Čerenkov Radiation

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Čerenkov radiation is investigated for the following cases: charged particles and neutrons moving in isotropic crystals; charged particles moving in an arbitrary direction with respect to the optic axis in uniaxial crystals.

GINSBURG[1,2] investigated Čerenkov radiation of uniaxial crystals when an electron moves either along or perpendicular to the optic axis, using the Hamilton method in electrodynamics of anisotropic media.

Using Ginsburg's formalism, we obtain for the energy radiated by the particle within an angle range ϕ, $\phi+d\phi$, and per unit path length:[3]

$$\frac{dH_i}{dl}=\frac{(2\pi)^6}{v_0^2c^22\pi}\int_0^{2\pi}\int_{\beta n_{\lambda i}>1}^{\infty}(\mathbf{j}_k\cdot\mathbf{a}_{\lambda i})^2w_{\lambda i}\times n_{\lambda i}^2(\theta_0,\phi)dw_{\lambda i}d\phi\cdots,\quad(1)$$

where ϕ and θ_0 determine the relative orientation of the propagation vector $\mathbf{k}_\lambda$ and the uniform velocity of the particle $\mathbf{v}_0$, ϕ is the angle the projection of $\mathbf{k}_\lambda$ on the XZ plane makes with the X axis, $\mathbf{a}_{\lambda i}$ is the component of the properly normalized vectors of electric field strength, $\mathbf{j}_k$ is the Fourier component of the charge density, and $n_{\lambda i}$ is the index of refraction.[4] The i index corresponds to two directions of polarization, the angle θ_0 is given by

$$\cos\theta_{0i}=\frac{c}{n_{\lambda i}v_0}=\frac{1}{n_{\lambda i}\beta},\quad(2)$$

and $w_{\lambda i}$ is given by

$$w_{\lambda i}^2=\frac{k_\lambda^2c^2}{n_{\lambda i}^2}=c^2\{(\mathbf{k}_\lambda^2\cdot\mathbf{a}_{\lambda i}^2)-(\mathbf{k}_\lambda\mathbf{a}_{\lambda i})^2\}.\quad(3)$$

In isotropic crystals, let $\mathbf{a}_{\lambda 1}$ be in the plane determined by $\mathbf{v}_0$ and $\mathbf{k}_\lambda$. Integrating (1) with respect to ϕ,

$$\frac{dH_1}{dl}=\frac{(2\pi)^6}{v_0^2c^2}\int_{\beta n_\lambda>1}\mathbf{j}_k^2\left(1-\frac{1}{n_\lambda^2(w)\beta^2}\right)wdw,\quad(4)$$

$$dH_2/dl=0.$$

For charged particles,

$$\mathbf{j}_k=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\mathbf{j}(r)e^{-i\mathbf{k}_\lambda\cdot\mathbf{r}}d\mathbf{r}=\frac{e\mathbf{v}_0}{(2\pi)^3},\quad(5)$$

as $\mathbf{k}\cdot\mathbf{r}\ll 1$ over the particle. Substituting (5) into (4), we get

$$\frac{dH_1}{dl}=\frac{e^2}{c^2}\int_{\beta n_\lambda>1}^{\infty}\left(1-\frac{1}{n_\lambda^2(w)\beta^2}\right)wdw,\quad(6)$$

which is the Frank and Tamm formula[5] giving the total energy radiated by a charged particle through the surface of a cylinder per unit length, the axis of which coincides with the line of motion of the particle. For neutrons,

$$\mathbf{j}_k=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\mathbf{j}(r)e^{-i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r}=\frac{c}{(2\pi)^3}\int_{-\infty}^{\infty}\nabla\times\mathbf{m}(r)e^{-i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r}=\frac{c}{(2\pi)^3}M\times\mathbf{k},\quad(7)$$

where M is the magnetic moment of the neutron.

Putting (7) into (4), assuming neutrons are oriented randomly in space, we find:[6]

$$\frac{dH_1}{dl}=\frac{M^2}{3v^2}\int_{\beta n_\lambda>1}^{\infty}\frac{w^2n_\lambda^2}{c^2}\left(1-\frac{1}{n_\lambda^2(w)\beta^2}\right)wdw.\quad(8)$$

This distribution depends on two extra powers of frequency compared to that of charged particles.

Comparing (6) and (8), the ratio of $dH/dwdl$ for neutrons to that of charged particles is of the order 10^{-17} for the visible region.

With reference to Fig. 1, we choose the electron velocity vector $\mathbf{v}_0$ so that it determines the YZ plane. The optic axis is taken along the Z direction. Indices o and e refer to the ordinary and extraordinary rays, respectively.

$$\epsilon_o=\epsilon_x=\epsilon_y,\quad \epsilon_o=\epsilon_x.$$

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[1] V. L. Ginsburg, J. Phys. (U.S.S.R.) 3, 95 (1940).
[2] V. L. Ginsburg, J. Phys. (U.S.S.R.) 3, 101 (1940).
[3] The reader is referred to references 1 and 2 for the method of calculation.
[4] The results of a special case are given by K. Tanaka, University of California Radiation Laboratory Report UCRL-1286, May 1951 (unpublished).
[5] I. Tamm, J. Phys. (U.S.S.R.) 1, 439 (1939).
[6] An alternative calculation obtained by expanding the field variables of Maxwell's equation in four-dimensional Fourier series and expressing the Poynting vector in terms of the generalized current density was reported at the Cambridge Meeting of the American Physical Society, January, 1953, by K. Tanaka [Phys. Rev. 90, 358 (1953)]. The results agree with those of the present paper.

