to take into account, and by which his theory differs from ours. That the contribution from I_4 is negligible will be shown by considering the orders of magnitude of the various contributions to the matrix element.

[41] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (The MacMillan Company, New York, 1946), p. 282, reference 39.

[42] We are indebted to R. R. Wilson for this remark.

We have discussed orders of magnitude already in Sec. VII. As in that section, we consider $\epsilon \gg 1$, and the angles θ_1, θ_2 of order $1/\epsilon$. Then, as is shown above Eq. (7.9), D_1, D_2, μ, u, v, etc., are of order unity. Also q is in general of this order but it can be as small as $O(1/\epsilon)$.

The integral I_1, involving the main part of the wave function, ψ_a, is of order ϵ, while the integrals $\mathbf{I}_2$, $\mathbf{I}_3$, involving the correction ψ_b to the wave function, are of order 1, Eq. (7.10). The matrix factor b, multiplying I_1, is of order $1/\epsilon$, as shown below (7.13), while the matrix factor $\mathbf{c}$, multiplying $\mathbf{I}_2$ and $\mathbf{I}_3$, is of order 1.

Now ψ_b satisfies the differential equation

$$(\nabla^2+p^2+2\epsilon a/r)\psi_b = ia\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}(1/r)\psi_a, \tag{9.3}$$

which is similar to (9.1) in that the right-hand side is of order ψ_a/r^2 in both equations. The order of magnitude of ψ_e must therefore be expected to be the same as that of ψ_b, and the order of I_4 the same as I_2 and I_3, namely unity. In Subsection *c*, we shall prove this statement in detail; but in Bess's paper I_4 comes out to be of order ϵ, like I_1, which is an unreasonable result.

Now the wave function ψ_e, like ψ_a, obviously contains no operators $\boldsymbol{\alpha}$, and therefore the integral I_4 will simply be multiplied by the matrix factor b, Eq. (7.13).[43] In Sec. VII, b was shown to be of order $1/\epsilon$; therefore, if I_4 is of order 1, its contribution to the matrix element is of order $1/\epsilon$ and hence negligible.

(b) The Order of Magnitude of I_1 and I_2

It is useful to discuss in a simple way the orders of magnitude of the principal integrals I_1 and $\mathbf{I}_2$. The first purpose is to check the validity of order-of-magnitude arguments which will later (Subsection c) be used to estimate I_4. Secondly, we shall show that there are no unsuspected cancellations by interference, and finally the arguments will give added insight into the mathematics.

We shall use for the integration variables both polar coordinates r, θ and cylindrical coordinates z, ρ, and use the fact that $q_z \approx \delta \approx O(1/\epsilon)$, while $q_0 \equiv q_\perp$ is in general of order 1. It can be shown that the main contribution to the integral I_1 comes from regions of space in which the hypergeometric functions F in the matrix element (5.6) can be replaced by their asymptotic expressions. This is permitted if the arguments of the hypergeometric functions are large compared with 1, *viz.*,

$$p_1 r + \mathbf{p}_1\cdot\mathbf{r} \gg 1. \tag{9.4}$$

Putting the z axis in the direction of $-\mathbf{p}_1$, this means

$$p_1(r-z) \approx \tfrac{1}{2}p_1 r\theta^2 \gg 1, \tag{9.5}$$

or

$$\theta \gg \theta_0 \equiv (p_1 r)^{-\frac{1}{2}}. \tag{9.6}$$

If $r \leqslant \epsilon$, then θ_0 as defined by (9.6) is larger than or equal to $1/\epsilon$, i.e., to the order of magnitude of θ_1 and θ_2. It is then justified, in first approximation, to consider the vectors $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{k}$ to be essentially in the same direction, $-z$.

