complex potential of the form

$$V=-V_0(1+i\zeta) \quad \text{for} \quad r<R,$$
$$V=0 \quad \text{for} \quad r>R,$$

where $V_0=19$ Mev and $\zeta=0.05$. Comparison of the measured cross sections with the calculated values shown in the lower part of Fig. 2 shows that the gradual change in cross section is reproduced by this theory. In the region investigated in this experiment the broad maximum at 1-Mev energy, evident in the Nd curve, slowly disappears with increasing atomic weight, a behavior which is predicted by the theory.

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Inelastic Scattering of Protons and Neutrons by Deuterons

R. M. FRANK AND J. L. GAMMEL
University of California, Los Alamos Scientific Laboratory, Los Alamos, New Mexico
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If it is assumed that the range of nuclear forces is small compared to the size of the deuteron and the wavelength of the incoming particle ("zero-range" approximation) and something like the impulse approximation is used, a connection between the cross sections for elastic and inelastic *p-d* or *n-d* scattering is derived.

The results of this simple theory are compared with available experimental data, and the agreement between the two is much better than the crude assumptions of the theory merit.

An interesting consequence of the theory is that the connection between the elastic and inelastic scattering cross sections is independent of the properties of the two-body forces. The elastic scattering cross section depends hardly at all on the exchange properties of the two-body *n-p* potential (provided it gives the binding energy of the deuteron correctly) and the *n-n* potential, and the connection suggests that the same is true of the inelastic scattering cross section. This is in disagreement with more elaborate calculations of Bransden and Burhop.

I. THE BORN APPROXIMATION FOR THE INELASTIC SCATTERING CROSS SECTION

A FORMULATION of the Born approximation for the inelastic scattering cross section has been given in several papers.[1,2] The initial and final states are the following:

Initial State. A neutron (say) is incident on a deuteron. The wave vector of the neutron is $\mathbf{k}$:

$$\mathbf{k}=\frac{M}{\hbar}\mathbf{v}, \quad k^2=\frac{4}{3}\frac{M}{\hbar^2}E, \tag{1}$$

where $\mathbf{v}$ is the velocity of the neutron in the center of mass system and E is the kinetic energy of *both* particles in the center-of-mass system.

Final State. A neutron is ejected with wave vector $\mathbf{k}'$ (the "scattered" neutron). A deuteron remains in an excited (continuum) state described by a wave vector $\mathbf{k}''$:

$$\begin{aligned}\mathbf{k}'&=\frac{M}{\hbar}\mathbf{v}', & k'^2&=\frac{4}{3}\frac{M}{\hbar^2}E',\\ \mathbf{k}''&=\frac{M}{\hbar}\mathbf{v}'', & k''^2&=\frac{M}{\hbar^2}E'',\end{aligned} \tag{2}$$

where $\mathbf{v}'$ and E' have the same meaning as $\mathbf{v}$ and E, $\mathbf{v}''$ is the velocity of the neutron (say) belonging to the deuteron in the center-of-mass system of the two particles forming the deuteron, and E'' is the excitation energy of the deuteron. Thus, with the usual definition,

$$\alpha^2=\frac{M}{\hbar^2}E_b, \tag{3}$$

where E_b is the binding energy of the deuteron, conservation of energy

$$E'+E''=E-E_b \tag{4}$$

gives

$$k'^2+(4/3)k''^2=k^2-(4/3)\alpha^2. \tag{5}$$

The differential cross section for the scattering process corresponding to a transition from this initial to this final state is [reference 1, Eq. (13b)]:

$$d\sigma=\frac{1}{4}\frac{1}{(2\pi)^5}\frac{4M^2}{3\hbar^2}\frac{k'}{k}|M|^2d\Omega'd\mathbf{k}'', \tag{6}$$

or, equivalently [reference 2, after Eq. (51)],

$$d\sigma=\frac{1}{4}\frac{1}{(2\pi)^5}\frac{4M^2}{3\hbar^2}\frac{1}{kk'}\times\delta(k'-[k^2-(4/3)k''^2-(4/3)\alpha^2]^{\frac{1}{2}})|M|^2d\mathbf{k}'d\mathbf{k}'', \tag{7}$$

[1] R. L. Gluckstern and H. A. Bethe, Phys. Rev. **81**, 761 (1951).
[2] Ta-You Wu and Julius Ashkin, Phys. Rev. **73**, 986 (1948).

where [reference 1, Eq. (25)]

$$M=\{(1-P_{13})\psi_f\chi_f,\ [V_{nn}(13)+V_{np}(12)]\psi_i\chi_i\}. \quad (8)$$

Whereas for elastic scattering Eq. (8) leads to three kinds of integrals,[3] it now leads to six kinds of integrals, because the deuteron wave function which appears in ψ_f depends on the direction of $\mathbf{k}''$. These integrals are

$$J_1^+=\int \varphi_{\mathbf{k}''}(23)f'(1)U(13)\varphi(23)f(1)d\tau_1 d\tau_2 d\tau_3,$$

$$J_2^+=\int \varphi_{\mathbf{k}''}(13)f'(2)U(13)\varphi(23)f(1)d\tau_1 d\tau_2 d\tau_3, \quad (9)$$

$$J_3^+=\int \varphi_{\mathbf{k}''}(23)f'(1)U(13)\varphi(12)f(3)d\tau_1 d\tau_2 d\tau_3,$$

and three more, J_1^-, J_2^-, and J_3^-, obtained by reversing the argument of $\varphi_{\mathbf{k}''}$; for example,

$$J_2^-=\int \varphi_{\mathbf{k}''}(31)f'(2)U(13)\varphi(23)f(1)d\tau_1 d\tau_2 d\tau_3. \quad (10)$$