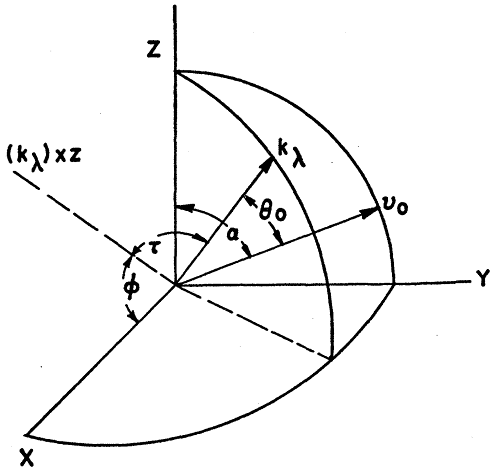

FIG. 1. Relation between the optic axis Z, propagation vector $\mathbf{k}$, and electron velocity vector $\mathbf{v}$.

The results of a calculation for $n_{\lambda i}^2$ and $(2\pi)^6(\mathbf{j}_k\cdot\mathbf{a}_{\lambda i})^2=e^2(\mathbf{v}_o\cdot\mathbf{a}_{\lambda i})^2$ that are to be substituted into Eq. (1) are as follows:

$$n_{\lambda o}^2=\epsilon_o, \tag{9a}$$

$$n_{\lambda e}^2=\frac{\epsilon_e\epsilon_o}{\epsilon_o\cos^2\phi+\epsilon_e\sin^2\phi+(\epsilon_o-\epsilon_e)\sin^2\phi\sin^2\tau}, \tag{9b}$$

where

$$\sin\tau=\frac{\cos\theta_i\sin\alpha\pm\sin\phi\cos\alpha(\sin^2\alpha-\cos^2\theta_i+\cos^2\alpha\sin^2\phi)^{\frac{1}{2}}}{\sin^2\alpha+\cos^2\alpha\sin^2\phi}, \tag{10}$$

$$\cos\theta_0=1/\epsilon_o^{\frac{1}{2}}\beta, \tag{11a}$$

$$\cos\theta_e=(Q/R)^{\frac{1}{2}}, \tag{11b}$$

$$Q=\frac{1}{(\epsilon_o-\epsilon_e)^2\sin^4\phi}\{\beta^2\epsilon_e\epsilon_o e(d\sin^2\alpha+\epsilon_o\cos^2\alpha\sin^2\phi)$$
$$+(\epsilon_o-\epsilon_o)\sin^2\phi(\epsilon_o\sin^2\phi\cos^2\alpha-d\sin^2\alpha)$$
$$+2(\epsilon_o-\epsilon_e)[\epsilon_e\epsilon_o(d\sin^2\alpha+\epsilon_o\cos^2\alpha\sin^2\phi)]^{\frac{1}{2}}$$
$$\times\beta\cos\alpha\sin\alpha\sin^3\phi\},$$

$$R=\left(\frac{\beta^2\epsilon_e\epsilon_o}{\epsilon_0-\epsilon_e}\right)^2\frac{e^2}{\sin^4\phi}-2(\sin^2\alpha-\sin^2\phi\cos^2\alpha)$$
$$\times\frac{\beta^2\epsilon_e\epsilon_o}{(\epsilon_o-\epsilon_e\sin^2\phi)}+1,$$

$$d=\epsilon_o\cos^2\phi+\epsilon_e\sin^2\phi,$$

$$e=\sin^2\alpha+\cos^2\alpha\sin^2\phi.$$

$$(\mathbf{v}_0\cdot\mathbf{a}_{\lambda_0})^2=v_0^2\frac{\cos^2\phi\sin^2\alpha(1-\sin^2\tau)}{\epsilon_0(\cos^2\phi+\sin^2\tau\sin^2\phi)} \tag{12a}$$

$$(\mathbf{v}_0\cdot\mathbf{a}_{\lambda e})^2=N/D, \tag{12b}$$

where

$$N=v_0^2\{(-\sin^2\alpha\sin^2\phi+\cos^2\alpha\sin^4\phi)\sin^4\tau$$
$$+(\sin^2\phi\sin^2\alpha+2\cos^4\alpha\sin^2\phi)\sin^2\tau+\cos^6\alpha$$
$$\pm2(\epsilon_o/\epsilon_e)\sin\alpha\cos\alpha(\cos^2\phi+\sin^2\tau\sin^2\phi)$$
$$\times\sin\tau[\sin^2\phi(1-\sin^2\tau)]^{\frac{1}{2}}\},$$

$$D=\epsilon_o\left\{\left(\sin^2\phi+\frac{\epsilon_o}{\epsilon_e}\cos^2\phi\right)-\left(1-\frac{\epsilon_o}{\epsilon_e}\right)\sin^2\phi\sin^2\tau\right\}$$
$$\times\{\cos^2\phi+\sin^2\tau\sin^2\phi\},$$

and $\sin\tau$ is given by (10).

One may easily find Ginsburg's results when the charged particle is traveling along and perpendicular to the optic axis, by taking the angle which determines the position of the velocity $\alpha=0$ and $\alpha=\pi/2$.

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