(infinite) set of graphs: the ladder approximation arising from use of the meson-nucleon equation.[1] Such a calculation represents another extreme in approximation schemes: it drops terms with many mesons in the field in favor of higher order (iterated) scattering graphs. However, any method which arbitrarily neglects an infinite number of relevant terms is, of course, open to the objection that inclusion of these other graphs could vitiate all its conclusions.

ACKNOWLEDGMENT

I wish to thank Professor J. Schwinger for suggesting this investigation and for many helpful conversations during its progress.

PHYSICAL REVIEW VOLUME 93, NUMBER 3 FEBRUARY 1, 1954

The Theory of Quantized Fields. V

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(Received October 26, 1953)

The Dirac field, as perturbed by a time-dependent external electromagnetic field that reduces to zero on the boundary surfaces, is the object of discussion. Apart from the modification of the Green's function, the transformation function differs in form from that of the field-free case only by the occurrence of a field-dependent numerical factor, which is expressed as an infinite determinant. It is shown that, for the class of fields characterized by finite space-time integrated energy densities, a modification of this determinant is an integral function of the parameter measuring the strength of the field and can therefore be expressed as a power series with an infinite radius of convergence. The Green's function is derived therefrom as the ratio of two such power series. The transformation function is used as a generating function for the elements of the occupation number labelled scattering matrix S and, in particular, we derive formulas for the probabilities of creating n pairs, for a system initially in the vacuum state. The general matrix element of S is presented, in terms of the classification that employs a time-reversed description for the negative frequency modes, with the aid of a related matrix Σ, which can be viewed as describing the development of the system in proper time. The latter is characterized as indefinite unitary, in contrast with the unitary property of S, which is verified directly. Two appendices are devoted to determinantal properties.

TIME-DEPENDENT ELECTROMAGNETIC FIELDS

THE previous paper in this series[1] dealt with the Dirac field, as coupled to a second prescribed Dirac field. We shall now discuss the effect of coupling with a prescribed Bose-Einstein field, using the example of the electromagnetic field. The Lagrange function and field equations of this system are presented in Eqs. (IV. 1, 2).

The simplest extension of the work of IV is obtained by supposing that the external electromagnetic field vanishes in the vicinity of the boundary surfaces σ_1 and σ_2, while assuming arbitrary values in the interior of this region. We shall retain the gauge $A_\mu=0$ to describe a zero field. The decomposition of the Dirac field into positive and negative frequency components on σ_1 and σ_2 can be performed as in IV, and the history of the system between σ_1 and σ_2 will be described by the transformation function $(\chi^{(-)\prime}\sigma_1|\chi^{(+)\prime}\sigma_2)$. The source substitution (IV. 33) produces the latter from the transformation function referring to zero eigenvalues,

$$(0\sigma_1|0\sigma_2)=\exp(i\mathcal{W}_0). \qquad (1)$$

The dependence of $\mathcal{W}_0$ upon the source is expressed by [(IV. 37)]

$$\delta\mathcal{W}_0=\int_{-\infty}^{\infty}(dx)[\delta\bar{\eta}\langle\psi\rangle+\langle\bar{\psi}\rangle\delta\eta], \qquad (2)$$

where now

$$\begin{aligned}\gamma_\mu[-i\partial_\mu-eA_\mu(x)]\langle\psi(x)\rangle+m\langle\psi(x)\rangle&=\eta(x),\\ [i\partial_\mu-eA_\mu(x)]\langle\bar{\psi}(x)\rangle\gamma_\mu+m\langle\bar{\psi}(x)\rangle&=\bar{\eta}(x),\end{aligned} \qquad (3)$$

are the equations to be solved in conjunction with the boundary conditions (IV. 40, 41). The associated Green's function [Eqs. (IV. 42, 43)] thus obeys the differential equations

$$\begin{aligned}&\gamma_\mu[-i\partial_\mu-eA_\mu(x)]G_+(x,x')+mG_+(x,x')\\ &=[i\partial_\mu'-eA_\mu(x')]G_+(x,x')\gamma_\mu+mG_+(x,x')=\delta(x-x'),\end{aligned} \qquad (4)$$

and the boundary condition that G_+ as a function of x, shall contain only positive frequencies for $x_0>x_0'$, A, and only negative frequencies for $x_0<x_0'$, A. We have indicated by $x_0>A$ and $x_0<A$ that the domain of nonvanishing field is confined, respectively, to earlier or later times than x_0. The same statements apply with x and x' interchanged.

The compatibility of these two forms of the boundary condition, for arbitrary A_μ, can be ascribed to the charge symmetry of the theory. If the second differ-

[1] J. Schwinger, Phys. Rev. 92, 1283 (1953).

ential equation for $G_+(x,x')$ is transposed with respect to the spinor indices and x interchanged with x', we obtain the differential equation for the charge conjugate Green's function

$$\gamma_\mu[-i\partial_\mu+eA_\mu(x)]\ {}_cG_+(x,x')+m\ {}_cG_+(x,x')=\delta(x-x'), \quad (5)$$

where

$${}_cG_+(x,x')=CG_+{}^{tr}(x',x)C^{-1} \quad (6)$$

and

$$\gamma_\mu{}^{tr}=-C^{-1}\gamma_\mu C. \quad (7)$$

The charge conjugate Green's function is thus obtained from G_+ by the substitution $A_\mu\rightarrow -A_\mu$. The relation inverse to (6),

$$G_+(x,x')=C\ {}_cG_+{}^{tr}(x',x)C^{-1}, \quad (8)$$

now shows that the boundary condition obeyed by $G_+(x,x')$, in its dependence on x', is the same as that of $G_+(x',x)$.

The integration of the differential expression (2) yields

$$\mathcal{W}_0=\int_{-\infty}^{\infty}(dx)(dx')\bar{\eta}(x)G_+(x,x')\eta(x')+w, \quad (9)$$

where w is the constant of integration, which characterizes the null eigenvalue transformation function in the absence of sources. We can no longer argue that this constant vanishes, in view of the presence of the external electromagnetic field. The resulting transformation function differs in form from (IV. 46, 47) only by the addition of w to $\mathcal{W}$. In particular,

$$(\chi^{(-)'}\sigma_1|\chi^{(+)'}\sigma_2)]_0=\exp\left[iw+i\oint d\sigma_\mu\oint d\sigma_{\nu'}\bar{\psi}'(x)\times\gamma_\mu G_+(x,x')\gamma_\nu\psi'(x')\right], \quad (10)$$

where $]_0$ indicates the restriction to zero sources.

The dependence of w upon the electromagnetic field is expressed by

$$\delta_A w=\left\langle\int_{\sigma_2}^{\sigma_1}(dx)\delta_A\mathfrak{L}(x)\right\rangle\Big]_0=\int_{-\infty}^{\infty}(dx)\delta A_\mu(x)\langle j_\mu(x)\rangle\Big]_0, \quad (11)$$

where

$$j_\mu(x)=e\tfrac{1}{2}[\bar{\psi}(x),\gamma_\mu\psi(x)]=-e\ \mathrm{tr}\gamma_\mu(\psi(x)\bar{\psi}(x'))_+\epsilon(x,x')]_{x'\rightarrow x}, \quad (12)$$

and we have used the notation of (IV. 35). In the second version of the current vector, it is understood that an average is taken of the two limiting forms obtained with $x_0-x_0'\rightarrow\pm 0$. The symbol tr expresses diagonal summation with respect to the spinor indices. Now

$$\delta_\eta\langle\psi(x)\rangle]_0=\int_{\sigma_2}^{\sigma_1}(dx')G_+(x,x')\delta\eta(x')=i\left\langle\int_{\sigma_2}^{\sigma_1}(dx')(\psi(x)\bar{\psi}(x')\delta\eta(x'))_+\right\rangle\Big]_0, \quad (13)$$

so that

$$i\langle(\psi(x)\bar{\psi}(x'))_+\rangle\epsilon(x,x')]_0=G_+(x,x'), \quad (14)$$

which reduces to (IV. 83) in the absence of an electromagnetic field. Accordingly,

$$\langle j_\mu(x)\rangle]_0=ie\ \mathrm{tr}\gamma_\mu G_+(x,x), \quad (15)$$

where $G_+(x,x)$ is defined by the same averaged limit as in (12). To construct w, we must integrate the differential expression

$$\delta w=\int_{-\infty}^{\infty}(dx)\ \mathrm{tr}ie\gamma_\mu\delta A_\mu(x)G_+(x,x), \quad (16)$$

with the initial condition that $w=0$ for $A_\mu=0$.

It should be noted that w is an even function of the external field, and therefore is an even function of e. This aspect of charge symmetry follows from the observation

$$\begin{aligned}\mathrm{tr}\gamma_\mu G_+(x,x)&=\mathrm{tr}\gamma_\mu{}^{tr}G_+{}^{tr}(x,x)\\&=-\mathrm{tr}\gamma_\mu\ {}_cG_+(x,x),\end{aligned} \quad (17)$$

which shows that

$$\langle j_\mu(x)\rangle]_0=\tfrac{1}{2}ie\ \mathrm{tr}\gamma_\mu(G_+(x,x)-{}_cG_+(x,x)) \quad (18)$$

is an odd function of the external field. Hence, w is an even function. It is also worth remarking that w is a gauge invariant, Lorentz invariant quantity.

Infinite Determinants

To obtain explicit formal expressions for w, we introduce a notation in which the spinor indices and space-time coordinates are regarded as matrix indices. Thus, the differential equations for the Green's functions are written as

$$[\gamma(p-eA)+m]G_+=G_+[\gamma(p-eA)+m]=1, \quad (19)$$

and the differential expression for w becomes

$$\delta w=Tr(ie\gamma\delta AG_+)=Tr(G_+ie\gamma\delta A), \quad (20)$$

where Tr represents the complete diagonal summation, with respect to continuous coordinates and discrete spinor indices.