Performing the spin sums in Eq. (8) and defining α_+ to be the coefficient of J_1^+, etc., as in reference 3, Eq. (9) and Eq. (10), we find for $S=3/2$, triplet continuum states:

$$\left.\begin{aligned}
\alpha_+&={}^3V_{nn}^-,\\
\alpha_-&=\tfrac{1}{2}{}^3V_{np}^+ +\tfrac{1}{2}{}^3V_{np}^-,\\
\beta_+&=-\tfrac{1}{2}{}^3V_{np}^+ +\tfrac{1}{2}{}^3V_{np}^-,\\
\beta_-&=-\tfrac{1}{2}{}^3V_{np}^+ -\tfrac{1}{2}{}^3V_{np}^-,\\
\gamma_+&=-{}^3V_{nn}^-,\\
\gamma_-&=\tfrac{1}{2}{}^3V_{np}^+ -\tfrac{1}{2}{}^3V_{np}^-;
\end{aligned}\right\} \quad (11)$$

for $S=1/2$, triplet continuum states:

$$\left.\begin{aligned}
\alpha_+&=\tfrac{3}{4}{}^1V_{nn}^+ +\tfrac{1}{4}{}^3V_{nn}^-,\\
\alpha_-&=\tfrac{1}{8}{}^3V_{np}^+ +\tfrac{1}{8}{}^3V_{np}^- +\tfrac{3}{8}{}^1V_{np}^+ +\tfrac{3}{8}{}^1V_{np}^-,\\
\beta_+&=\tfrac{1}{4}{}^3V_{np}^+ -\tfrac{1}{4}{}^3V_{np}^-,\\
\beta_-&=\tfrac{1}{4}{}^3V_{np}^+ +\tfrac{1}{4}{}^3V_{np}^-,\\
\gamma_+&=\tfrac{3}{4}{}^1V_{nn}^+ -\tfrac{1}{4}{}^3V_{nn}^-,\\
\gamma_-&=\tfrac{1}{8}{}^3V_{np}^+ -\tfrac{1}{8}{}^3V_{np}^- +\tfrac{3}{8}{}^1V_{np}^+ -\tfrac{3}{8}{}^1V_{np}^-;
\end{aligned}\right\} \quad (12)$$

[3] R. S. Christian and J. L. Gammel, Phys. Rev. **91**, 100 (1953). We follow the notation of this paper in the following.

for $S=1/2$, singlet continuum states:

$$\left.\begin{aligned}
\alpha_+&=-\frac{\sqrt{3}}{4}({}^1V_{nn}^+ -{}^3V_{nn}^-),\\
\alpha_-&=\frac{\sqrt{3}}{8}(-{}^3V_{np}^+ -{}^3V_{np}^- +{}^1V_{np}^+ +{}^1V_{np}^-),\\
\beta_+&=-\frac{\sqrt{3}}{4}({}^1V_{np}^+ -{}^1V_{np}^-),\\
\beta_-&=-\frac{\sqrt{3}}{4}({}^1V_{np}^+ +{}^1V_{np}^-),\\
\gamma_+&=-\frac{\sqrt{3}}{4}({}^1V_{nn}^+ +{}^3V_{nn}^-),\\
\gamma_-&=\frac{\sqrt{3}}{8}(-{}^3V_{np}^+ +{}^3V_{np}^- +{}^1V_{np}^+ -{}^1V_{np}^-),
\end{aligned}\right\} \quad (13)$$

and [compare reference 1, Eq. (31)]

$$|M|^2=\tfrac{2}{3}|M(S=3/2)|^2+\tfrac{1}{3}|M(S=1/2,\text{ triplet states})|^2 +\tfrac{1}{3}|M(S=1/2,\text{ singlet states})|^2. \quad (14)$$

Equations (11) and (12) can be compared with Eqs. (9) and (10) of reference 3. Were the deuteron wave function in ψ_f the ground-state wave function, interchanging the particles in its argument would make no difference, so that for elastic scattering $J_1^+=J_1^-$, for example. In fact, comparing the two sets of equations, we see that

$$\alpha=\alpha_+ +\alpha_-, \quad (15)$$

for example, where α is given by Eq. (9) of reference 3.

II. COORDINATE SYSTEM

If 1 is the "incoming" neutron (say) and 2 and 3 are the proton and neutron in the deuteron, respectively, we use

$$\mathbf{r}_3-\mathbf{r}_2=\mathbf{r},\quad -\mathbf{r}_1+\tfrac{1}{2}(\mathbf{r}_2+\mathbf{r}_3)=\mathbf{q}. \quad (16)$$

III. CONTINUUM STATES OF THE DEUTERON

In the following, we use zero-range potentials for the two-body potentials. As explained in reference 3, it is not necessary to think of this as a strict zero-range approximation. It is perhaps better to imagine that is an approximate way of evaluating the integrals in Eq. (9) in which it is assumed that the range of the two-body nuclear forces is short compared to the size of the deuteron and the wavelength of the incident particle, and then it does not seem such a crude way of proceeding.

Consistent with this approximation, we take for the wave function of the ground state of the deuteron

$$\varphi(r)=\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\frac{\exp(-\alpha r)}{r}, \tag{17}$$

and for the triplet continuum states[4]

$$\varphi_{\mathbf{k}''}(r)=\exp(i\mathbf{k}''\cdot\mathbf{r})+\frac{\{\exp(2i\delta_0(k''))\}-1}{2ik''}\frac{\exp(ik''r)}{r}. \tag{18}$$

Equations (17) and (18) indicate that the wave functions reach their asymptotic form at once. The condition that $\varphi_{f''}$ and φ be orthogonal requires that

$$k''\cot\delta_0=-\alpha, \tag{19}$$

as may be calculated from Eqs. (17) and (18) by setting

$$\int\varphi\varphi_{\mathbf{k}''}d\mathbf{r}=0. \tag{20}$$

A further assumption contained in Eq. (18) is that in a two body n-p collision only the S-state interaction is important and no interaction takes place in states with $l\geqslant 1$. This is known to be very nearly the case, especially for low energies.

