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Transport Processes as Foundations of the Heisenberg and Obukhoff Theories of Turbulence

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The transfer of energy across the spectrum is investigated by the method of the harmonic analysis of the Navier-Stokes equation. The phase correlation which enters into the transfer is determined by the statistical considerations of transport processes. The results obtained allow a theoretical explanation to be given to the Heisenberg and Obukhoff postulates, delimiting the conditions of their applicability. By extending the above considerations of transport processes to shear flow, the production of turbulent energy from the mean flow is determined in spectral terms. As an application, the spectral laws for energy and shear are derived, and compared with measurements.

I. INTRODUCTION

IN the metamorphosis of turbulence, we generally distinguish three phenomena: production, transfer, and dissipation. Recent theories confine themselves mostly to the study of transfer and dissipation, because they are controlled by fine patterns of turbulence which present an infinite number of degrees of freedom, and are, therefore, suitable for the attack by similarity and statistical methods. Of the two phenomena of fine patterns, the most characteristic one in turbulent motion is the transfer. The turbulent energy produced in the form of relatively big patterns is transferred to smaller and smaller patterns, because of the interaction between eddies, increasing in this way the local turbulent gradient, and hence promoting the dissipation by viscosity. It is this transfer, nonlinear in nature, which is responsible for the change of profiles of eddies, and hence for the transmission of energy through the spectrum.

The expression for the viscous dissipation is well known. In theories on the spectrum of turbulence the difficulty lies in the formulation of an expression of transfer, called transfer function. Owing to analytical difficulties relating to the interaction mechanism of eddies, the transfer function has been assumed usually on dimensional grounds. Since it controls the rate of decrease of energy, it has the dimension of u^2t^{-1} or u^3l^{-1}, or in spectral terms $(Fk)^{3/2}k$. Here $\frac{1}{2}u^2$ is the kinetic energy per unit mass, t is the age of coherence, l is the size of the domain of coherence, F is the spectral function, such that Fdk is the kinetic energy associated with wave numbers between k and $k+dk$. With such dimensional compositions, Heisenberg[1] writes the transfer function W_k in the form

$$W_k=2\nu_k\int_0^k dk'k'^2F(k'). \tag{1a}$$

Here W_k is the transfer of energy from wave numbers smaller than k to wave numbers larger than k; ν_k is the turbulent viscosity defined by

$$\nu_k=\kappa\int_k^\infty dn\left(\frac{F(n)}{n^3}\right)^{\frac{1}{2}} \tag{1b}$$

on dimensional grounds, and κ is a numerical constant. In this form, the transfer is expressed as a turbulent dissipation which, in a form similar to the viscous dissipation, is the product of the turbulent viscosity by the square of the vorticity. By means of general dimensional considerations, von Karman[2] writes a general expression for W_k which comprises (1a) as a special form.

In another way Obukhoff[3] writes

$$W_k=\kappa\int_k^\infty dnF(n)\left[2\int_0^k dk'k'^2F(k')\right]^{\frac{1}{2}}. \tag{2}$$

The numerical constant κ may not be identical in (1b) and (2). In the form (2) the transfer is considered as a production of energy by shearing stresses. Thus it is proportional to the product of a Reynolds stress (first integral) by the vorticity (second integral).

The two expressions (1a) and (2) represent two different theories of turbulence, the one based on the turbulent dissipation, the other on the turbulent shear. While the dimensional reasonings present the transfer in a simple way and is rich in possible applications, the physical foundations of the basic mechanism remain rather obscure. It is to be remarked that any dimensionally correct expression of transfer is expected to lead to the same "5/3" law of spectrum in the inertial range. This is the reason why the spectral laws based on the Heisenberg and Obukhoff hypotheses agree with the Kolmogoroff theory[4] in that range, although not in other ranges. We shall come back later to discuss further these hypotheses.

[1] W. Heisenberg, Z. Physik **124**, 628 (1948); see also S. Chandrasekhar, Astrophys. J. **110**, 329 (1949).

[2] T. von Karman, Proc. Natl. Acad. Sci. U. S. **34**, 530 (1948).