We shall employ the Green's function for the zero field situation, which obeys

$$(\gamma p+m)G_+{}^0=G_+{}^0(\gamma p+m)=1. \quad (21)$$

If we multiply the two equations of (19) with G_+^0, on the left and right, respectively, we get the integral equations

$$(1-G_+^0 e\gamma A)G_+ = G_+(1-e\gamma A G_+^0) = G_+^0, \qquad (22)$$

which have the symbolic solution

$$G_+ = (1-G_+^0 e\gamma A)^{-1}G_+^0 = G_+^0(1-e\gamma A G_+^0)^{-1}. \qquad (23)$$

Hence,

$$\begin{aligned} i\delta w &= Tr[(1-e\gamma A G_+^0)^{-1}\delta(-e\gamma A G_+^0)] \\ &= Tr[(1-G_+^0 e\gamma A)^{-1}\delta(-G_+^0 e\gamma A)]. \end{aligned} \qquad (24)$$

We thus encounter the differential expression

$$TrX^{-1}\delta X = \delta(\log \det X), \qquad (25)$$

which, together with the initial condition, $\det 1 = 1$, completely defines the determinant[2] of a matrix (or operator). Therefore,

$$e^{iw} = \det(1-e\gamma A G_+^0) = \det(1-G_+^0 e\gamma A). \qquad (26)$$

The identity of the two forms expresses the determinantal property

$$\begin{aligned} \det(1-G_+^0 e\gamma A) &= \det[G_+^0(1-e\gamma A G_+^0)(G_+^0)^{-1}] \\ &= \det(1-e\gamma A G_+^0). \end{aligned} \qquad (27)$$

We have shown that w is an even function of e. Hence, we must have

$$e^{iw} = \det(1+e\gamma A G_+^0), \qquad (28)$$

which can also be derived directly from the transposition property of determinants. An explicitly even formula now follows from the multiplication property,

$$e^{2iw} = \det(1-e^2\gamma A G_+^0 \gamma A G_+^0), \qquad (29)$$

which would be the result of constructing w from (18). The relation between the values of e^{iw} for two different fields can also be obtained by determinant multiplication. We first observe that (26) can be written

$$e^{iw} = \det[(G_+)^{-1}G_+^0]. \qquad (30)$$

Hence,

$$\begin{aligned} \exp(iw^{(1)})/\exp(iw^{(2)}) \\ &= \det[(G_+^{(1)})^{-1}G_+^0]\det[(G_+^0)^{-1}G_+^{(2)}] \\ &= \det[(G_+^{(1)})^{-1}G_+^{(2)}] \\ &= \det[1-e\gamma(A^{(1)}-A^{(2)})G_+^{(2)}]. \end{aligned} \qquad (31)$$

The matrices that enter in these determinants are of the form $1+\lambda K$. An infinitesimal change in the parameter λ yields, according to (25),

$$\delta \log \det(1+\lambda K) = Tr(1+\lambda K)^{-1}\delta\lambda K. \qquad (32)$$

If we define the logarithm of the matrix $1+\lambda K$ by

$$\log(1+\lambda K) = \int_0^\lambda (1+\lambda' K)^{-1} d\lambda' K, \qquad (33)$$

we find that

$$\det(1+\lambda K) = \exp[Tr \log(1+\lambda K)]. \qquad (34)$$

Under appropriate circumstances, the matrix $\log(1+\lambda K)$ can be expanded in powers of λ and

$$Tr \log(1+\lambda K) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}\lambda^n K_n, \qquad (35)$$

where

$$K_n = TrK^n. \qquad (36)$$

If we then expand the exponential in (34), we get

$$\det(1+\lambda K) = \sum_{n=0}^{\infty}\lambda^n d_n, \qquad (37)$$

with

$$d_n = \sum_k \frac{(K_1)^{k_1}}{k_1!}\frac{(-\frac{1}{2}K_2)^{k_2}}{k_2!}\frac{(\frac{1}{3}K_3)^{k_3}}{k_3!}\cdots, \qquad (38)$$

in which the summation is extended over all non-negative integers k_1, k_2, $\cdots$ such that

$$n = k_1 + 2k_2 + 3k_3 + \cdots. \qquad (39)$$

The direct expansion[3] of the determinant is expressed by the coefficient

$$d_n = \frac{1}{n!}\int (dx_1)\cdots(dx_n)\det{}_{(n)}K(x_i,x_j), \qquad (40)$$

where the summation over spinor indices is understood. The identity of the two expressions, (38) and (40), is established by remarking that each of the $n!$ terms in the development of $\det_{(n)}K(x_i,x_j)$ consists of one or more cycles in the variables. Thus, for $n=6$, one of the terms is

$$-[K(x_1,x_1)][K(x_2,x_3)K(x_3,x_2)][K(x_4,x_5)K(x_5,x_6)K(x_6,x_4)].$$

On carrying out the integrations, this will yield the product of three traces, $-K_1K_2K_3$. In general, we shall find k_1 unary cycles, k_2 binary cycles, etc., and these integers are related to n by (39). The number of terms that have the magnitude $K_1^{k_1}K_2^{k_2}\cdots$ is

$$\frac{n!}{k_1!2^{k_2}k_2!3^{k_3}k_3!\cdots}, \qquad (41)$$

[2] The equivalence of this definition with the customary one is shown in Appendix A.

[3] See, for example, E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Cambridge, 1927), Sec. 11.2. A derivation, with the aid of operator methods, will be found in Appendix B.

where the powers of 2, 3, $\cdots$ express the cyclic symmetry of the traces. A binary cycle contains an odd permutation, and in general the sign factor is

$$(-1)^{k_2+k_4+\cdots}. \tag{42}$$

The resulting expression for (40) is just (38).

The quantity

$$\det{}'(1+\lambda K)=e^{-Tr\lambda K}\det(1+\lambda K) \tag{43}$$

is evidently obtained from $\det(1+\lambda K)$ by placing $K_1=0$. Accordingly, in the power series expansion

$$\det{}'(1+\lambda K)=\sum_{n=0}^{\infty}\lambda^n d_n{}', \tag{44}$$

the coefficients

$$d_n{}'=\frac{1}{n!}\int(dx_1)\cdots(dx_n)\det{}_{(n)}{}'K(x_i,x_j) \tag{45}$$

contain determinants which are obtained from those of d_n by omitting all unary cycles. This is equivalent to striking out the diagonal elements of the n-dimensional matrix. Similarly,

$$\det{}''(1+\lambda K)=\exp[-Tr\lambda K+\tfrac{1}{2}Tr\lambda^2K^2]\times\det(1+\lambda K) \tag{46}$$

has a power series expansion in terms of determinants from which all unary and binary cycles are omitted. However, the latter process cannot be represented by simply omitting elements of the matrix.

The modified determinant $\det'(1+\lambda K)$ obeys the fundamental inequality

$$|\det{}'(1+\lambda K)|^2\leq\exp[Tr\lambda K(\lambda K)^\dagger]. \tag{47}$$

To prove this, we first remark that

$$\begin{aligned}|\det{}'(1+\lambda K)|^2&=\exp[-Tr\lambda K]\det(1+\lambda K)\exp[-Tr(\lambda K)^\dagger]\times\det(1+\lambda K)^\dagger\\&=\exp[Tr\lambda K(\lambda K)^\dagger]\det{}'[(1+\lambda K)(1+\lambda K)^\dagger].\end{aligned} \tag{48}$$

If $H=1+A$ represents a Hermitian, non-negative matrix, we have

$$\det{}'H=\exp[Tr(\log(1+A)-A)]\leq 1, \tag{49}$$

since, for every eigenvalue $1+A'\geq 0$, there exists the inequality

$$\log(1+A')-A'\leq 0. \tag{50}$$

Hence,

$$\det{}'[(1+\lambda K)(1+\lambda K)^\dagger]\leq 1, \tag{51}$$

which establishes[4] (47).

We conclude from this inequality that $\det'(1+\lambda K)$ is devoid of singularities throughout the finite λ plane, provided that

$$TrKK^\dagger<\infty. \tag{52}$$

The modified determinant is then an integral function of λ, and we can assert that the Taylor series (44) converges for all λ. The same statements apply to $\det''(1+\lambda K)$, since

$$|TrK^2|\leq TrKK^\dagger. \tag{53}$$

The dependence of $\det'(1+\lambda K)$ upon the elements of the matrix K is described by

$$\begin{aligned}\delta_K\det{}'(1+\lambda K)&=\det{}'(1+\lambda K)Tr[((1+\lambda K)^{-1}-1)\lambda\delta K]\\&=-\det{}'(1+\lambda K)Tr[(1+\lambda K)^{-1}\lambda K\lambda\delta K],\end{aligned} \tag{54}$$

which indicates that the quantities

$$\begin{aligned}\det{}'(1+\lambda K)(x|(1+\lambda K)^{-1}\lambda K|x')\\=-(\delta/\delta\lambda K(x',x))\det{}'(1+\lambda K)\end{aligned} \tag{55}$$

are integral functions of λ. Now

$$(\delta/\delta K(x',x))d_2{}'=-K(x,x'), \tag{56}$$

and

$$\begin{aligned}(\delta/\delta K(x',x))d_n{}'&=\frac{1}{n!}\int(dx_1)\cdots(dx_n)\sum_{i\neq j}\delta(x'-x_i)\delta(x-x_j)\\&\qquad\times(\partial/\partial K(x_i,x_j))\det{}_{(n)}{}'K(x_i,x_j)\\&=\frac{1}{(n-2)!}\int(dx_3)\cdots(dx_n)[(\partial/\partial K(x_2,x_1))\\&\qquad\times\det{}_{(n)}{}'K(x_i,x_j)]_{x_1=x,x_2=x'},\end{aligned} \tag{57}$$

so that

$$\begin{aligned}\det{}'(1+\lambda K)(x|(1+\lambda K)^{-1}\lambda K|x')\\=\lambda K(x,x')+\sum_{n=1}^{\infty}\lambda^{n+1}\frac{1}{n!}\int(dx_1)\cdots(dx_n)\\\times\det{}_{(n+1)}{}'\begin{bmatrix}K(x,x'), & K(x,x_j)\\K(x_i,x'), & K(x_i,x_j)\end{bmatrix}\end{aligned} \tag{58}$$

is a power series with an infinite radius of convergence, under the stated condition on K. We have employed a determinantal notation which corresponds to the partitioning of the $n+1$ dimensional array with respect to the elements of the first row and of the first column. The discarding of unary cycles here implies the omission of all diagonal elements save the first, $K(x,x')$.