It should be noted that we are not making the usual assumption that the wave functions for continuum states of the deuteron are plane waves.

IV. THE ZERO RANGE APPROXIMATION

With these coordinates and wave functions, we find

$$J_1^+=\int\varphi_{\mathbf{k}''}^*(\mathbf{r})\exp(-i\mathbf{k}'\cdot\mathbf{q})\times U(|\mathbf{q}+\tfrac{1}{2}\mathbf{r}|)\varphi(r)\exp(i\mathbf{k}\cdot\mathbf{q})d\mathbf{r}d\mathbf{q},$$

$$J_2^+=\int\varphi_{\mathbf{k}''}^*(\mathbf{q}+\tfrac{1}{2}\mathbf{r})\exp[-i\mathbf{k}'\cdot(-\tfrac{1}{2}\mathbf{q}+\tfrac{3}{4}\mathbf{r})]\times U(|\mathbf{q}+\tfrac{1}{2}\mathbf{r}|)\varphi(r)\exp(i\mathbf{k}\cdot\mathbf{q})d\mathbf{r}d\mathbf{q}, \tag{21}$$

$$J_3^+=\int\varphi_{\mathbf{k}''}^*(\mathbf{r})\exp(-i\mathbf{k}'\cdot\mathbf{q})U(|\mathbf{q}+\tfrac{1}{2}\mathbf{r}|)\times\varphi(|\mathbf{q}-\tfrac{1}{2}\mathbf{r}|)\exp(-i\mathbf{k}\cdot(\tfrac{1}{2}\mathbf{q}+\tfrac{3}{4}\mathbf{r}))d\mathbf{r}d\mathbf{q}.$$

The sign of the argument of $\varphi_{\mathbf{k}''}$ is changed for J_1^-, J_2^-, J_3^-. With a zero-range potential,

$$U(|\mathbf{q}+\tfrac{1}{2}\mathbf{r}|)=U_0\delta(|\mathbf{q}+\tfrac{1}{2}\mathbf{r}|), \tag{22}$$

integration over $\mathbf{r}$ gives

$$\mathbf{r}=-2\mathbf{q}, \tag{23}$$

so that

$$J_1^+=\frac{4\pi}{3}r_0^3U_0\int\varphi_{\mathbf{k}''}^*(-2\mathbf{q})\times\exp(-i\mathbf{k}'\cdot\mathbf{q})\varphi(2q)\exp(i\mathbf{k}\cdot\mathbf{q})d\mathbf{q},$$

$$J_2^+=\frac{4\pi}{3}r_0^3U_0\varphi_{\mathbf{k}''}(r_0)\int\exp(2i\mathbf{k}'\cdot\mathbf{q})\varphi(2q)\exp(i\mathbf{k}\cdot\mathbf{q})d\mathbf{q},$$

$$J_3^+=\frac{4\pi}{3}r_0^3U_0\int\varphi_{\mathbf{k}''}^*(-2\mathbf{q})\times\exp(-i\mathbf{k}'\cdot\mathbf{q})\varphi(2q)\exp(i\mathbf{k}\cdot\mathbf{q})d\mathbf{q}.$$

Since

$$\lim_{r_0\to 0}r_0^2U_0 \tag{24}$$

is finite,

$$\lim_{r_0\to 0}r_0^3U_0 \tag{25}$$

vanishes, and J_1^+ and J_3^+ vanish. But

$$\lim_{r_0\to 0}r_0^3\varphi_{\mathbf{k}''}(r_0)U_0=\lim_{r_0\to 0}r_0\varphi_{\mathbf{k}''}(r_0)\lim_{r_0\to 0}(r_0^2U_0)$$

$$=\frac{\{\exp 2i\delta_0(k'')\}-1}{2ik''}\lim_{r_0\to 0}r_0^2U_0. \tag{26}$$

This is independent of the direction of $\mathbf{k}''$, so that in this approximation

$$J_2^+=J_2^-=J_2=\frac{4\pi}{3}\frac{\{\exp 2i\delta_0(k'')\}-1}{2ik''}\cdot\lim_{r_0\to 0}r_0^2U_0\times\int\exp(2i\mathbf{k}'\cdot\mathbf{q})\varphi(2q)\exp(i\mathbf{k}\cdot\mathbf{q})d\mathbf{q}, \tag{27}$$

and all other integrals vanish.

For elastic scattering, instead of Eq. (27) we find, in the same way,

$$J_2=\frac{4\pi}{3}\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\lim_{r_0\to 0}r_0^2U_0\times\int\exp(2i\mathbf{k}'\cdot\mathbf{q})\varphi(2q)\exp(i\mathbf{k}\cdot\mathbf{q})d\mathbf{q}. \tag{28}$$

Using Eqs. (11), (12), (13), and (14), we find for the

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[4] Very accurate continuum wave functions for the deuteron may be obtained for a Yukawa potential in this same way. For the ground state a well-known approximation is [see for example G. F. Chew, Phys. Rev. **74**, 809 (1948)]

$$\varphi=\left(\frac{\alpha\beta(\alpha+\beta)}{2\pi(\beta-\alpha)^2}\right)^{\frac{1}{2}}\frac{\{\exp(-\alpha r)\}-\{\exp(-\beta r)\}}{r}.$$

For the approximate continuum wave function,

$$\varphi_{k''}(r)=\exp(i\mathbf{k}''\cdot\mathbf{q})+\frac{\{\exp(2i\delta_0(k''))\}-1}{2ik''r}(\{\exp(2ik''r)\}-\{\exp(-\beta r)\}),$$