[3] A. Obukhoff, Compt. rend. acad. sci. U. R. S. S. **32**, 19 (1941); see also Bull. acad. sci. U. R. S. S. Sér. géograph. et géophys. **32**, 453 (1941).

[4] A. N. Kolmogoroff, Compt. rend. acad. sci. U. R. S. S. **30**, 301 (1941) and **32**, 16 (1941).

By leaving the dimensional method, Burgers'[5] theory uses, as a model of turbulence, a simple nonlinear partial differential equation having the essential properties of the hydrodynamical equations. The model shows saw-tooth profiles. The study of the law of successions of the segments can give interesting insights into the transfer mechanism and the spectrum of turbulence.

In the following pages an attempt will be made to study the mechanism of transfer by starting from the hydrodynamical equations, and hence to give a physical foundation to the hypotheses of Heisenberg and Obukhoff about the transfer function, delimiting the circumstances under which the theories are valid. The analysis of the turbulent velocity into modes, by applying the Navier-Stokes equations, leads to a dissipation equation, similar to the Boltzmann equation of velocity distribution (see Sec. 2). The nonlinear term, responsible for the transfer of turbulent energy between various modes, is comparable to the collision term of Boltzmann, the latter being, however, much simpler. Here the nonlinear term represents the mechanism of merging of one element into another, when they reach maximum steepness. After coalescence of elements, the one disappears and the other acquires the sum of momenta, in a manner analogous to the cascade processes advanced by Onsager.[6] Thus there results a transfer of momentum, for the study of which a statistical method is introduced in Sec. 3, describing the transport processes in turbulence.

As a result of the transport processes, and by the use of characteristic functions (Sec. 4), a relation of phase correlation can be obtained and gives a general expression for the transfer function, from which the Heisenberg and the Obukhoff formulas are derived as special cases (Sec. 5). At the same time this gives the opportunity of discussing the conditions of their applicability.

Finally, the considerations of transport processes are extended to shear flow, so that, in addition to the transfer function, the production function plays an important role. It is determined in spectral terms by the relation of phase correlation again. As an application, the spectral laws for energy and shear can be derived (Sec. 6).

II. HARMONIC ANALYSIS OF AN IRREGULAR MOTION AND MODULATION DUE TO THE TRANSFER OF ENERGY

For an isotropic and homogeneous turbulent field, the Navier-Stokes equation for the velocity fluctuations u_i is

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_j^2}. \tag{3}$$

$u_i(t, x)$ must satisfy the condition of continuity

$$\partial u_i/\partial x_j=0. \tag{4}$$

Here p is pressure, ρ is density, ν is kinematic viscosity, t is time, and $\mathbf{x}$ is vector position, with components x_i. A summation is understood whenever an index repeats.

Let us apply the Fourier transform to the above equations. The amplitude functions are

$$a_i(\mathbf{k})=\frac{1}{8\pi^3}\int_{\mathbf{x}-\mathbf{X}}^{\mathbf{x}+\mathbf{X}} d\mathbf{x}' u_i(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}'}, \tag{5a}$$

with the wave-number vector $\mathbf{k}$ as an argument, while

$$d\mathbf{x}'=dx_1'dx_2'dx_3'$$

is the elementary volume. In order to secure convergence in the Fourier analysis, the functions u_i are supposed *truncated*, i.e., the values of u_i are restricted within a finite element of volume, say a finite parallelepiped of sides $2X_1$, $2X_2$, $2X_3$. After Fourier transforms, Eqs. (3) and (4) become, respectively,

$$\frac{\partial a_i(\mathbf{k})}{\partial t}=-\nu k^2 a_i(\mathbf{k})-\int_{-\infty}^{\infty} d\mathbf{n} i n_j a_j(\mathbf{k}-\mathbf{n}) \times\left[a_i(\mathbf{n})-\frac{k_i k_s}{k^2}a_s(\mathbf{n})\right] \tag{6}$$

and

$$k_i a_i(\mathbf{k})=0. \tag{7}$$

Equation (6) indicates that the rate of change of the amplitude functions $a_i(\mathbf{k})$ is governed by two factors: a viscous dissipation proportional to k^2, and a nonlinear interaction between the mode $\mathbf{k}$ and all other modes $\mathbf{n}$ extending from $-\infty$ to ∞. In these notations, the energy equation can be written as follows:

$$\frac{\partial}{\partial t}a_i(\mathbf{k})a_i(-\mathbf{k}) =-2\nu k^2 a_i(\mathbf{k})a_i(-\mathbf{k})-\int_{-\infty}^{\infty} d\mathbf{n}q(\mathbf{k},\mathbf{n}). \tag{8a}$$