In a similar way, we have

$$\begin{aligned}\delta_K\det{}''(1+\lambda K)&=\det{}''(1+\lambda K)Tr[((1+\lambda K)^{-1}-1+\lambda K)\lambda\delta K]\\&=\det{}''(1+\lambda K)Tr[(1+\lambda K)^{-1}\lambda^2K^2\lambda\delta K],\end{aligned} \tag{59}$$

[4] One can also prove (47) with the aid of Hadamard's inequality, R. Courant and D. Hilbert, *Methoden der Mathematischen Physik* (Interscience Publishers, Inc., New York, 1943), p. 31.

so that

$$\det{}''(1+\lambda K)(x|(1+\lambda K)^{-1}\lambda^2K^2|x') = (\delta/\delta\lambda K(x',x))\det{}''(1+\lambda k) \quad (60)$$

is an integral function of λ, which is represented by the convergent power series

$$\det{}''(1+\lambda K)(x|(1+\lambda K)^{-1}\lambda^2K^2|x') = -\sum_{n=1}^{\infty}\lambda^{n+1}\frac{1}{n!}\int(dx_1)\cdots(dx_n) \times \det{}_{(n+1)}''\begin{bmatrix} 0 & K(x,x_j) \\ K(x_i,x'), & K(x_i,x_j)\end{bmatrix}. \quad (61)$$

The omission of $K(x,x')$, in comparison with the series (58), corresponds to its origin by differentiation from the binary cycle term $\frac{1}{2}TrK^2$.

To employ these results, we first place

$$\lambda K = -e^2\gamma AG_+^0\gamma AG_+^0. \quad (62)$$

We shall show later that the condition (52) on K holds for a certain class of electromagnetic fields. Then

$$\det{}'(1-e^2\gamma AG_+^0\gamma AG_+^0) = \exp[e^2Tr\gamma AG_+^0\gamma AG_+^0]e^{2iw}$$

is an integral function of e, or of the parameter measuring the strength of the field. But, according to (26) and (46), we also have

$$\det{}''(1-e\gamma AG_+^0) = \exp[\tfrac{1}{2}e^2Tr\gamma AG_+^0\gamma AG_+^0]e^{iw}, \quad (64)$$

in which we have used the fact that

$$Tr\gamma AG_+^0 = -Tr\gamma A\ {}_cG_+^0 = 0, \quad (65)$$

since $G_+^0 = {}_cG_+^0$. Hence

$$\det{}'(1-e^2\gamma AG_+^0\gamma AG_+^0) = [\det{}''(1-e\gamma AG_+^0)]^2, \quad (66)$$

which shows that the power series expansion for $\det''(1-e\gamma AG_+^0)$ cannot have a finite radius of convergence, since this would contradict the integral function property of the left side in (66). Therefore (64) is an integral function of e, which is represented by the convergent power series

$$\exp(-ie^2w_1)e^{iw} = \det{}''(1-e\gamma AG_+^0) = 1+\sum_{n=2}^{\infty}e^{2n}\frac{1}{(2n)!}\int(dx_1)\cdots(dx_{2n}) \times \mathrm{tr}\left\{\prod_{i=1}^{2n}(\gamma_iA(x_i))\ \det{}_{(2n)}''G_+^0(x_i,x_j)\right\}. \quad (67)$$

We have exploited the known even character of this function, and written

$$w_1 = \tfrac{1}{2}iTr\gamma AG_+^0\gamma AG_+^0 \quad (68)$$

for the coefficient of e^2 in the expansion of w. Incidentally, since iw is the logarithm of an integral function, its Taylor series will, in general, have a finite radius of convergence, which is the magnitude of the smallest root possessed by the integral function (67).

With the knowledge that $\det''(1-e\gamma AG_+^0)$ is an integral function, we turn to (61) which provides information about the elements of the matrix

$$\begin{aligned}(1-e\gamma AG_+^0)^{-1}-1-e\gamma AG_+^0 &= e\gamma A[G_+^0(1-e\gamma AG_+^0)^{-1}-G_+^0] \\ &= e\gamma A(G_+-G_+^0).\end{aligned} \quad (69)$$

On removing the factor $e\gamma A(x)$ from both sides, we obtain the power series representation of the integral function

$$\det{}''(1-e\gamma AG_+^0)[G_+(x,x')-G_+^0(x,x')] = \sum_{n=1}^{\infty}(-e)^n\frac{1}{n!}\int(dx_1)\cdots(dx_n)\ \mathrm{tr}\left\{\prod_{i=1}^{n}(\gamma_iA(x_i)) \times\det{}_{(n+1)}''\begin{bmatrix}0, & G_+^0(x,x_j)\\ G_+^0(x_i,x'), & G_+^0(x_i,x_j)\end{bmatrix}\right\}. \quad (70)$$

The Green's function $G_+(x,x')$ is thus obtained as the ratio of two integral functions,[5] which are each exhibited explicitly as infinite series. We can also present (70) in terms of the matrix I, defined by

$$G_+ = G_+^0 + G_+^0IG_+^0, \quad (71)$$

or

$$G_+(x,x') = G_+^0(x,x') + \int(dx'')(dx''')G_+^0(x,x'') \times I(x'',x''')G_+^0(x''',x'). \quad (72)$$

Thus,

$$\det{}''(1-e\gamma AG_+^0)I(x,x') = \sum_{n=1}^{\infty}(-e)^n\frac{1}{n!}\int(dx_1)\cdots(dx_n)\ \mathrm{tr}\left\{\prod_{i=1}^{n}(\gamma_iA(x_i)) \times\det{}_{(n+1)}''\begin{bmatrix}0, & \delta(x-x_j)\\ \delta(x_i-x'), & G_+^0(x_i,x_j)\end{bmatrix}\right\} \quad (73)$$

in which $\delta(x-x_j)$ and $\delta(x_i-x')$ also imply delta symbols for the suppressed spinor indices.

In discussing the condition (52), where λK is given by (62), it is well to notice that the value of $\det'(1+\lambda K)$ is not affected by replacing K with $f^{-1}Kf$, where f is an essentially arbitrary function of x, concerning which we only require that it exhibit the space-time localiza-

[5] This result can be identified with the solution of the integral equations (22) by the methods of Fredholm, Hilbert, and Poincaré. We have, in effect, developed the requisite parts of this theory directly from the differential form (25). Applications of the Fredholm procedure to the theory of scattering have been made by several authors [R. Jost and A. Pais, Phys. Rev. **82**, 840 (1951); A. Salam and P. T. Matthews, Phys. Rev. **90**, 690 (1953)].

tion of $A_\mu(x)$, consistent with the same property for $f(x)^{-1}A_\mu(x)$. Thus, we may say that $f(x)=(A(x))^{\frac{1}{2}}$, where $A(x)$ measures the magnitude of the potential vector for the external field. Then

$$Tr\lambda K(\lambda K)^\dagger = e^4\int (dx)(dx')(dx_1)(dx_2)$$
$$\times \mathrm{tr}[f(x)^{-1}\gamma A(x)G_+{}^0(x,x_1)\gamma A(x_1)G_+{}^0(x_1,x')$$
$$\times f(x')^2\gamma_0 G_-{}^0(x',x_2)\gamma A(x_2)$$
$$\times G_-{}^0(x_2,x)\gamma A(x)f(x)^{-1}\gamma_0], \quad (74)$$

and the integrations are all confined to the interior of the region occupied by the field. Conditions for the finiteness of this integral thus refer to the possible singular character of the field in the vicinity of a point (or points). Let us suppose that

$$A(x)\sim[(x-x_0)^2]^{-\frac{1}{2}(1-\beta)}=|x-x_0|^{-(1-\beta)} \quad (75)$$

in the vicinity of x_0. The convergence of the integral involves the behavior of $G_\pm{}^0(x,x')$ for $x\sim x'$. Since the Green's functions satisfy a first-order differential equation with a four-dimensional delta-function inhomogeneity, we have

$$x\sim x': \quad G_\pm{}^0(x,x')\sim|x-x'|^{-3}. \quad (76)$$

The sixteen fold integral (74) will converge about the point x_0, despite the singularities of the four A and the four G^0 provided that $16>4(1-\beta)+12$, or

$$\beta>0. \quad (77)$$

Hence $F(x)$, the measure of the field strength,

$$x\sim x_0: \quad F(x)\sim|x-x_0|^{-(2-\beta)}, \quad (78)$$

cannot be as singular as $|x-x_0|^{-2}$, which implies the integrability of $F(x)^2$. Accordingly, our discussion applies to those electromagnetic fields for which the space-time integrated energy density is finite.[6]

We must observe, finally, that

$$w_1=\tfrac{1}{2}i\int (dx)(dx')\ \mathrm{tr}[\gamma A(x)G_+{}^0(x,x')$$
$$\times\gamma A(x')G_+{}^0(x',x)]=\infty, \quad (79)$$

independently of the field. This stems from the coincidence of the singularities of $G_+{}^0(x,x')$ and $G_+{}^0(x',x)$ at $x=x'$. Since w_1 must be gauge invariant, the potential vectors always occur differentiated, and the singularity of each Green's function is effectively reduced to $|x-x'|^{-2}$. Hence the integral (79) is logarithmically divergent at $x-x'=0$. The explicit result obtained elsewhere[7] can be transformed without difficulty into

$$w_1=-\int(dx)(dx')\tfrac{1}{4}F_{\mu\nu}(x)w_1(x-x')F_{\mu\nu}(x'), \quad (80)$$

where

$$w_1(x-x')=\frac{1}{4\pi^2}\int_{2m}^{\infty}\kappa d\kappa[(1-(2m/\kappa)^2)^{\frac{1}{2}}$$
$$-\tfrac{1}{3}(1-(2m/\kappa)^2)^{\frac{3}{2}}]\mathcal{G}_+(x-x',\kappa), \quad (81)$$

and

$$\mathcal{G}_+(x-x',\kappa)=\frac{1}{(2\pi)^4}\int(dk)\frac{e^{ik(x-x')}}{k^2+\kappa^2-i\epsilon}, \quad \epsilon\to+0 \quad (82)$$

is the outgoing wave Green's function associated with the differential equation

$$(-\partial_\mu{}^2+\kappa^2)\mathcal{G}_+(x-x',\kappa)=\delta(x-x'). \quad (83)$$

The quantity κ, which appears as a mass parameter in the last equation, ranges from $2m$ to infinity, thereby producing the logarithmic divergence that we have already recognized.