If the argument $u_1 = p_1(r-z)$ is large, the hypergeometric function is given by the asymptotic formula

$$F_1 = \frac{e^{\pi a_1/2}}{\Gamma(1+ia_1)} e^{ia_1 \log u_1}\left(1-\frac{ia_1^2}{u_1}\right) - \frac{ie^{\pi a_1/2}}{\Gamma(-ia_1)} - \frac{e^{iu_1 - ia_1 \log u_1}}{u_1} + O\left(\frac{1}{u_1^2}\right). \tag{9.7}$$

Through u_1, this expression depends on

$$\rho = r\theta; \tag{9.8}$$

we have

$$u_1 = p_1(r-z) \approx p_1\rho^2/2z. \tag{9.9}$$

The other factor depending substantially on ρ is

$$G \equiv \exp(iq_\perp\rho). \tag{9.10}$$

It is easy to see that there is no important constructive interferences between F_1, F_2, and G.

The integral over ρ arises mainly from values of ρ up to

$$\rho_1 = 1/q, \tag{9.11}$$

because at larger values of ρ, rapid oscillations of G set in. Since the volume element is $2\pi\rho d\rho dz$ and since the first term of (9.7) does not contain any power of ρ as a factor, the ρ integration gives a result of the order

$$1/q^2. \tag{9.12}$$

The integral over z contains only one important z-dependent factor, *viz.*,

$$\exp(iq_z z), \tag{9.13}$$

which shows that values of $z \approx r$ up to $1/q_z$ contribute, and this quantity is of order ϵ. Hence we find from a very simple consideration of orders of magnitude that

$$I_1 \sim \epsilon/q^2, \tag{9.14}$$

which agrees with the explicit evaluation in (7.10).

In $\mathbf{I}_2$ and $\mathbf{I}_3$, we need the derivative dF/du which is in the region of validity of the asymptotic expression (9.7):

$$\frac{dF_1}{du} = e^{\pi a_1/2}\frac{1}{u_1}\left(\frac{e^{ia_1 \log u_1}}{\Gamma(ia_1)} + \frac{e^{iu_1 - ia_1 \log u_1}}{\Gamma(-ia_1)}\right) + O\left(\frac{1}{u_1^2}\right), \tag{9.15}$$

where the first and second term correspond to the two first terms in (9.7).

We need consider only those components of $\mathbf{I}_2$ and $\mathbf{I}_3$ which have a matrix factor of order unity, i.e., (from (6.2)) are in the x or y direction. We have from (6.3a), for example,

$$I_{2x} \sim \frac{1}{\epsilon_1}\int e^{i\mathbf{q}\cdot\mathbf{r}} F_2^* \frac{dF_1}{du_1}(\nabla u_1)_x d^3r. \tag{9.16}$$

[43] In this point, we agree with Bess.

We use the asymptotic expressions (6.30) and (9.15) for F_2^* and dF_1/du_1 [from which F_2^* is essentially of $O(1)$ and dF_1/du_1 is essentially of $O(1/u_1)$, when $\rho>\rho_0=r\theta_0=(r/p_1)^{\frac{1}{2}}$],[44] and the small angle approximations (9.9) and

$$(\nabla u_1)_x \approx p_1\theta \approx p_1\rho/z, \tag{9.17}$$

[note (9.8)]. The integrals over z and ρ are, then, in view of the remark following (9.9),

$$\int \exp(iq_\perp\rho)d\rho \sim 1/q_\perp, \tag{9.18}$$

and

$$\int \exp(iq_z z)dz \sim 1/q_z = O(\epsilon), \tag{9.19}$$

so that

$$I_{2x} \sim 1/q_\perp = O(1), \tag{9.20}$$

in agreement with the explicit result in (7.10).

(c) The Main Term in the Neglected Part of the Wave Function

In this subsection, we shall discuss the integral I_4, Eq. (9.2), which arises from the main term in the neglected part of the wave function. We shall show that this is of order 1, i.e., of the same order as I_2.