Eqs. (19) and (20) lead to

$$k''\cot\delta_0(k'')\left[\frac{1}{\alpha^2+k''^2}-\frac{1}{\beta^2+k''^2}\right]=\frac{1}{\alpha+\beta}-\frac{1}{2\beta}+\frac{\beta}{\beta^2+k''^2}-\frac{\alpha}{\alpha^2+k''^2},$$

which corresponds to a scattering length of 5.26×10^{-13} cm as compared to the accepted value $5.29\pm 0.04\times 10^{-13}$ cm. For the 90-Mev n-p scattering it gives a ^3S phase shift of 54° as compared to the accepted value 60°±5°.

inelastic scattering

$$|M|^2=\tfrac{2}{3}|J_2|^2+\tfrac{1}{12}|J_2|^2+\tfrac{1}{4}({}^1V_{np}{}^+)^2|J_2(\text{singlet})|^2, \tag{29}$$

and for the elastic scattering

$$|M|^2=\tfrac{3}{4}|J_2|^2. \tag{30}$$

V. CONNECTION BETWEEN THE INELASTIC AND ELASTIC SCATTERING CROSS SECTIONS

Equations (27) and (28) are very much alike. The $\mathbf{k}''$s which appear in them are different, because in Eq. (28) (elastic scattering)

$$|\mathbf{k}'|=|\mathbf{k}|,$$

whereas for Eq. (27) (inelastic scattering), Eq. (5) applies. However, following the procedures of the impulse approximation,[5] we overlook this, so that

$$\frac{J_2(\text{inelastic})}{J_2(\text{elastic})}=\left(\frac{2\pi}{\alpha}\right)^{\frac{1}{2}}\frac{\{\exp(2i\delta_0(k''))\}-1}{2ik''}, \tag{31}$$

and

$$\frac{|J_2(\text{inelastic})|^2}{|J_2(\text{elastic})|^2}=\frac{2\pi}{\alpha}\frac{\sin^2\delta_0(k'')}{(k'')^2}. \tag{32}$$

Then Eqs. (29) and (30) and Eq. (6) and the corresponding equation for elastic scattering give

$$d\sigma_{\text{in}}=\sigma_{\text{el}}\frac{2\pi}{\alpha}\frac{k'}{k}\sin^2\delta_0(k'')\frac{4\pi}{(2\pi)^3}dk''d\Omega'. \tag{33}$$

σ_{el} depends on the angle between $\mathbf{k}'$ and $\mathbf{k}$. The last 4π in the numerator comes from $d\mathbf{k}''=4\pi k''^2dk''$, and the $1/(2\pi)^3$ comes from the fact that the cross section for elastic scattering has a $1/(2\pi)^2$ instead of the $1/(2\pi)^5$ in Eq. (6). Sometimes a $1/(2\pi)^{\frac{3}{2}}$ is put in front of Eq. (18) to make the equations corresponding to Eq. (6) the same for elastic and inelastic scattering; but, however we do it, we have the $1/(2\pi)^3$ at the end.

Still following the ideas of the impulse approximation, we proceed as follows.[6] To calculate an inelastic cross section we put the *experimental* *n-d* angular distribution into Eq. (33) and values of $\sin^2\delta_0(k'')$ calculated from

$$\sin^2\delta_0(k'')=\frac{k''^2}{[k''\cot\delta_0(k'')]^2+k''^2} \tag{34}$$

and

$$k''\cot\delta_0(k'')=-\frac{1}{a}+\frac{1}{2}r_0(k'')^2, \tag{35}$$

where we use the best values of the scattering lengths and effective range. Of course, in view of Eq. (29), part of the time we use the triplet scattering length and effective range and, part of the time, the singlet scattering length and effective range. Equation (33) written more fully, with Eq. (29) taken into account becomes

$$d\sigma_{\text{in}}=\frac{\sigma_{\text{el}}}{\pi\alpha}\frac{k'}{k}k''^2\left\{\frac{1}{(k''\cot\delta)^2_{\text{triplet}}+k''^2}+\frac{({}^1V_{np}{}^+)^2}{3}\frac{1}{(k''\cot\delta)^2_{\text{singlet}}+k''^2}\right\}dk''d\Omega', \tag{36a}$$

or alternately, according to Eq. (7),

$$d\sigma_{\text{in}}=\frac{\sigma_{\text{el}}}{4\pi^2\alpha}\frac{1}{kk'}\left\{\qquad\right\}\times\delta(k'-[k^2-(4/3)k''^2-(4/3)\alpha^2]^{\frac{1}{2}})d\mathbf{k}'d\mathbf{k}''. \tag{36b}$$

VI. EMISSION OF NEUTRONS AND PROTONS IN *n-d* INELASTIC SCATTERING

In *n-d* inelastic scattering, two neutrons (a "scattered" neutron and a neutron "ejected" from the deuteron) appear and one proton (a proton "ejected" from the deuteron).

The wave-number vectors of these particles are as follows:

"scattered" neutron: $\mathbf{k}'$, (37a)

"ejected" neutron: $\mathbf{S}_n=-\tfrac{1}{2}\mathbf{k}'+\mathbf{k}''$, (37b)

"ejected" proton: $\mathbf{S}_p=-\tfrac{1}{2}\mathbf{k}'-\mathbf{k}''$. (37c)

$\mathbf{k}''$ may be eliminated from $d\sigma_{\text{in}}$, Eq. (36b), by the use of one of Eqs. (37b) or (37c).

If we want the energy and angular distribution of the "scattered" neutron ($d\sigma_{\text{scatt}}$), we express Eq. (36b) in terms of $\mathbf{k}'$ and $\mathbf{S}_n$ (say) and integrate out the directions of $\mathbf{S}_n$. [The magnitude of $\mathbf{S}_n$ is fixed by Eq. (37b) and Eq. (5).] This is easy to do since it is equivalent to putting k'' and dk'' from Eq. (5) into Eq. (36a), which gives the energy and angular distribution of the "scattered" neutrons at once.