The divergent part is separated by writing

$$\mathcal{G}_+(x-x',\kappa)=\kappa^{-2}\delta(x-x')+\kappa^{-2}\partial_\mu{}^2\mathcal{G}_+(x-x',\kappa), \quad (84)$$

which yields

$$w_1(x-x')=C\delta(x-x')+\partial_\mu{}^2\omega(x-x'), \quad (85)$$

where

$$C=\frac{1}{6\pi^2}(\log(\kappa/m)-\tfrac{5}{6})]_{\kappa\to\infty}, \quad (86)$$

and

$$\omega(x-x')=\frac{1}{4\pi^2}\int_{2m}^{\infty}\frac{d\kappa}{\kappa}[(1-(2m/\kappa)^2)^{\frac{1}{2}}$$
$$-\tfrac{1}{3}(1-(2m/\kappa)^2)^{\frac{3}{2}}]\mathcal{G}_+(x-x',\kappa). \quad (87)$$

Thus,

$$w_1=-C\int(dx)\tfrac{1}{4}F_{\mu\nu}{}^2(x)+\tfrac{1}{2}\int(dx)(dx')$$
$$\times\partial_\nu F_{\mu\nu}(x)\omega(x-x')\partial_\lambda{}'F_{\mu\lambda}(x'), \quad (88)$$

and the second term is finite for the class of fields to which our results refer. This follows from the remark that $\mathcal{G}_+(x-x',\kappa)\sim|x-x'|^{-2}f(\kappa|x-x'|)$, for $x\sim x'$. If the derivatives of the fields have the singularity $|x-x_0|^{-(3-\beta)}$, $\beta>0$, the integral

$$\int(dx)(dx')\partial_\nu F_{\mu\nu}(x)\mathcal{G}_+(x-x',\kappa)\partial_\lambda F_{\mu\lambda}(x')$$

is convergent, and behaves like $\kappa^{-2\beta}$ for $\kappa\to\infty$. The final integration is therefore convergent.

[6] This criterion is also stated by A. Salam and P. T. Matthews, reference 5.

[7] J. Schwinger, Phys. Rev. **82**, 664 (1951). In Eqs. (6.26) and (6.29), the denominators printed as $4\pi^2$ should be $(4\pi)^2$.

Although w is divergent,

$$|e^{iw}|^2=e^{-2Imw} \tag{89}$$

is finite, since the divergent parameter C is real. We shall derive a useful determinantal expression for this quantity. The matrix I, which has been introduced in (71), is also defined by

$$e\gamma AG_+=IG_+{}^0,\quad G_+e\gamma A=G_+{}^0I, \tag{90}$$

or

$$I=e\gamma A(1-G_+{}^0e\gamma A)^{-1}=(1-e\gamma AG_+{}^0)^{-1}e\gamma A. \tag{91}$$

The adjoint matrix

$$\bar{I}=\gamma_0 I^\dagger\gamma_0 \tag{92}$$

is characterized by the same equations, but with $G_-{}^0$, G_- replacing $G_+{}^0$, G_+. If we solve (91) and the analogous adjoint equations for $e\gamma A$, we find that

$$\begin{aligned}e\gamma A&=(1+IG_+{}^0)^{-1}I=I(1+G_+{}^0I)^{-1}\\&=\bar{I}(1+G_-{}^0\bar{I})^{-1}=(1+\bar{I}G_-{}^0)^{-1}\bar{I},\end{aligned} \tag{93}$$

which yields the important equations

$$I-\bar{I}=I(G_+{}^0-G_-{}^0)\bar{I}=\bar{I}(G_+{}^0-G_-{}^0)I. \tag{94}$$

The zero-field retarded and advanced Green's functions are given by

$$\begin{aligned}G_{\text{ret}}{}^0(x,x')&=G_+{}^0(x,x')+iS^{(-)}(x,x')\\&=G_-{}^0(x,x')+iS^{(+)}(x,x')\\&=\begin{cases}i[S^{(+)}(x,x')+S^{(-)}(x,x')], & x_0>x_0'\\ 0, & x_0\leq x_0',\end{cases}\end{aligned} \tag{95}$$

and

$$\begin{aligned}G_{\text{adv}}{}^0(x,x')&=G_+{}^0(x,x')-iS^{(+)}(x,x')\\&=G_-{}^0(x,x')-iS^{(-)}(x,x')\\&=\begin{cases}0, & x_0\geq x_0'\\ -i[S^{(+)}(x,x')+S^{(-)}(x,x')], & x_0<x_0',\end{cases}\end{aligned} \tag{96}$$

in which we have written

$$\begin{aligned}S^{(+)}(x,x')&=\sum_{+,p}\psi_{\lambda p}(x)\bar{\psi}_{\lambda p}(x'),\\ S^{(-)}(x,x')&=\sum_{-,p}\psi_{\lambda p}(x)\bar{\psi}_{\lambda p}(x').\end{aligned} \tag{97}$$

Now

$$\begin{aligned}1-G_{\text{ret}}{}^0e\gamma A&=(1-iS^{(-)}I)(1-G_+{}^0e\gamma A)\\&=(1+iS^{(+)}\bar{I})(1-G_-{}^0e\gamma A),\end{aligned} \tag{98}$$

and

$$\begin{aligned}1-G_{\text{adv}}{}^0e\gamma A&=(1+iS^{(+)}I)(1-G_+{}^0e\gamma A)\\&=(1+iS^{(-)}\bar{I})(1-G_-{}^0e\gamma A),\end{aligned} \tag{99}$$

from which we obtain

$$\begin{aligned}\det(1-G_{\text{ret}}{}^0e\gamma A)&=\det(1-iS^{(-)}I)\det(1-G_+{}^0e\gamma A)\\&=\det(1-iS^{(+)}\bar{I})\det(1-G_-{}^0e\gamma A)\end{aligned} \tag{100}$$

and

$$\begin{aligned}\det(1-G_{\text{adv}}{}^0e\gamma A)&=\det(1+iS^{(+)}I)\det(1-G_+{}^0e\gamma A)\\&=\det(1+iS^{(-)}\bar{I})\det(1-G_-{}^0e\gamma A).\end{aligned} \tag{101}$$

But

$$\det(1-G_{\text{ret}}{}^0e\gamma A)=\det(1-G_{\text{adv}}{}^0e\gamma A)=1, \tag{102}$$

since all cycles vanish when the matrix $K(x,x')$ is proportional to $G_{\text{ret}}{}^0(x,x')$ or $G_{\text{adv}}{}^0(x,x')$. In particular, unary cycles do not occur, in view of the definitions[8] of these Green's functions at $x_0=x_0'$.
Therefore,

$$\begin{aligned}e^{iw}=\det(1-G_+{}^0e\gamma A)&=[\det(1-iS^{(-)}I)]^{-1}\\&=[\det(1+iS^{(+)}I)]^{-1}\end{aligned} \tag{103}$$

and

$$\begin{aligned}e^{-iw^*}=\det(1-G_-{}^0e\gamma A)&=[\det(1+iS^{(-)}\bar{I})]^{-1}\\&=[\det(1-iS^{(+)}\bar{I})]^{-1}.\end{aligned} \tag{104}$$

On multiplying together corresponding expressions in (103) and (104) we encounter determinants of the matrices

$$\begin{aligned}(1-iS^{(-)}I)(1+iS^{(-)}\bar{I})&=1-iS^{(-)}(I-\bar{I})+S^{(-)}IS^{(-)}\bar{I}\\&=1+S^{(-)}IS^{(+)}\bar{I},\end{aligned} \tag{105}$$

and

$$(1+iS^{(+)}I)(1-iS^{(+)}\bar{I})=1+S^{(+)}IS^{(-)}\bar{I}, \tag{106}$$

in which we have used (94), written as

$$I-\bar{I}=iI(S^{(+)}-S^{(-)})\bar{I}=i\bar{I}(S^{(+)}-S^{(-)})I, \tag{107}$$

in virtue of the relation

$$G_+{}^0-G_-{}^0=i(S^{(+)}-S^{(-)}). \tag{108}$$

Hence

$$\begin{aligned}|e^{iw}|^2&=[\det(1+S^{(-)}IS^{(+)}\bar{I})]^{-1}\\&=[\det(1+S^{(+)}IS^{(-)}\bar{I})]^{-1}.\end{aligned} \tag{109}$$

Only matrix elements of I and $\bar{I}$ connecting positive and negative frequency modes are involved here. With an evident notation to designate these submatrices, we have

$$\begin{aligned}|e^{iw}|^2&=[\det(1+I_{-+}\bar{I}_{+-})]^{-1}\\&=[\det(1+I_{+-}\bar{I}_{-+})]^{-1}\end{aligned} \tag{110}$$

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[8] The arbitrariness of the definitions for $x_0=x_0'$ is without influence on the final result. Let $G_{\text{ret}}{}^0(x,x')$, for example, be more generally defined at $x_0-x_0'=0$ as the numerical multiple $\mu(0\leq\mu\leq 1)$ of the value for $x_0-x_0'=+0$, namely,

$$i\,\Sigma_{\lambda p}\,\psi_{\lambda p}(x)\bar{\psi}_{\lambda p}(x')[=i\gamma_0\delta(\mathbf{x}-\mathbf{x}')].$$

Then

$$\begin{aligned}\det(1-G_{\text{ret}}{}^0e\gamma A)&=\exp\left[-\int(dx)\,\operatorname{tr}G_{\text{ret}}{}^0(x,x)e\gamma A(x)\right]\\&=\exp\left[-i\mu\sum_{\lambda p}\int(dx)\bar{\psi}_{\lambda p}(x)e\gamma A(x)\psi_{\lambda p}(x)\right],\end{aligned}$$

which is of absolute value unity. This is the property actually employed in deriving (109).

and

$$\bar{I}_{+-}=I_{-+}^*, \quad \bar{I}_{-+}=I_{+-}^*. \tag{111}$$

Hence $I_{-+}\bar{I}_{+-}$ and $I_{+-}\bar{I}_{-+}$ are nonnegative Hermitian matrices, which possess nonnegative real eigenvalues, and

$$|e^{iw}|^2\leq 1. \tag{112}$$

The Scattering Matrix

The history of the Dirac field under the influence of the external electromagnetic field is given by the transformation function (10), which we now employ as a generating function for the occupation number transformation function $(n\sigma_1|n'\sigma_2)$, according to

$$(\chi^{(-)'}\sigma_1|\chi^{(+)'}\sigma_2)=\sum_{n,n'}(\chi^{(-)'}|n)(n\sigma_1|n'\sigma_2)(n'|\chi^{(+)'}). \tag{113}$$

We shall represent this transformation function in terms of the matrix, referred to the standard surface, of a unitary operator S,

$$(n\sigma_1|n'\sigma_2)=\exp(iP(n)x_1)(n|S|n')\exp(-iP(n')x_2). \tag{114}$$

We can regard S as describing an equivalent disturbance which is localized on the standard surface.