As the amplitudes of a_i may be also functions of time, a time average may be operated on the correlations between the amplitudes. Such time averages will be understood here and in the following, without additional symbols. The integral term of the right-hand

[5] Prof. J. M. Burgers has considered to a large extent the application of a mathematical model to the statistical theory of turbulence in the papers: Proc. Acad. Sci. Amsterdam **43**, 8 (1940), and Advances in Applied Mechanics **1**, 182 (1948). He considered the formation of vortex sheets and the correlation problems in the model of turbulence in the papers: Proc. Acad. Sci. Amsterdam **53**, 122, 247, 393, 718, 732 (1950).

[6] L. Onsager, Nuovo cimento, Suppl. 6, Ser. 9, No. 2, 279 (1949); see also Phys. Rev. **68**, 286 (1945).

side is the basic term for the formulation of the transfer function. It is

$$\int_{-\infty}^{\infty} d\mathbf{n}q(\mathbf{k},\mathbf{n})=\int_{-\infty}^{\infty} d\mathbf{n}in_j a_i(\mathbf{n})[a_j(\mathbf{k}-\mathbf{n})a_i(-\mathbf{k})+a_j(-\mathbf{k}-\mathbf{n})a_i(\mathbf{k})]$$
$$=-\int_{-\infty}^{\infty} d\mathbf{n}ik_j a_i(\mathbf{k})\cdot a_j(\mathbf{n}-\mathbf{k})a_i(-\mathbf{n})+\int_{-\infty}^{\infty} d\mathbf{n}ik_j a_i(-\mathbf{k})\cdot a_j(-\mathbf{n}+\mathbf{k})a_i(\mathbf{n}). \quad (8b)$$

The second integral is the conjugate part of the first integral. q is called modulation function. Its role is to excite other modes when one mode is produced. As easily checked it has the following properties:

(a) q is antisymmetrical, i.e., $q(n,k)+q(k,n)=0$. This condition imposes the conservation of modulated energy, and can be easily verified by (7) and (8b).

(b) The three arguments entering in the three amplitude functions have a sum zero.

(c) $\mathbf{k}$ should not be parallel to $\mathbf{n}$ in order to have a nonvanishing q.

It does not look easy to find a solution for a_i, or for $|a_i|^2$, which is controlled by a modulation mechanism having all those properties. From a general standpoint, the dynamical equations (6) show a certain analogy with the Boltzmann equation for the velocity distribution. The integral term of (6) represents nonlinear interactions between modes, comparable to the collision term of Boltzmann which was also in the form of an integral. However, Eqs. (6) and (7) are much more complicated, and the method of successive approximations, often applied to the Boltzmann equation, cannot be used here. In order to obtain full use of isotropy, the energy Eq. (8a) can be expressed in terms of a spectral function. Before going to this end, let us first find the spectral decomposition of the kinetic energy. As a Fourier inversion of (5a), u_i can be written in terms of a_i as follows:

$$u_i(\mathbf{x})=\int_{-\infty}^{\infty} d\mathbf{k}a_i(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (5b)$$

Its mean square value over a volume $V=2X_1 2X_2 2X_3$ gives

$$\langle u_i^2\rangle=\frac{8}{V}\int_{-\infty}^{\infty} d\mathbf{k}'\int_{-\infty}^{\infty} d\mathbf{k}''a_i(\mathbf{k}')a_i(\mathbf{k}'')\times\frac{\sin(k_1'+k_1'')X_1}{k_1'+k_1''}\frac{\sin(k_2'+k_2'')X_2}{k_2'+k_2''}\frac{\sin(k_3'+k_3'')X_3}{k_3'+k_3''}.$$