Graphs of $d\sigma_{\text{scattered}}$ as functions of E_n, the energy of the scattered neutron in the laboratory system, and θ_n, the angle through which it is scattered, are presented in Fig. 1.[7]

[5] G. F. Chew and G. C. Wick, Phys. Rev. **85**, 636 (1952).

[6] Gluckstern and Bethe (reference 1, p. 770) also replace terms occurring in a result calculated in Born approximation by observed values wherever possible.

[7] Transformation to the laboratory system is accomplished as follows. First, we express Eq. (36a) in terms of energies by using Eqs. (1)–(5). We find

$$d\sigma=\frac{1}{2\pi}\frac{\sigma_{\text{el}}}{\sqrt{E_b}}\frac{\sqrt{E'}}{\sqrt{E_{\text{lab}}}}(E_{\text{lab}}-\tfrac{3}{2}E'-\tfrac{3}{2}E_b)^{\frac{1}{2}}\times\left\{\frac{1}{D_{\text{triplet}}}+\frac{({}^1V_{np}{}^+)^2}{3}\frac{1}{D_{\text{singlet}}}\right\}dE'd\Omega',$$

where

$$D=\frac{\hbar^2}{M}[(k''\cot\delta)^2+k''^2]$$

and E_{lab} is the energy of the incident particle in the laboratory system. This may be expressed in terms of laboratory energies

If we want the energy and angular distribution of the "ejected" neutron ($d\sigma_{\text{ejected}}$), we express Eq. (36b) in terms of $\mathbf{k}'$ and $\mathbf{S}_n$ and integrate out the directions of $\mathbf{k}'$ [the magnitude of $\mathbf{k}'$ is fixed by Eq. (37b) and Eq. (5)]. This time there is no easy way to do the calculation. The procedure for carrying it through is described in reference 2, paragraph C, p. 996.

Let E_n be the energy of the ejected neutron and θ_n the angle it makes with $\mathbf{k}$ (both measured in the laboratory system). The wave-number vector of the ejected neutron in the laboratory system is

$$\mathbf{p}_n = \mathbf{S}_n + \tfrac{1}{2}\mathbf{k}, \tag{38}$$

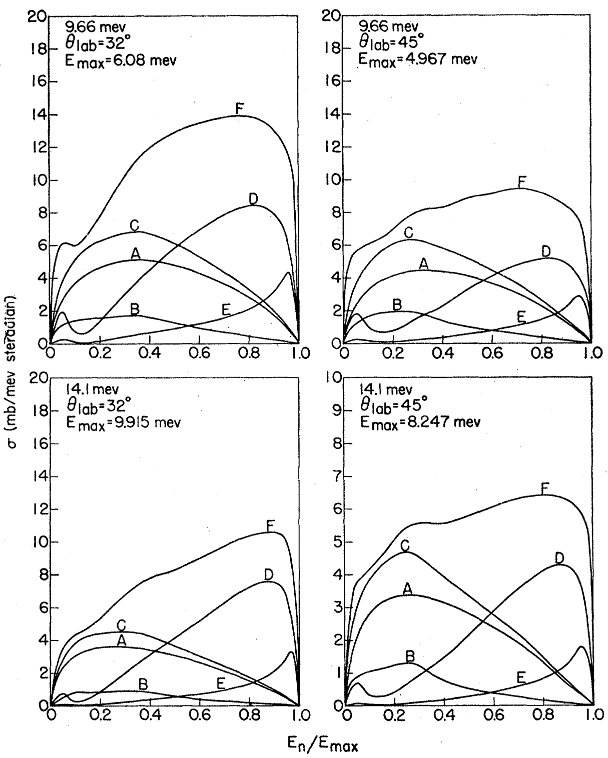

FIG. 1. Energy distributions of inelastically scattered particles at two angles. The contributions of triplet deuteron continuum states to the cross sections for emission of protons and scattering of neutrons are labeled A and D, respectively. The corresponding curves for singlet deuteron continuum states are labeled B and E. The cross sections for emission of protons and neutrons are labeled C and F, respectively.

and angles by using

$$\cos(k',k) = \frac{2\sqrt{E_n}\cos\theta_n - \sqrt{E_0}}{2(E_n + \frac{1}{4}E_0 - \sqrt{E_0 E_n}\cos\theta_n)^{\frac{1}{2}}},$$

$$E' = \tfrac{3}{2}[E_n + \tfrac{1}{4}E_0 - (E_0 E_n)^{\frac{1}{2}}\cos\theta_n],$$

and

$$dE' d\cos(\mathbf{k}',\mathbf{k}) = |J|\, dE_n d\cos\theta_n,$$

where E_0 is $4E_{\text{lab}}/9$ and J is the Jacobian of the transformation:

$$J = \frac{3}{2}\left(\frac{3}{2}\frac{E_n}{E'}\right)^{\frac{1}{2}}.$$

Numerically, with the energies in Mev,

$$D_{\text{triplet}} = 1.431 + 0.455(E_{\text{lab}} - 1.5E' - 1.5E_b) + 0.00782(\quad)^2,$$
$$D_{\text{singlet}} = 0.0743 + 0.743(\quad\quad) + 0.0195(\quad)^2.$$

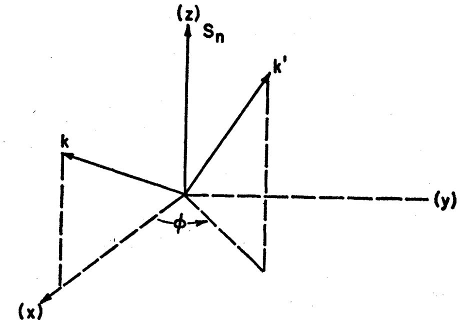

FIG. 2. Coordinate system to which $\mathbf{k}'$ is referred in performing the integrations to obtain angular and energy distributions of ejected particles.