On introducing the matrix I, according to (72), we get

$$\oint d\sigma_\mu \oint d\sigma_\nu'\bar{\psi}'(x)\gamma_\mu G_+(x,x')\gamma_\nu\psi'(x')$$
$$=\oint d\sigma_\mu \oint d\sigma_\nu'\bar{\psi}'(x)\gamma_\mu G_+{}^0(x,x')\gamma_\nu\psi'(x')$$
$$+\int(dx)(dx')\bar{\psi}'(x)I(x,x')\psi'(x'), \tag{115}$$

in which we have written at interior points, as in IV,

$$\psi'(x)=i\oint d\sigma_\mu' G_+{}^0(x,x')\gamma_\mu\psi'(x'),$$
$$\bar{\psi}'(x)=-i\oint d\sigma_\mu'\bar{\psi}'(x')\gamma_\mu G_+{}^0(x',x), \tag{116}$$

and, it should be noted, the Green's functions are those for zero external field. Hence,

$$(\chi^{(-)'}\sigma_1|\chi^{(+)'}\sigma_2)=(\chi^{(-)'}\sigma_1|\chi^{(+)'}\sigma_2)]_0 e^{iw}$$
$$\times\exp\left[i\int(dx)(dx')\bar{\psi}'(x)I(x,x')\psi'(x')\right], \tag{117}$$

where, in this section, $]_0$ designates the absence of an external electromagnetic field. It will be recalled that

$$(\chi^{(-)'}\sigma_1|\chi^{(+)'}\sigma_2)]_0$$
$$=\exp[\textstyle\sum_{\lambda p}\chi_{\lambda p}{}^{(-)'}e^{ipx_1}e^{-ipx_2}\chi_{\lambda p}{}^{(+)'}], \tag{118}$$

and that

$$\psi'(x)=\textstyle\sum_+\psi_{\lambda p}(x)e^{-ipx_2}\chi_{\lambda p}{}^{(+)'}+\sum_-\psi_{\lambda p}(x)e^{ipx_1}\chi_{\lambda p}{}^{(-)'},$$
$$\bar{\psi}'(x)=\textstyle\sum_-\bar{\psi}_{\lambda p}(x)e^{-ipx_2}\chi_{\lambda p}{}^{(+)'}+\sum_+\bar{\psi}_{\lambda p}(x)e^{ipx_1}\chi_{\lambda p}{}^{(-)'}. \tag{119}$$

On incorporating the phase factors e^{ipx_1}, e^{-ipx_2} into the eigenvalues, we arrive at the generating function for the scattering matrix, $(n|S|n')$,

$$\sum(\chi^{(-)'}|n)(n|S|n')(n'|\chi^{(+)'})$$
$$=e^{iw}\exp[\sum_{++}\chi_{\lambda p}{}^{(-)'}(\delta_{\lambda p,\lambda'p'}+iI(\lambda p,\lambda'p'))\chi_{\lambda'p'}{}^{(+)'}$$
$$+\sum_{--}\chi_{\lambda p}{}^{(-)'}(\delta_{\lambda p,\lambda'p'}-iI(\lambda'p',\lambda p))\chi_{\lambda'p'}{}^{(+)'}$$
$$+\sum_{+-}\chi_{\lambda p}{}^{(-)'}iI(\lambda p,\lambda'p')\chi_{\lambda'p'}{}^{(-)'}$$
$$+\sum_{-+}\chi_{\lambda p}{}^{(+)'}iI(\lambda p,\lambda'p')\chi_{\lambda'p'}{}^{(+)'}], \tag{120}$$

where

$$I(\lambda p,\lambda'p')=\int(dx)(dx')\bar{\psi}_{\lambda p}(x)I(x,x')\psi_{\lambda'p'}(x'). \tag{121}$$

A simplified generating function that describes the transitions of a system known to be initially in the vacuum state, is obtained by placing the eigenvalues $\chi^{(+)'}$ equal to zero,

$$\textstyle\sum_n(\chi^{(-)'}|n)(n|S|0)$$
$$=e^{iw}\exp[i\sum_{+-}\chi_{\lambda p}{}^{(-)'}I(\lambda p,\lambda'p')\chi_{\lambda'p'}{}^{(-)'}]. \tag{122}$$

Hence,

$$(0|S|0)=e^{iw}, \tag{123}$$

and

$$(n_+n_-|S|0)=\delta_{n_+,n_-}e^{iw}i^{n^2}\det{}_{(n)}I(+,-), \tag{124}$$

where $n=n_+=n_-$ is the number of pairs of oppositely charged particles that are created, and the n-dimensional determinant is constructed from the elements $I(\lambda p,\lambda'p')$, where the row index λp ranges, in standard order, over the positive frequency modes that are occupied in the final state, and the column index similarly refers to the negative frequency modes. The factor $i^{n^2}(=1,\ n$ even; $=i,\ n$ odd) arises from i^n combined with $(-1)^{\frac{1}{2}n(n-1)}$, the latter being introduced on bringing the eigenvalues into standard order. The determinant can also be written as

$$\det{}_{(n)}I(+,-)=\frac{1}{(n!)^2}\int(dx_1)\cdots(dx_n)(dx_1')\cdots(dx_n')$$
$$\times(\det{}_{(n)}\bar{\psi}_+(x_i))(\det{}_{(n)}I(x_i,x_j'))$$
$$\times(\det{}_{(n)}\psi_-(x_j')). \tag{125}$$

The probability that the system persist in the vacuum state is thus given by

$$p(0,0)=|(0|S|0)|^2=|e^{iw}|^2, \tag{126}$$

while the probability for creating n pairs of particles in specified modes is

$$p(n_+n_-,0) = |e^{iw}|^2 |\det{}_{(n)} I(+,-)|^2. \quad (127)$$

The total probability for the creation of n pairs is, therefore,

$$p_{n,0} = \sum_{n_+=n_-=n} p(n_+n_-,0) = |e^{iw}|^2 \frac{1}{(n!)^2} \sum_{+-} (\det{}_{(n)} I(+,-)) \times (\det{}_{(n)} \bar{I}(-,+)), \quad (128)$$

in which the latter summation is extended independently over the n positive frequency modes, and the n negative frequency modes. The factor $1/(n!)^2$ thus removes the repetitious counting of final states. We insert (125), together with the analogous expression for $\det_{(n)} \bar{I}(-,+)$, and employ determinantal relations of the type

$$\frac{1}{n!} \sum_- (\det{}_{(n)} \psi_-(x_i))(\det{}_{(n)} \bar{\psi}_-(x_j)) = \det{}_{(n)} S^{(-)}(x_i,x_j'), \quad (129)$$

and

$$\frac{1}{n!} \int (dx_1') \cdots (dx_n')(\det{}_{(n)} I(x_i,x_j)) \times (\det{}_{(n)} S^{(-)}(x_j',x_k'')) = \det{}_{(n)} (x_i|IS^{(-)}|x_k''). \quad (130)$$

This yields

$$p_{n,0} = |e^{iw}|^2 \frac{1}{n!} \int (dx_1) \cdots (dx_n) \times \det{}_{(n)} (x_i|S^{(+)}IS^{(-)}\bar{I}|x_j). \quad (131)$$

The probability for encountering the system in some final state is thus

$$\sum_{n=0}^{\infty} p_{n,0} = |e^{iw}|^2 \sum_{n=0}^{\infty} \frac{1}{n!} \int (dx_1) \cdots (dx_n) \times \det{}_{(n)} (x_i|S^{(+)}IS^{(-)}\bar{I}|x_j) = |e^{iw}|^2 \det(1+S^{(+)}IS^{(-)}\bar{I}) = 1, \quad (132)$$

in virtue of the expression (109) for the quantity which is the vacuum-vacuum transition probability.

The general matrix element of S is advantageously presented in terms of the classification introduced in IV, which employs a time-reversed description for the negative frequency modes. We take the generating function (113) for the matrix of S and write

$$(\chi^{(-)\prime}|S|\chi^{(+)\prime}) = e^{iw}[\bar{\psi}'|\Sigma|\psi'], \quad (133)$$

so that

$$[\bar{\psi}'|\Sigma|\psi'] = \exp\left[i \oint d\sigma_\mu \oint d\sigma_\nu' \bar{\psi}'(x)\gamma_\mu \times G_+(x,x')\gamma_\nu\psi'(x') \right], \quad (134)$$

and it is understood that the eigenvalues on the right are to be referred to the standard surface. This expression serves as the generating function for the occupation number matrix of Σ, according to

$$[\bar{\psi}'|\Sigma|\psi'] = \sum_{N,N'} [\bar{\psi}'|N][N|\Sigma|N'][N'|\psi']. \quad (135)$$

In virtue of the relation

$$(\chi^{(-)\prime}|n)(n'|\chi^{(+)\prime}) = (-1)^{N_-}(-1)^{\frac{1}{2}(N_--N_-')(N_-+1-N_-')} \times [\bar{\psi}'|N][N'|\psi'], \quad (136)$$

we have the following connection between the matrices of S and Σ:

$$(n|S|n') = e^{iw}(-1)^{N_-}(-1)^{\frac{1}{2}(N_--N_-')(N_-+1-N_-')} \times [N|\Sigma|N']. \quad (137)$$

The occupation numbers $N = n_+, n_-'$ are associated with modes that propagate out of the region bounded by σ_1 and σ_2 (positive frequencies on σ_1, negative frequencies on σ_2), while the occupation numbers $N' = n_+', n_-$ are those of modes that propagate into the region (positive frequencies on σ_2, negative frequencies on σ_1). Hence Σ connects an "initial" state described by incoming fields with a "final" state specified by outgoing fields. The sense of development is that of time for the positive frequency modes, but is reversed for the negative frequency modes. Without attempting to justify the term here, we speak of Σ as describing the development of the system in proper time. It should be noted that the number of "particles" is conserved, $N = N'$, as is evident from the structure of the generating function (134). Thus, $n_+ + n_-' = n_+' + n_-$, or $n_+ - n_- = n_+' - n_-'$; the conservation of particles in proper time is equivalent to the conservation of charge is conventional time.[9]

In the absence of an external field, we have $S=1$, and $\Sigma = \epsilon$, where, according to (IV. 96)

$$[\bar{\psi}'|\epsilon|\psi'] = \exp[\sum \bar{\psi}_{\lambda p}'\epsilon(\lambda)\psi_{\lambda p}'] = \sum [\bar{\psi}'|N](-1)^{N_-}[N|\psi'], \quad (138)$$

so that

$$[N|\epsilon|N'] = \delta_{N,N'}(-1)^{N_-}. \quad (139)$$

This also follows from (137). On using (115) and (138), the generating function (134) assumes the form

$$[\bar{\psi}'|\Sigma|\psi'] = \exp[\sum \bar{\psi}_{\lambda p}'\sigma(\lambda p, \lambda' p')\psi_{\lambda' p'}'], \quad (140)$$

with

$$\sigma(\lambda p, \lambda' p') = \epsilon(\lambda)\delta_{\lambda p, \lambda' p'} + iI(\lambda p, \lambda' p'). \quad (141)$$

[9] See R. P. Feynman, Phys. Rev. 76, 749 (1949).