We shall evaluate I_4 by replacing ψ_{2a} by the plane wave, $e^{i\mathbf{p}_2\cdot\mathbf{r}}$. This amounts to replacing N_2F_2 by unity, and thus to neglecting some interference effects. The arguments of Subsection b have shown that the elimination of interference effects will not reduce the order of magnitude of the integrals. Furthermore, we take into account that N_1 and N_2 [Eq. (6.31) and remarks preceding Eq. (8.16)] are of order unity.
Then

$$I_4 = \int e^{-i(\mathbf{p}_2+\mathbf{k})\cdot\mathbf{r}}\psi_{1e}d^3r, \tag{9.21}$$

i.e., apart from a constant factor, I_4 is the Fourier component of ψ_{1e}, corresponding to the momentum $\mathbf{p}_2+\mathbf{k}$, or it is the momentum-space wave function. To calculate this, we multiply (9.1) on both sides by $e^{-i(\mathbf{p}_2+\mathbf{k})\cdot\mathbf{r}}$ and integrate over space. One of the integrals thus occurring we integrate by parts:

$$\int e^{-i(\mathbf{p}_2+\mathbf{k})\cdot\mathbf{r}}\nabla^2\psi_{1e}d^3r = -(\mathbf{p}_2+\mathbf{k})^2\int e^{-i(\mathbf{p}_2+\mathbf{k})\cdot\mathbf{r}}\psi_{1e}d^3r. \tag{9.22}$$

Another integral on the left-hand side, *viz.*,

$$2\epsilon a\int e^{-i(\mathbf{p}_2+\mathbf{k})\cdot\mathbf{r}}\psi_{1e}d^3r/r, \tag{9.23}$$

we neglect; this is an approximation similar to the replacement of ψ_{2a} by a plane wave (the error is probably smaller if the Coulomb field is neglected in *both* ψ_1 and ψ_2). Then we get

$$I_4 = -\frac{a^2}{p_1^2-(\mathbf{p}_2+\mathbf{k})^2}\int e^{i(\mathbf{p}_1-\mathbf{p}_2-\mathbf{k})\cdot\mathbf{r}}\frac{d^3r}{r^2}, \tag{9.24}$$

where we have also replaced ψ_{1a} by the plane wave $e^{i\mathbf{p}_1\cdot\mathbf{r}}$ on the right-hand side of (9.1).

Now

$$p_1^2-(\mathbf{p}_2+\mathbf{k})^2 = 2k(\epsilon_2-p_2\cos\theta_2) = D_2, \tag{9.25}$$

and is of $O(1)$, according to (8.7). Therefore, using the definition of $\mathbf{q}$, Eq. (8.3), the order of magnitude is

$$I_4 \sim \int e^{i\mathbf{q}\cdot\mathbf{r}}d^3r/r^2 \sim 1/q. \tag{9.26}$$

As expected, this is of the same order as I_2, namely independent of ϵ, and has also the same q-dependence as I_2. In Bess' calculation, the integral corresponding to I_4 turned out to be of the same order as I_1 which is unreasonable.

Together with the arguments about the matrix factor in Subsection a, this shows that the contribution of I_4 is of order $1/\epsilon$, and that therefore ψ_e can actually be neglected.

(d) Other Contributions

It is rather evident that the contributions from still higher approximations to the wave function are negligible, and that the same is true of cross terms containing "small parts" of both the initial and final wave function, such as

$$I_5 = \int \psi_{2b}^* e^{-i\mathbf{k}\cdot\mathbf{r}}\psi_{1b}d\tau. \tag{9.27}$$

This particular term gives a contribution of order $\epsilon^{-2}\log\epsilon$ to the matrix element, as shown below.