so that

$$p_n^2 = (2M/\hbar^2)E_n. \tag{39}$$

In integrating out the directions of $\mathbf{k}'$, we refer to $\mathbf{S}_n$ as polar axis ($\mu = \cos\mathbf{k}', \mathbf{S}_n$), the azimuth, φ, being measured from the plane of $\mathbf{k}$ and $\mathbf{S}_n$ (see Fig. 2). The angle between $\mathbf{k}'$, $\mathbf{k}$ which appears in σ_{el} in Eq. (36b) must be expressed in terms of variables appropriate to this reference system.

$$\cos(\mathbf{k}',\mathbf{k}) = \sin(\mathbf{k},\mathbf{S}_n)(1-\mu^2)^{\frac{1}{2}}\cos\varphi + \mu\cos(\mathbf{k},\mathbf{S}_n) \equiv y. \tag{40}$$

Actually, Eq. (37b) and Eq. (5) leave two values of k' possible:

$$k_+' = \tfrac{1}{2}[-S_n\mu + (S_n^2\mu^2 - 4S_n^2 + 3k^2 - 4\alpha^2)^{\frac{1}{2}}], \tag{41a}$$

$$k_-' = \tfrac{1}{2}[-S_n\mu - (S_n^2\mu^2 - 4S_n^2 + 3k^2 - 4\alpha^2)^{\frac{1}{2}}]. \tag{41b}$$

Only real positive values of k_+' and k_-' are permitted. This restricts the allowed values of μ for a given S_n. For

$$0 \leqq S_n^2 \leqq \tfrac{3}{4}k^2 - \alpha^2,$$

k_+' is real and positive for all μ. k_-' is negative for all μ and thus is excluded. For

$$\tfrac{3}{4}k^2 - \alpha^2 < S_n^2 \leqq k^2 - (4/3)\alpha^2,$$

both k_+' and k_-' are real and positive when

$$-1 \leqq \mu \leqq -2\left(1 - \frac{3}{4}\frac{k^2}{S_n^2} + \frac{\alpha^2}{S_n^2}\right)^{\frac{1}{2}} = -\nu$$

and are either complex or negative when $\mu > -\nu$. For

$$S_n^2 > k^2 - (4/3)\alpha^2,$$

both k_+' and k_-' are complex for all μ, and this region is excluded. (This is an obvious consequence of conservation of energy.)

Any function $f(k')$ of k' labeled $f^+(k')$ or $f^-(k')$ means $f(k_+')$ or $f(k_-')$ in the following. Let

$$I = \frac{\sigma_{\text{el}}(y)}{4\pi^2\alpha}\frac{1}{(k''\cot\delta)^2 + k''^2}, \tag{42}$$

where $k''\cot\delta$ is given by Eq. (35) expressed in terms

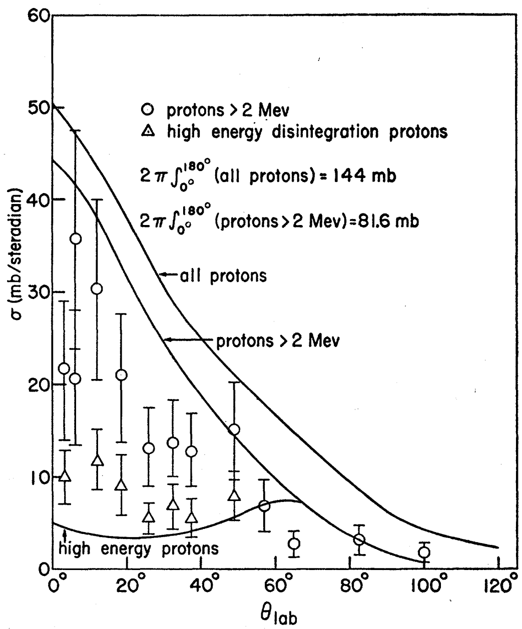

FIG. 3. Angular distribution of disintegration protons from 14.1-Mev inelastic n-d scattering.

of $\mathbf{k}'$ and S_n through Eq. (37b). Then Eqs. (55) and (56) of reference 2 become

$$d\sigma_{\text{ejected}}=\frac{M}{\hbar^2}\frac{p_n}{k}I(E_n,\theta_n)dE_n d\Omega_n. \tag{43}$$

For

$$0<S_n^2<\tfrac{3}{4}k^2-\alpha,$$

$$I(E_n,\theta_n)=\frac{3}{2}\int_0^{2\pi}d\varphi\int_{-1}^{1}d\mu \times\frac{k_+'^2}{(3k^2-4\alpha^2-4S_n^2+S_n^2\mu^2)^{\frac{1}{2}}}I^+. \tag{44a}$$

For

$$\tfrac{3}{4}k^2-\alpha^2<S_n^2<k^2-(4/3)\alpha^2,$$

$$I(E_n,\theta_n)=\frac{3}{2}\int_0^{2\pi}d\varphi\int_{-1}^{-\nu}d\mu\frac{k_+'^2}{(3k^2-4\alpha^2-4S_n^2+S_n^2\mu^2)^{\frac{1}{2}}}I^+ +\frac{3}{2}\int_0^{2\pi}d\varphi\int_{-1}^{-\nu}d\mu\frac{k_-'^2}{(3k^2-4\alpha^2-4S_n^2+S_n^2\mu^2)^{\frac{1}{2}}}I^-. \tag{44b}$$

Graphs of $d\sigma_{\text{ejected}}$ as a function of E_n and θ_n are presented in Fig. 1. It can be seen that the energy and angular distribution of the ejected neutron and ejected proton are the same in the present theory because Eq. (36b) depends only on the magnitude of $\mathbf{k}''$, so that expressing (36b) in terms of k' and S_n or k' and S_p give the same result.