The general occupation number matrix element of $\sum$ is thereby obtained as

$$[N|\sum|N']=\delta_{N,N'}\det{}_{(N)}\sigma(\lambda p,\lambda' p'), \qquad (142)$$

where λp and $\lambda' p'$ range, in standard order, over the occupied modes of the "final" and "initial" states, respectively.

We shall now use the connection between I and $\bar{I}$, Eq. (107), to prove that $\sum$ satisfies the relations

$$\sum{}^{\dagger}\epsilon\sum=\sum\epsilon\sum{}^{\dagger}=\epsilon. \qquad (143)$$

This indefinite-unitary property will first be established for $\sigma(\lambda p,\lambda' p')$, which represents the sub-matrix of $\sum$ for single "particle" states. We combine (141) with

$$\sigma^{\dagger}(\lambda p,\lambda' p')=\epsilon(\lambda)\delta_{\lambda p,\lambda' p'}-i\bar{I}(\lambda p,\lambda' p'), \qquad (144)$$

and construct

$$\sum_{\lambda'' p''}\sigma^{\dagger}(\lambda p,\lambda'' p'')\epsilon(\lambda'')\sigma(\lambda'' p'',\lambda' p')=\epsilon(\lambda)\delta_{\lambda p,\lambda' p'} +i[I(\lambda p,\lambda' p')-\bar{I}(\lambda p,\lambda' p')] +\sum_{\lambda'' p''}\bar{I}(\lambda p,\lambda'' p'')\epsilon(\lambda'')I(\lambda'' p'',\lambda' p'). \qquad (145)$$

But (107) implies that

$$I(\lambda p,\lambda' p')-\bar{I}(\lambda p,\lambda' p') =i\sum_{\lambda'' p''}\bar{I}(\lambda p,\lambda'' p'')\epsilon(\lambda'')I(\lambda'' p'',\lambda' p'). \qquad (146)$$

Therefore

$$\sum_{\lambda'' p''}\sigma^{\dagger}(\lambda p,\lambda'' p'')\epsilon(\lambda'')\sigma(\lambda'' p'',\lambda' p')=\epsilon(\lambda)\delta_{\lambda p,\lambda' p'}, \qquad (147)$$

or, in matrix notation,

$$\sigma^{\dagger}\epsilon\sigma=\epsilon. \qquad (148)$$

Since the fundamental relation between I and $\bar{I}$ is unaltered by the substitution $I\leftrightarrow -\bar{I}$, we also have

$$\sigma\epsilon\sigma^{\dagger}=\epsilon. \qquad (149)$$

The general statement can be deduced from (142). With

$$[N|\sum{}^{\dagger}|N']=\delta_{N,N'}\det{}_{(N)}\sigma^{\dagger}(\lambda p,\lambda' p'), \qquad (150)$$

we get

$$\begin{aligned}\sum_{N''}[N|\sum{}^{\dagger}\epsilon|N''][N''|\sum|N'] &=\delta_{N,N'}\frac{1}{N!}\sum_{\lambda'' p''}\det{}_{(N)}(\sigma^{\dagger}(\lambda p,\lambda'' p'')\epsilon(\lambda''))\times\det{}_{(N)}(\sigma(\lambda'' p'',\lambda' p'))\\ &=\delta_{N,N'}\det{}_{(N)}\Big(\sum_{\lambda'' p''}\sigma^{\dagger}(\lambda p,\lambda'' p'')\epsilon(\lambda'')\sigma(\lambda'' p'',\lambda' p')\Big)\\ &=\delta_{N,N'}(-1)^{N_-}=[N|\epsilon|N'], \end{aligned}\qquad (151)$$

which establishes the first part of (143). The proof of the second part is based analogously upon (149). In a closely related derivation, we combine the partial generating functions

$$\sum_{N'}[N|\sum|N'][N'|\psi']=\prod_{\lambda p}\Big(\sum_{\lambda' p'}\sigma(\lambda p,\lambda' p')\psi_{\lambda' p'}{}'\Big)^{N}, \qquad (152)$$

and

$$\sum_{N'}[\bar{\psi}'|N'][N'|\sum{}^{\dagger}\epsilon|N] ='\prod_{\lambda p}\Big(\sum_{\lambda' p'}\bar{\psi}_{\lambda' p'}{}'\sigma^{\dagger}(\lambda' p',\lambda p)\epsilon(\lambda)\Big)^{N}, \qquad (153)$$

into

$$\begin{aligned}[\bar{\psi}'|\sum{}^{\dagger}\epsilon\sum|\psi'] &=\exp[\sum\bar{\psi}_{\lambda p}{}'\sigma^{\dagger}(\lambda p,\lambda'' p'')\epsilon(\lambda'')\sigma(\lambda'' p'',\lambda' p')\psi_{\lambda' p'}{}']\\ &=\exp[\sum\bar{\psi}_{\lambda p}{}'\epsilon(\lambda)\psi_{\lambda p}{}']\\ &=[\bar{\psi}'|\epsilon|\psi'].\end{aligned} \qquad (154)$$

The indefinite-unitary property of $\sum$ implies, in particular, that

$$\sum{}_N(-1)^{N_- - N_-'}|[N|\sum|N']|^2=1, \qquad (155)$$

which indicates that $|[N|\sum|N']|^2$ is not necessarily less than unity. But this is to be anticipated from the significance of these quantities as relative, rather than absolute probabilities,

$$|[N|\sum|N']|^2=p(n,n')/p(0,0). \qquad (156)$$

Consider, for example, the probability for no change in a single particle state, relative to the probability for the maintainance of the vacuum state,

$$p(1_{\lambda p},1_{\lambda p})/p(0,0)=|\sigma(\lambda p,\lambda p)|^2. \qquad (157)$$

The one-particle version of (155),

$$\sum_{\lambda' p'}\epsilon(\lambda)\epsilon(\lambda')|\sigma(\lambda' p',\lambda p)|^2=1, \qquad (158)$$

asserts that

$$|\sigma(\lambda p,\lambda p)|^2=1-\sum_{\lambda' p'\neq\lambda p}\epsilon(\lambda)\epsilon(\lambda')|\sigma(\lambda' p',\lambda p)|^2. \qquad (159)$$

Thus, for $\lambda>0$, say, we have

$$p(1_{\lambda p},1_{\lambda p})/p(0,0)=1-\sum{}_+|I(\lambda' p',\lambda p)|^2 +\sum{}_-|I(\lambda' p',\lambda p)|^2, \qquad (160)$$

where the positive frequency mode summation omits λp. These oppositely signed summations express the changes in the probability ratio, relative to unity, produced by transitions of the particle to other positive frequency modes, and by the exclusion principle suppression of pair creation in which a particle would have occupied the mode λp.

We shall now supply the explicit verification that S is a unitary operator. The proof is more involved than the elementary demonstration of the indefinite-unitary property possessed by $\sum$. This is attributable to the non-conservation of the number of particles, as compared with the number of "particles." Our procedure begins with (117), written as

$$(\Phi(\chi^{(-)\prime})S\Psi(\chi^{(+)\prime}))=(\Phi(\chi^{(-)\prime})\Psi(x^{(+)\prime}))e^{iw}\times\exp\left[i\int(dx)(dx')\bar{\psi}'(x)I(x,x')\psi'(x')\right],\quad(161)$$

with the usual understanding on referring eigenvalues to the standard surface. By introducing the operators that possess the eigenvalues $\chi_{\lambda p}{}^{(-)\prime}$ we can exhibit the vector $S\Psi(\chi^{(+)\prime})$, and thereby the adjoint vector $\Phi(\chi^{(-)\prime})S^\dagger$. We must then prove that the product $(\chi^{(-)\prime}|S^\dagger S|\chi^{(+)\prime})$ equals $(\chi^{(-)\prime}|\chi^{(+)\prime})$. The evaluation of this product is accomplished by bringing the operators to bear upon their eigenvectors, which will yield a function of the eigenvalues, multiplied into $(\chi^{(-)\prime}|\chi^{(+)\prime})$. This function of the eigenvalues can also be obtained by making the substitution $\chi_{\lambda p}{}^{(\pm)}\rightarrow\chi_{\lambda p}{}^{(\pm)}+\chi_{\lambda p}{}^{(\pm)\prime}$, and evaluating the null eigenvalue matrix element. Thus, the demonstration of $S^\dagger S=1$ is reduced to the verification that

$$|e^{iw}|^2(0|F|0)=1,\quad(162)$$

where

$$F=\exp\left\{-i\int(dx)(dx')[\bar{\psi}'(x)+\textstyle\sum_-\bar{\psi}_{\lambda p}(x)\chi_{\lambda p}{}^{(+)}]\times\bar{I}(x,x)[\psi'(x')+\textstyle\sum_+\psi_{\lambda'p'}(x')\chi_{\lambda'p'}{}^{(+)}]\right\}\times\exp\left\{i\int(dx)(dx')[\bar{\psi}'(x)+\textstyle\sum_+\bar{\psi}_{\lambda p}(x)\chi_{\lambda p}{}^{(-)}]\times I(x,x')[\psi'(x')+\textstyle\sum_-\psi_{\lambda'p'}(x')\psi_{\lambda'p'}{}^{(-)}]\right\}.\quad(163)$$