The significant contribution from I_5 will come from those components with matrix factor of order 1, which result from choosing the z component of ψ_{1b} and a component of ψ_{2b}^* perpendicular to z, or *vice versa*. Following the order of magnitude estimation of I_{2x}, we use the asymptotic expression (9.14) and a similar one for dF_2^*/du_2 and consider

$$I_{5xz} \sim \frac{1}{\epsilon_1\epsilon_2}\int e^{i\mathbf{q}\cdot\mathbf{r}}\frac{(\nabla u_1)_x}{u_1}\frac{(\nabla u_2)_z}{u_2}d^3r. \tag{9.28}$$

Noting that

$$(\nabla u_2)_z/u_2 \approx 1/r, \quad (\nabla u_1)_x \approx p_1\theta, \quad u_1 \approx p_1r\theta^2/2, \tag{9.29}$$

[44] Since the integral in (9.16) converges with $dF_1/du_1 \sim 1/u_1$, it will certainly converge in the region $\rho<\rho_0$ where the asymptotic expression (9.15) is not valid and dF_1/du_1 is of order 1 or less.

and using spherical coordinates, we have

$$I_{5xz}\sim\frac{1}{\epsilon^2}\int e^{iq\ r+iq_\perp r\theta\cos\phi}drd\theta d\phi, \tag{9.30}$$

the azimuthal angle ϕ being measured from the direction of $\mathbf{q}$. The integral over ϕ gives $J_0(q_\perp r\theta)$ and that over θ is then $1/q_\perp r$ for $q_\perp r\gtrsim O(1)$, and of $O(1)$ otherwise. The major contribution to the integral over r is thus in the region $1\leqslant r\leqslant 1/q_z$ from which we have

$$\int_1^{1/q_z}(q_\perp r)^{-1}\exp(iq_z r)dr\sim\frac{\log q_z}{q_\perp}=O(\log\epsilon), \tag{9.31}$$

so that I_{5xz} is of order

$$\epsilon^{-2}\log\epsilon. \tag{9.32}$$

This shows that I_5 is smaller than I_4 which we previously discussed.

The integrals which arise from higher corrections have their integrands more and more concentrated at small r. Ultimately, the main contribution to the integrals comes from values of r of the order of one wavelength, i.e., $1/p$. These integrals will then be of order $1/\epsilon^3$. Now it might be possible, with sufficient mathematical skill, to actually calculate the contributions of order $1/\epsilon$, $\epsilon^{-2}\log\epsilon$ and $1/\epsilon^2$ to the matrix element by explicit evaluation of some of the integrals we have neglected. But there would be very little hope for the numerous contributions of order $1/\epsilon^3$; these can probably be evaluated only in polar coordinates.

X. ESTIMATE OF ERROR

In Sec. IX we found that the main term neglected in calculating the matrix element comes from I_4 and is of relative order $1/\epsilon$. We believe that there are no cancellations from other neglected terms and that the error is indeed of order $1/\epsilon$.

Errors are introduced both in the initial and final state wave function. Since $\epsilon_2<\epsilon_1$ for bremsstrahlung, the error in the cross section must be estimated to be of order $1/\epsilon_2$. It has long been known that the Bethe-Heitler treatment is wrong near the upper limit of the bremsstrahlung spectrum, and corrections have been proposed by Heitler.[45] Probably this problem could now be solved, either using the wave function of this paper or a method similar to that of Harvey Hall.[12]

No approximation is made in the wave function of the quantum, $e^{i\mathbf{k}\cdot\mathbf{r}}$. Hence the theory is expected to remain correct for small k.

The total cross section for pair production (or the total energy radiated in bremsstrahlung) is obtained by integration over ϵ_2. Since all energies ϵ_2 between 1 and ϵ_1 contribute about equally, the relative error in the total cross section is about

$$\Phi=\int_1^{\epsilon_1}\varphi(\epsilon_2)d\epsilon_2/\epsilon_1, \tag{10.1}$$

where $\varphi(\epsilon_2)$ is the relative error at ϵ_2. Thus, if

$$\varphi\sim 1/\epsilon_2,\quad \Phi\sim\epsilon_1^{-1}\log\epsilon_1. \tag{10.2}$$

For pair production, k should be substituted for ϵ_1.

[45] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1944).