The cross section for the emission of neutrons is composed of two parts:

$$d\sigma_n=d\sigma_{\text{scattered}}+d\sigma_{\text{ejected}}. \tag{45}$$

The total cross section for the scattered neutrons obtained by integrating $d\sigma_{\text{scattered}}$ over E_n and θ_n must be the same as the total cross section for the ejected neutrons because every time a neutron is scattered one is ejected. That we must get this result follows on a moment's reflection. We have integrated Eq. (36b), expressed in terms of $\mathbf{k}'$ and $\mathbf{S}_n$, in only two different orders, so that we ought to get the same result from them. This requirement serves as a useful check on the numerical work.

Since two neutrons appear in the final state,

$$\begin{aligned}\sigma_{\text{inelastic, total}}&=\tfrac{1}{2}(\sigma_{\text{scattered, total}}+\sigma_{\text{ejected, total}})\\&=\sigma_{\text{scattered, total}}=\sigma_{\text{ejected, total}},\end{aligned} \tag{46}$$

whereas the total cross section for emission of neutrons is larger than this by a factor of two, whereas the cross section for emission of protons is equal to it. The total scattering cross section is

$$\sigma_{\text{total}}=\sigma_{\text{elastic total}}+\sigma_{\text{inelastic total}}. \tag{47}$$

$d\sigma_{\text{scattered}}$ and $d\sigma_{\text{ejected}}$ cannot be observed separately, of course [without auxiliary assumptions like $d\sigma_{\text{ejected}}(\text{proton})=d\sigma_{\text{ejected}}(\text{neutron})$].

In Sec. VII, the triplet and singlet cross sections are combined by taking ${}^1V_{np}{}^+=0.69$ in Eq. (36), and the scattered and ejected neutrons are combined by using Eq. (45).

VII. COMPARISON WITH EXPERIMENTS

A. 14.1 Mev

The cross section for emission of protons is compared with the results of Allred, Armstong, and Rosen[8] in Fig. 3. The theoretical curve has been modified to take account of the fact that only protons of energy greater than 2 Mev were observed:

$$\sigma(\theta)=\int_{2\text{ Mev}}^{E_{\max}}\sigma(E,\theta)dE. \tag{48}$$

Allred, Armstrong, and Rosen also observed high-energy disintegration protons (protons whose range in the emulsion is greater than the range in the emulsion of deuterons elastically scattered at the same angle). Figure 4 shows $E_p(\theta_{\text{lab}})$, the energy of a proton whose range in the emulsion is the same as the range in the emulsion of a deuteron elastically scattered at the same angle, as a function of θ_{lab}. Then the cross section for emission of high-energy disintegration protons is

$$\sigma(\theta)=\int_{E_p(\theta)}^{E_{\max}}\sigma(E,\theta)dE. \tag{49}$$

The total cross section at 14.1 Mev has been measured with precision by the transmission method, and is 802 mb.[9] It is extremely doubtful that the total cross

[8] Allred, Armstrong, and Rosen, Phys. Rev. **91**, 90 (1953).
[9] Poss, Salant, Snow, and Yuan, Phys. Rev. **87**, 116 (1952).

section for elastic scattering is greater than 650 mb.[3] This leaves 150 mb for the total cross section for emission of protons. Allred, Armstrong, and Rosen observed 57 ± 13 mb for the total cross section for emission of protons of energy greater than 2 Mev. This work was undertaken partly in order to determine whether the cross section for emission of protons of energy less than 2 Mev could be sufficiently large to explain the difference between 150 mb and 57 ± 13 mb. This appears to be the case. We find 144 mb for the total cross section for emission of protons and 80 mb for the cross section for emission of protons of energy greater than 2 Mev.

On the whole the theoretical results seem in good agreement with the experimental results. From Fig. 3 it might be concluded that the theoretical energy distributions have too many low-energy protons and too few high-energy ones, but in view of the experimental uncertainties, this conclusion is not certain.

B. 9.66 Mev

At this energy we may compare our results with the experimental results of Juanita H. Gammel.[10] She observed protons of energy greater than 1.3 Mev emitted in p-d scattering (the names "proton" and "neutron" have to be interchanged for comparison with the calculated n-d angular distributions). This was allowed for as in Eq. (48). Figure 5 shows that theory and experiment are in excellent agreement (no doubt fortuitously). She found

$$\int_{20°}^{130°} \sin\theta d\theta \int_{1.3\text{ Mev}}^{E_{\max}} dE\sigma(E,\theta)=114\text{ mb},$$

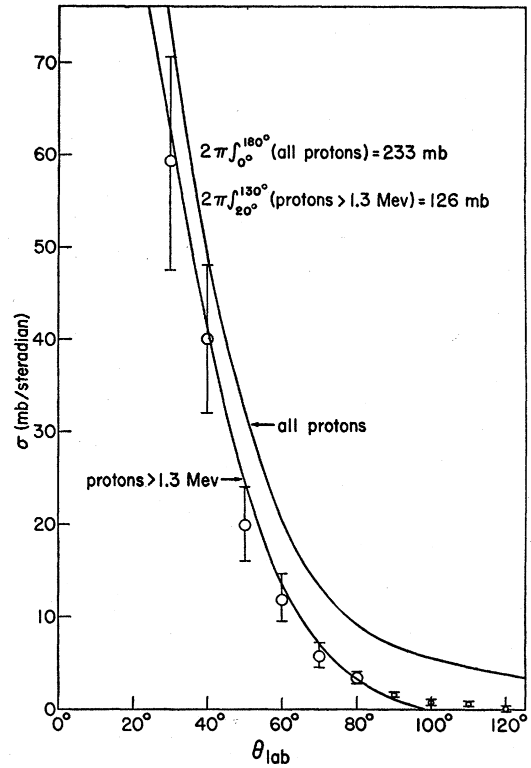

FIG. 5. Angular distribution of protons emitted in 9.66-Mev inelastic p-d scattering.

and we find 126 mb for the same quantity. It is interesting to note that the total cross section for emission of protons is more than twice this, or 233 mb. About one-half of the protons have an energy less than 1.3 Mev.