By increasing X_i indefinitely, we obtain

$$\langle u_i^2\rangle=\frac{8\pi^3}{V}\int_{-\infty}^{\infty} d\mathbf{k}a_i(\mathbf{k})a_i(-\mathbf{k}). \quad (9)$$

Since $\langle u_i^2\rangle$ must be independent of V, it follows that the absolute value of a_i^2 must be proportional to V. As a_i is obtained by the summation of a large number of variables like

$$u_i(\mathbf{x}')e^{i\mathbf{k}\cdot\mathbf{x}'}$$

according to (5a), the fact that its absolute value is proportional to V must be related to the same results found in problems of random walk, even though the paths may be bound by partial correlations.[7]

Instead of extending the integration in (9) over the whole $\mathbf{k}$ space, we can integrate over a spherical shell of radii between k and $k+dk$, and obtain

$$2F(k)=\frac{8\pi^3}{V}\int_0^{2\pi} d\varphi\int_0^{\pi} d\theta\,\sin\theta\, k^2 a_i(\mathbf{k})a_i(-\mathbf{k}).$$

For isotropic turbulence, F is reduced to the expression

$$F(k)=\frac{8\pi^3}{V}2\pi k^2 a_i(\mathbf{k})a_i(-\mathbf{k}). \quad (10)$$

The wave-number vector $\mathbf{k}$ has a magnitude k. The kinetic energy is given by the integral

$$\rho\int_0^{\infty} dkF(k)=\rho\langle u_i^2\rangle/2.$$

With the use of definition (10), we are now able to express the energy equation (8a) in terms of the spectral function F, by integrating with respect to k for the magnitude between 0 and k. We obtain

$$\frac{\partial}{\partial t}\int_0^k dkF(k)=2\nu\int_0^k dk'k'^2F(k')+\frac{8\pi^3}{V}\int_{0\leq k'\leq k} d\mathbf{k}'\int_{-\infty}^{\infty} d\mathbf{n}\tfrac{1}{2}q(\mathbf{k}',\mathbf{n}). \quad (11)$$

The double integral of the right-hand side of (11) is called the transfer function and is denoted by W_k. By the reason of antisymmetry of $q(k',n)$, the integrations can be extended from 0 to k for the variable k', and from k to ∞ for the variable n, instead of from $-\infty$ to ∞ for n. Hence we can write the transfer function of (11) in the following general form:

$$W_k=-\frac{8\pi^3}{V}\int_{0\leq k'\leq k} d\mathbf{k}'\int_{-\infty}^{\infty} d\mathbf{n}\tfrac{1}{2}q(\mathbf{n},\mathbf{k}')$$
$$=-\frac{1}{2}\frac{8\pi^3}{V}\int_{0\leq k'\leq k} d\mathbf{k}'ik_j'a_i(\mathbf{k}')\times\int_{k\leq n\leq\infty} d\mathbf{n}a_i(-\mathbf{n})a_j(\mathbf{n}-\mathbf{k}')+\frac{1}{2}\frac{8\pi^3}{V}\int_{0\leq k'\leq k} d\mathbf{k}'ik_j'a_i(-\mathbf{k}')\times\int_{k\leq n\leq\infty} d\mathbf{n}a_i(\mathbf{n})a_j(-\mathbf{n}+\mathbf{k}'). \quad (12)$$

[7] C. M. Tchen, J. Chem. Phys. 20, 214 (1952).

Several important features of the transfer can be noted from this expression as follows:

(i) In the form (12), the transfer of energy from wave numbers smaller than k to wave numbers larger than k is ensured by the action of shearing stresses (second integral) in a background of turbulent vorticity (first integral), in the same way by which the energy of the turbulent motion is produced by the gradient of the mean motion through the action of the Reynolds stresses.