The commutation properties of the operators $\chi_{\lambda p}{}^{(\pm)}$ should be noted. Since

$$(\chi^{(-)\prime}|\chi^{(+)\prime})=\exp(\sum_{\lambda p}\chi_{\lambda p}{}^{(-)\prime}\chi_{\lambda p}{}^{(+)\prime}),\quad(164)$$

we have

$$\delta(\chi^{(-)\prime}|\chi^{(+)\prime})=(\chi^{(-)\prime}|\chi^{(+)\prime})\sum_{\lambda p}(\delta\chi_{\lambda p}{}^{(-)\prime}\chi_{\lambda p}{}^{(+)\prime}+\chi_{\lambda p}{}^{(-)\prime}\delta\chi_{\lambda p}{}^{(+)\prime})=(\chi^{(-)\prime}|\sum_{\lambda p}(\delta\chi_{\lambda p}{}^{(-)}\chi_{\lambda p}{}^{(+)}+\chi_{\lambda p}{}^{(-)}\delta\chi_{\lambda p}{}^{(+)})|\chi^{(+)\prime}),\quad(165)$$

and therefore

$$G_{\chi^{(+)}}=i\sum_{\lambda p}\chi_{\lambda p}{}^{(-)}\delta\chi_{\lambda p}{}^{(+)},\quad G_{\chi^{(-)}}=i\sum_{\lambda p}\chi_{\lambda p}{}^{(+)}\delta\chi_{\lambda p}{}^{(-)}.\quad(166)$$

The implied commutation relations are

$$\{\chi_{\lambda p}{}^{(+)},\chi_{\lambda'p'}{}^{(+)}\}=\{\chi_{\lambda p}{}^{(-)},\chi_{\lambda'p'}{}^{(-)}\}=0,\quad\{\chi_{\lambda p}{}^{(+)},\chi_{\lambda'p'}{}^{(-)}\}=\delta_{\lambda p,\lambda'p'}.\quad(167)$$

Naturally, the same results are obtained from the operator properties of ψ and $\bar{\psi}$.

Our first task is to demonstrate that the left side of (162) is independent of the eigenvalues. Now

$$(\delta_l/\delta\bar{\psi}'(x))F=-i\int(dx')\bar{I}(x,x')(\psi'(x')+\textstyle\sum_+\psi_{\lambda p}(x')\chi_{\lambda p}{}^{(+)})F+Fi\int(dx')I(x,x')(\psi'(x')+\textstyle\sum_-\psi_{\lambda p}(x')\chi_{\lambda p}{}^{(-)}).\quad(168)$$

Furthermore, for $\lambda>0$,

$$[\chi_{\lambda p}{}^{(+)},F]=(\partial_l/\partial\chi_{\lambda p}{}^{(-)})F=Fi\int(dx)(dx')\bar{\psi}_{\lambda p}(x)I(x,x')\times(\psi'(x')+\textstyle\sum_-\psi_{\lambda'p'}(x')\chi_{\lambda'p'}{}^{(-)}),\quad(169)$$

and, with $\lambda<0$,

$$[F,\chi_{\lambda p}{}^{(-)}]=(\partial_r/\partial\chi_{\lambda p}{}^{(+)})=i\int(dx)(dx')\bar{\psi}_{\lambda p}(x)\bar{I}(x,x')\times(\psi'(x')+\textstyle\sum_+\psi_{\lambda'p'}(x')\chi_{\lambda'p'}{}^{(+)})F.\quad(170)$$

The latter results can be expressed by

$$[\psi'(x)+\textstyle\sum_+\psi_{\lambda p}(x)\chi_{\lambda p}{}^{(+)},F]=F\int(dx')(x|iS^{(+)}I|x')\times(\psi'(x')+\textstyle\sum_-\psi_{\lambda'p'}(x')\chi_{\lambda'p'}{}^{(-)}),\quad(171)$$

and

$$[F,\psi'(x)+\textstyle\sum_-\psi_{\lambda p}(x)\chi_{\lambda p}{}^{(-)}]=\int(dx')(x|iS^{(-)}\bar{I}|x')\times(\psi'(x')+\textstyle\sum_+\psi_{\lambda'p'}(x')\chi_{\lambda'p'}{}^{(+)})F,\quad(172)$$

where $S^{(\pm)}(x,x')$ are defined in (97). On introducing the notation

$$\begin{aligned}(0|(\psi'(x)+\textstyle\sum_+\psi_{\lambda p}(x)\chi_{\lambda p}{}^{(+)})F|0)&=(0|F|0)f_+(x),\\(0|F(\psi'(x)+\textstyle\sum_-\psi_{\lambda p}(x)\chi_{\lambda p}{}^{(-)})|0)&=(0|F|0)f_-(x),\end{aligned}\quad(173)$$

(168) asserts that

$$(\delta_l/\delta\bar{\psi}'(x))(0|F|0)=-(0|F|0)i\int(dx')[\bar{I}(x,x')f_+(x')-I(x,x')f_-(x')],\quad(174)$$

while (171) and (172) supply the information

$$f_+(x)-\int(dx')(x|iS^{(+)}I|x')f_-(x')=\psi'(x),$$
$$f_-(x)-\int(dx')(x|iS^{(-)}\bar{I}|x')f_+(x')=\psi'(x). \qquad (175)$$

By subtracting the two equations in (175), we get (suppressing the indices)

$$(1+iS^{(-)}\bar{I})f_+=(1+iS^{(+)}I)f_-. \qquad (176)$$

But, according to (107),

$$I(1+iS^{(-)}\bar{I})=(1+iIS^{(+)})\bar{I}, \qquad (177)$$

while

$$I(1+iS^{(+)}I)=(1+iIS^{(+)})I. \qquad (178)$$

Hence, if we multiply (176) with I, and remove the ensuing common factor, $1+iIS^{(+)}$, we are left with

$$\bar{I}f_+=If_-, \qquad (179)$$

which shows that

$$(\delta_l/\delta\bar{\psi}'(x))(0|F|0)=0. \qquad (180)$$

Evidently, $(0|F|0)$ is also independent of the eigenvalues $\psi'(x)$. The problem remaining is the demonstration of

$$|e^{iw}|^2(0|\exp[-i\sum_{-+}\chi_{\lambda p}{}^{(+)}\bar{I}(\lambda p,\lambda'p')\chi_{\lambda'p'}{}^{(+)}]$$
$$\times\exp[i\sum_{+-}\chi_{\lambda p}{}^{(-)}I(\lambda p,\lambda'p')\chi_{\lambda'p'}{}^{(-)}]|0)=1. \qquad (181)$$

But this is just the statement that

$$(0|S^\dagger S|0)=\sum_n|(n|S|0)|^2=1, \qquad (182)$$

the proof[10] of which has already been given, in (132). The substitution $I\leftrightarrow-\bar{I}$ convert this verification of $S^\dagger S=1$ into one for $SS^\dagger=1$.

The discussion of time-independent fields will be deferred to a subsequent paper.

APPENDIX A

We want to verify here that the differential form (25) correctly defines $\det X$. The integrability of this expression must be demonstrated first. By considering two independent variations, we confirm that

$$\delta_2 Tr(X^{-1}\delta_1 X)-\delta_1 Tr(X^{-1}\delta_2 X)$$
$$=-Tr(X^{-1}\delta_2 X\,X^{-1}\delta_1 X)$$
$$+Tr(X^{-1}\delta_1 X\,X^{-1}\delta_2 X)=0, \qquad (1)$$

in virtue of the fundamental trace property, $TrXY=TrYX$. With infinite matrices, the applicability of this property requires suitable convergence of the traces. On using a discrete labelling of the matrix elements, we can express (25) as

$$(\partial/\partial X_{ij})\det X=(X^{-1})_{ji}\det X. \qquad (2)$$

[10] For further discussion, see Appendix B.

A second differentiation with respect to an arbitrary element of the matrix yields

$$(\partial/\partial X_{ij})(\partial/\partial X_{kl})\det X=[(X^{-1})_{ji}(X^{-1})_{lk}$$
$$-(X^{-1})_{jk}(X^{-1})_{li}]\det X. \qquad (3)$$

The right side reverses sign on interchanging the row indices i, k, and on interchanging the column indices j, l. Hence, the second derivative vanishes for equal row, or equal column indices. In particular,

$$(\partial^2/\partial X_{ij}{}^2)\det X=0. \qquad (4)$$

We also conclude that

$$\sum_j X_{ij}(\partial/\partial X_{ij})\det X=\sum_j X_{ij}(X^{-1})_{ji}\det X$$
$$=\det X, \qquad (5)$$

and that

$$\sum_i X_{ij}(\partial/\partial X_{ij})\det X=\det X. \qquad (6)$$

Therefore $\det X$ is a linear, homogeneous function of the elements in each row, and in each column, which is antisymmetrical in the rows and in the columns of the matrix. This establishes the identity of $\det X$, as defined by (25) and the initial condition $\det 1=1$, with the conventional concept.

The multiplication property of determinants follows directly from the definition (25). Indeed,

$$Tr(XY)^{-1}\delta(XY)=TrX^{-1}\delta X+TrY^{-1}\delta Y \qquad (7)$$

states that

$$\delta(\log\det XY)=\delta(\log\det X)+\delta(\log\det Y), \qquad (8)$$

whence

$$\det XY=\det X\det Y. \qquad (9)$$

The constant of integration has been fixed by the initial condition. Note also that the trace property

$$TrX=TrX^{tr} \qquad (10)$$

leads immediately to

$$\det X=\det X^{tr}. \qquad (11)$$

APPENDIX B

In this section we discuss the connection between determinants and ordered operators. Let $\chi_r{}^{(\pm)}$ be a set of operators that satisfy

$$\{\chi_r{}^{(+)},\chi_s{}^{(+)}\}=\{\chi_r{}^{(-)},\chi_s{}^{(-)}\}=0,$$
$$\{\chi_r{}^{(+)},\chi_s{}^{(-)}\}=\delta_{rs}, \qquad (1)$$

and consider the ordered exponential

$$V=\exp[\sum_{rs}\chi_r{}^{(+)};\lambda K_{rs}\chi_s{}^{(-)}], \qquad (2)$$

in which we have used the notation

$$\exp[A;B]=\sum_{n=0}^{\infty}\frac{1}{n!}A^nB^n. \qquad (3)$$

We deduce the commutation properties

$$[\chi_r^{(+)},V]=-(\partial_r/\partial\chi_r^{(-)})V=-\sum_p \chi_p^{(+)}\lambda K_{pr}V, \quad (4)$$

and

$$[V,\chi_s^{(-)}]=-(\partial_l/\partial\chi_s^{(+)})V=-V\sum_t \lambda K_{st}\chi_t^{(-)}, \quad (5)$$

whence

$$\sum_p \chi_p^{(+)}(1+\lambda K)_{pr}V=V\chi_r^{(+)},\qquad V\sum_t(1+\lambda K)_{st}\chi_t^{(-)}=\chi_s^{(-)}V, \quad (6)$$

and

$$\chi_r^{(+)}V=V\sum_p \chi_p^{(+)}[1/(1+\lambda K)]_{pr},\qquad V\chi_s^{(-)}=\sum_t[1/(1+\lambda K)]_{st}\chi_t^{(-)}V. \quad (7)$$