Coulomb effects were neglected in the calculation. They might be taken into account approximately by use of the following as a Coulomb penetration factor:

$$C_0(k+\tfrac{1}{2}k)C_0(|\mathbf{k}'+\tfrac{1}{2}\mathbf{k}'+\mathbf{k}''|); \qquad (50)$$

that is, we use the relative momenta of the two protons in the initial and final states. Here, as usual,

$$C_0=2\pi\eta/[\exp(2\pi\eta)-1], \quad \eta=e^2/\hbar v_{\text{relative}}. \qquad (51)$$

For the initial state,

$$\tfrac{3}{2}k=Mv_{\text{relative}}/\hbar; \qquad (52)$$

and for the final state,

$$|\tfrac{3}{2}\mathbf{k}'+\mathbf{k}''|=Mv_{\text{relative}}/\hbar. \qquad (53)$$

C. Total Cross Section

A graph of the n-d inelastic cross section is presented in Fig. 6.

VIII. DISCUSSION

Bransden and Burhop have presented some calculations which show that the inelastic cross section

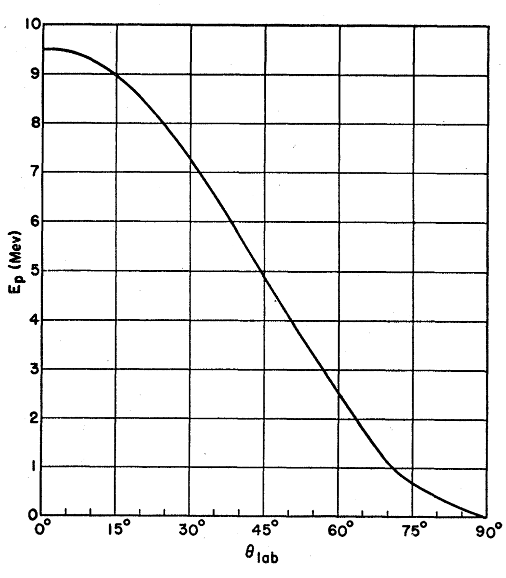

FIG. 4. The energy of a proton having the same range in the emulsion as a deuteron elastically scattered at the same angle as a function of the laboratory angle (14.1-Mev n-d scattering).

[10] Juanita H. Gammel (to be published).

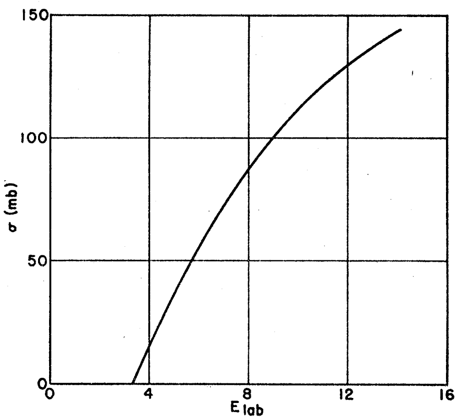

FIG. 6. Total disintegration cross section for *n-d* scattering as a function of energy.

depends sensitively on the nature of the nuclear forces.[11] This does not result from our calculation. The connection (36a) between the inelastic and elastic cross sections involves only ${}^1V_{np}{}^+$, the value of which is determined by the singlet *n-p* scattering length. Since Christian and Gammel[3] found that the elastic cross section depends hardly at all on the unknown properties of the nuclear forces, it follows from Eq. (36a) that the same is true of the inelastic cross section. The discrepancy between this result and that of Bransden and Burhop arises partly perhaps from the crudeness of our theory, but the problem should be studied more carefully if possible.

Another point of interest is the following. Use of plane-wave wave functions for the continuum states of the deuteron in calculating the integrals (the *J*'s) would have made them all vanish. One is led to believe, first, that the nonplane-wave parts of the deuteron-continuum states should make the most important contribution to the inelastic cross section even in calculations better than the one made here, and, second, that it is doubtful that the use of plane wave-continuum wave functions could give a better approximation to the total scattering cross section than the inelastic scattering cross section as is sometimes stated (one would hardly want to get zero for the total scattering cross section, especially after calculating a nonzero elastic scattering cross section as Christian and Gammel did using the same zero-range approximation used here).

It might not be out of place to point out one further motivation for carrying through such a long calculation based on such doubtful premises as the zero-range approximation, the impulse approximation, and the replacement of quantities calculated in the Born approximation by exact quantities.

The "phase shifts" which occur in an analysis of *n-d* elastic scattering at energies greater than 3.342 Mev must be complex numbers since inelastic scattering also takes place. We have developed a method of calculating complex phase shifts for which it is necessary to know the absorption from each partial wave by inelastic scattering. Since the connection Eq. (36a) between the inelastic and elastic cross sections is independent of the angle between $\mathbf{k}$ and $\mathbf{k}'$ and of the direction of $\mathbf{k}''$, the ratio $\sigma_{\mathrm{in},l}/\sigma_{\mathrm{el},l}$ is independent of l (l is the angular momentum of the partial wave) and is equal to $\sigma_{\mathrm{in}}/\sigma_{\mathrm{el}}$. This makes it simple to compute $\sigma_{\mathrm{in},l}$ when σ_{in}, σ_{el}, and $\sigma_{\mathrm{el},l}$ are known.

Thus the calculation was carried out partly in order to determine whether such a simple relation as Eq. (36a) could be in agreement with experimental evidence at low energies.

IX. ACKNOWLEDGMENTS

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We are also indebted to Dr. Charles Critchfield, Dr. Richard Christian, and Dr. Don Dodder for valuable discussions about the principles of the calculation.

[11] B. H. Bransden and E. H. S. Burhop, Proc. Phys. Soc. (London) **A63**, 1337 (1950).