(ii) Formula (12) indicates that for $k=0$, W_k vanishes. W_k also vanishes for $k=\infty$. This means that the total transfer across the spectrum must vanish. Consequently, the transfer is not a real dissipation of energy, in the sense that the energy is lost from the turbulent flow in the form of heat, but it promotes indirectly the dissipation by producing steep gradients which are more easily attacked by viscosity.

(iii) In order to obtain a transfer function in terms of spectral function, the first difficulty confronting us from (12) is to separate the three amplitude functions into two appropriate groups, each representing a well-defined physical entity of the transfer mechanism. Secondly, since according to (9) and (10), a^2 was proportional to V, the quantity a^3 should be proportional to $V^{3/2}$. It seems that the missing factor $(k^3V)^{\frac{1}{2}}$ in (12) should be embodied in the phase correlation of the three amplitude functions. The factor $k^{3/2}$ is written here to make $(k^3V)^{\frac{1}{2}}$ dimensionless. In this way we would arrive at

$$W_k \sim F^{3/2}k^{5/2}.$$

This expression gives the $k^{-5/3}$ law of the spectrum. The basic mechanism of phase interaction which would entail a factor $(k^3V)^{\frac{1}{2}}$ forms a difficult problem to which not much help can be found from the hydrodynamical equations. Here we must look for a statistical method, describing the transport processes by turbulence and serving as a basis of studying the phase correlation. We will come back to this matter in Sec. 3.

(iv) The integrands

$$a_i(-\mathbf{n})a_j(\mathbf{n}-\mathbf{k}'); \quad a_i(\mathbf{n})a_j(-\mathbf{n}+\mathbf{k}'),$$

of (12) can be considered as a result of the transport of momentum in the i direction by a cross motion in the j direction. Therefore, from its mean value as defined by the surface integral

$$\int_n ds(\mathbf{n})\cdots,$$

such that

$$\int_{k\leq n\leq\infty} d\mathbf{n}\cdots = \int_k^\infty dn \int_n ds(\mathbf{n})$$

for all directions of the vector n, the absolute value of n being confined between n and $n+dn$, a gradient,

$$-ik_j'a_i(\mathbf{k}') \quad \text{or} \quad ik_j'a_i(-\mathbf{k}'),$$

multiplied by a turbulent viscosity characterizing the transport process, must come out under certain circumstances which we shall discuss in detail in Sec. 4. Hence, a plausible formulation seems *a priori* to be as follows:

$$\left.\begin{aligned} &\int_{k\leq n\leq\infty} d\mathbf{n}a_i(\mathbf{n})a_j(\mathbf{k}'-\mathbf{n}) = -ik_j'a_i(\mathbf{k}')\nu_k, \\ &\text{with its conjugate} \\ &\int_{k\leq n\leq\infty} d\mathbf{n}a_i(-\mathbf{n})a_j(\mathbf{n}-\mathbf{k}') = ik_j'a_i(-\mathbf{k}')\nu_k. \end{aligned}\right\} \quad (13)$$

Substituting (13) into (12), we derive the transfer function (1a) postulated by Heisenberg. Equation (13) is the equation of phase correlation, and plays an important role in the determination of the transfer function. From the physical point of view, it expresses that, by interaction between $\mathbf{k}'$ and $\mathbf{n}$ $(n>k')$, the smaller element $\mathbf{n}$ is merged into the larger element $\mathbf{k}'$. Such an intermingling produces an exchange of momentum proportional to the average steepness of the resulting element, i.e., gradient of the larger element $\mathbf{k}'$, and to the diffusivity of the smaller submerged element $\mathbf{n}$. A proof and a generalization of (13) are found in Sec. 4.

(iv) On a pure dimensional ground, without consideration of phase interaction, the first integral of (12) could be considered as a turbulent gradient proportional to

$$\left[2\int_0^k dk'k'^2F(k')\right]^{\frac{1}{2}},$$

and the second integral could be considered as a stress tensor proportional to

$$\int_k^\infty dnF(n).$$

In this way we would arrive at the Obukhoff formula (2). The question may be asked whether some physical basis may be attributed to it. This problem together with the physical foundation of the Heisenberg formula will be studied in Sec. 5.