On differentiating V with respect to λ, we get

$$\begin{aligned}(\partial/\partial\lambda)V&=\sum K_{rs}\chi_r^{(+)}V\chi_s^{(-)}\\&=\sum V\chi_r^{(+)}(K/1+\lambda K)_{rs}\chi_s^{(-)}\\&=V\sum(K/1+\lambda K)_{rr}-\sum V\chi_s^{(-)}\chi_r^{(+)}(K/1+\lambda K)_{rs}\\&=VTr(K/1+\lambda K)-\sum\chi_s^{(-)}V\chi_r^{(+)}(K/(1+\lambda K)^2)_{rs},\end{aligned} \quad (8)$$

and the integration of the final form supplies the identity

$$\exp[\sum\chi_r^{(+)};\lambda K_{rs}\chi_s^{(-)}]=\det(1+\lambda K)\times\exp[-\sum\chi_s^{(-)};\chi_r^{(+)}(\lambda K/1+\lambda K)_{rs}]. \quad (9)$$

The $(\chi^{(-)\prime}||\chi^{(+)\prime})$ matrix element of this operator relation states that

$$(\chi^{(-)\prime}|\exp[\sum\chi_r^{(+)};\lambda K_{rs}\chi_s^{(-)}]|\chi^{(+)\prime})=\det(1+\lambda K)\exp\{\sum\chi_s^{(-)\prime}\chi_r^{(+)\prime}[1/(1+\lambda K)]_{rs}\}, \quad (10)$$

and, in particular, that

$$\det(1+\lambda K)=(0|\exp[\sum_{rs}\chi_r^{(+)};\lambda K_{rs}\chi_s^{(-)}]|0), \quad (11)$$

which exhibits $\det(1+\lambda K)$ as the matrix element of an operator. The derivation of this result requires no more than the first two rearrangements of (B8). Now

$$\exp[\sum\chi_r^{(+)};\lambda K_{rs}\chi_s^{(-)}]=\sum_{n=0}^{\infty}\lambda^n\frac{1}{n!}\sum_{r,s}\chi_{r_1}^{(+)}\cdots\chi_{r_n}^{(+)}\times\prod_{i=1}^{n}K_{r_is_i}\chi_{s_n}^{(-)}\cdots\chi_{s_1}^{(-)}, \quad (12)$$

and

$$\chi_{s_n}^{(-)}\cdots\chi_{s_1}^{(-)}\Psi(0)=\epsilon_{s_1\cdots s_n}\Psi(n'),\qquad \Psi(0)^\dagger\chi_{r_1}^{(+)}\cdots\chi_{r_n}^{(+)}=\epsilon_{r_1\cdots r_n}\Psi(n)^\dagger, \quad (13)$$

where the occupation number eigenvectors are preceded by alternating symbols, which are unity if the operators appear in some standard order. Therefore,

$$\det(1+\lambda K)=\sum_{n=0}^{\infty}\lambda^n\frac{1}{n!}\sum_r \det{}_{(n)}K_{r_ir_j}, \quad (14)$$

since the $s_1\cdots s_n$ must be some permutation of the $r_1\cdots r_n$, and the factor $\epsilon_{r_1\cdots r_n}\epsilon_{s_1\cdots s_n}$ is $+1$ or -1, according as the permutation is even or odd. The analogous result for a continuously labelled matrix is obtained with the substitution

$$K_{r_ir_j}\to(dx_i)^{\frac{1}{2}}K(x_i,x_j)(dx_j)^{\frac{1}{2}}, \quad (15)$$

namely,

$$\det(1+\lambda K)=\sum_{n=0}^{\infty}\lambda^n\frac{1}{n!}\int(dx_1)\cdots(dx_n)\times\det{}_{(n)}K(x_i,x_j). \quad (16)$$

It is of interest to consider the similar properties of ordered exponentials constructed from operators that obey the B.E. commutation relations

$$[\chi_r^{(+)},\chi_s^{(+)}]=[\chi_r^{(-)},\chi_s^{(-)}]=0,\qquad [\chi_r^{(+)},\chi_s^{(-)}]=\delta_{rs}. \quad (17)$$

Then the minus sign is to be omitted from (B4) and (B5), so that the sign of K is reversed in (B6) and (B7). This results in (B8) being replaced by

$$(\partial/\partial\lambda)V=VTr(K/1-\lambda K)+\sum\chi_s^{(-)}V\chi_r^{(+)}(K/(1-\lambda K)^2)_{rs}, \quad (18)$$

which yields the B.E. identity

$$\exp[\sum\chi_r^{(+)};\lambda K_{rs}\chi_s^{(-)}]=[1/\det(1-\lambda K)]\times\exp\{\sum\chi_s^{(-)};\chi_r^{(+)}[\lambda K/(1-\lambda K)]_{rs}\}. \quad (19)$$

Thus the B.E. analog of (B11) is

$$1/\det(1-\lambda K)=(0|\exp[\sum\chi_r^{(+)};\lambda K_{rs}\chi_s^{(-)}]|0). \quad (20)$$

In the expansion (B12), the order of left and right factors is now irrelevant, and repeated indices can occur. Thus (B13) is replaced by

$$\chi_{s_n}^{(-)}\cdots\chi_{s_1}^{(-)}\Psi(0)=(\prod n_k'!)^{\frac{1}{2}}\Psi(n'),\qquad \Psi(0)^\dagger\chi_{r_1}^{(+)}\cdots\chi_{r_n}^{(+)}=(\prod n_k!)^{\frac{1}{2}}\Psi(n)^\dagger. \quad (21)$$

The indices $s_1\cdots s_n$ must be some permutation of the $r_1\cdots r_n$, and each term carries the factor $\Pi n_k!$. Hence the expansion of (B20) is represented by

$$1/\det(1-\lambda K)=\sum_{n=0}^{\infty}\lambda^n\frac{1}{n!}\sum_r \mathrm{perm}_{(n)}K_{r_ir_j}, \quad (22)$$

where the so-called permanent is defined by

$$\mathrm{perm}_{(n)}K_{r_ir_j}=\sum_{s=r}\prod_{i=1}^{n}K_{r_is_i}, \quad (23)$$

and the summation with respect to the s_i is extended over all $n!$ permutations of the r_i. This quantity differs from the corresponding determinant by the omission of the alternating sign factor which, in the latter, enforces the nonoccurrence of identical rows or columns. In the continuous limit [Eq. (B15)], repeated indices produce no contribution and we get

$$1/\det(1-\lambda K)=\sum_{n=0}^{\infty}\lambda^n\frac{1}{n!}\int(dx_1)\cdots(dx_n)\times\text{perm}_{(n)}K(x_i,x_j). \quad (24)$$

This permanent coincides with a special case of the symmetrant defined in IV. The relation between the expansions (B14) and (B22) can be understood in terms of the expressions

$$\det(1+\lambda K)=\exp[Tr\log(1+\lambda K)] \quad (25)$$

and

$$1/\det(1-\lambda K)=\exp[-Tr\log(1-\lambda K)], \quad (26)$$

since the sign factors in (38) arise from the successive sign changes in the series (35), which are missing in

$$-Tr\log(1-\lambda K)=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^n K_n. \quad (27)$$

Finally, we shall use operator techniques to prove that

$$(0|F_0|0)=\det(1+S^{(+)}IS^{(-)}\bar{I}), \quad (28)$$

where F_0 represents (163) with zero eigenvalues, thus verifying (181). The dependence of F_0 upon the quantities $\bar{I}(x,x')$ is given by

$$(\delta/\delta\bar{I}(x,x'))F_0=-i\sum_-\times\bar{\psi}_{\lambda p}(x)\chi_{\lambda p}{}^{(+)}\sum_+\psi_{\lambda' p'}(x')\chi_{\lambda' p'}{}^{(+)}F_0. \quad (29)$$

Now, according to (171) and (172),

$$[\sum_+\psi_{\lambda p}(x)\chi_{\lambda p}{}^{(+)},F_0]=F_0\int(dx')(x|iS^{(+)}I|x')\sum_-\psi_{\lambda' p'}(x')\chi_{\lambda' p'}{}^{(-)} \quad (30)$$

and

$$[F_0,\sum_-\psi_{\lambda p}(x)\chi_{\lambda p}{}^{(-)}]=\int(dx')(x|iS^{(-)}\bar{I}|x')\sum_+\psi_{\lambda' p'}(x')\chi_{\lambda' p'}{}^{(+)}F_0, \quad (31)$$

which are combined in

$$\int(dx')(x|1+S^{(+)}IS^{(-)}\bar{I}|x')\sum_+\psi_{\lambda p}(x')\chi_{\lambda p}{}^{(+)}F_0=F_0\sum_+\psi_{\lambda p}(x)\chi_{\lambda p}{}^{(+)}+\int(dx')(x|iS^{(+)}I|x')\sum_-\psi_{\lambda p}(x')\chi_{\lambda p}{}^{(-)}F_0. \quad (32)$$

We deduce that

$$(0|\sum_-\bar{\psi}_{\lambda p}(x)\chi_{\lambda p}{}^{(+)}\sum_+\psi_{\lambda' p'}(x')\chi_{\lambda' p'}{}^{(+)}F_0|0)=(x'|(1+S^{(+)}IS^{(-)}\bar{I})^{-1}iS^{(+)}IS^{(-)}|x)(0|F_0|0), \quad (33)$$

and consequently

$$(\delta/\delta\bar{I}(x,x'))(0|F_0|0)=(x'|(1+S^{(+)}IS^{(-)}\bar{I})^{-1}S^{(+)}IS^{(-)}|x)(0|F_0|0). \quad (34)$$

We thus arrive at the differential expression

$$\delta\log(0|F_0|0)=Tr[(1+S^{(+)}IS^{(-)}\bar{I})^{-1}\times\delta(S^{(+)}IS^{(-)}\bar{I})], \quad (35)$$

which proves (B28).