III. TRANSPORT PROCESSES IN TURBULENCE

In the preceding section, we have studied the transfer function from the hydrodynamical equations, and found that the relation of phase correlation (13) is important in the derivation of the Heisenberg formula. The fact that the phase interactions should come out, indeed, in the form (13) involving a turbulent viscosity, must be examined from the statistical theory of transport processes in turbulence, which we shall study in some detail in the following pages. For this purpose, a transition probability controlling time and displacement will be used to describe the transport processes. Its properties have been studied in detail in connection with the motion of small particles suspended in a turbulent fluid,[8] and applied to the study of the configurations of long chain molecules.[9]

Let

$$p(t', \mathbf{x}'; t, \mathbf{x})d\mathbf{x}$$

be the probability for a fluid element, which at the instant t, started from the point $\mathbf{x}'$, to arrive in the region $\mathbf{x}$, $d\mathbf{x}$ (i.e., in an elementary volume between $x_1, x_1+dx_1; x_2, x_2+dx_2; x_3, x_3+dx_3$) at the instant t. The function p controls the dispersion of the fluid element, and is called dispersion function. It is supposed continuous, and satisfies the condition

$$\int_{-\infty}^{\infty} d\mathbf{x} p(t', \mathbf{x}'; t, \mathbf{x}) = 1. \tag{14a}$$

Here and in the following formulas the limits of integration for convenience are given as $-\infty, +\infty$; the meaning of the formulas is that the integration is carried out over the whole available domain of $\mathbf{x}$-values, which may be of limited extent. Formula (14a) expresses the condition that all fluid elements starting from $\mathbf{x}'$ must find their place in the totality of the region $d\mathbf{x}$ forming the available domain of the variable $\mathbf{x}$.

It is in the nature of the phenomena of motion, whether regular or irregular, that we must expect

$$\lim_{t=t'} p(t', \mathbf{x}'; t, \mathbf{x}) = \delta(\mathbf{x}-\mathbf{x}') = \begin{cases} 0, & \text{for } \mathbf{x}-\mathbf{x}' \neq 0 \\ \infty, & \text{for } \mathbf{x}-\mathbf{x}' = 0. \end{cases} \tag{15a}$$

In all dispersion problems an important role is played by the mean values of the displacement and of its second and higher powers. Introduce the notations

$$\langle \mathbf{l} \rangle = \int_{-\infty}^{\infty} d\mathbf{x}(\mathbf{x}-\mathbf{x}') p(t', \mathbf{x}'; t, \mathbf{x}), \tag{16a}$$

$$\langle l_\alpha l_\beta \rangle = \int_{-\infty}^{\infty} d\mathbf{x}(x_\alpha - x_\alpha')(x_\beta - x_\beta') p(t', \mathbf{x}'; t, \mathbf{x}), \tag{17a}$$

etc. The mean value of the displacement itself, $\langle \mathbf{l} \rangle$, will be zero in the case of a symmetrical dispersion; in other cases, it can be expected that $\langle \mathbf{l} \rangle$ varies as $t-t'$. The mean value of the square of the displacement, $\langle l^2 \rangle$, can be of the order of $t-t'$ for small values of this interval. This can be seen when as an example for p we take the Gaussian function. The Gaussian function, which usually is considered as a typical example of a dispersion function, has furthermore the property that the mean values of higher powers of the displacement, as $\langle l^3 \rangle$ etc., for small intervals $t-t'$ are small compared with $t-t'$. Following Kolmogoroff we shall assume this to be a general property of the dispersion function to be considered here. Kolmogoroff, moreover, assumes that the ratios

$$\langle \mathbf{l} \rangle / (t-t') \quad \text{and} \quad \langle l^2 \rangle / (t-t')$$

tend to constant values (independent of $t-t'$) as $t-t'$ is decreased indefinitely.

The dispersion function p which defines the motion of the fluid elements, controls also the exchange of some physical or mechanical property $\phi(t, \mathbf{x})d\mathbf{x}$ distributed in the space and carried by the fluid elements in the region $d\mathbf{x}$, at the instant t. As a condition of statistical conservation of ϕ, we can write

$$\phi(t, \mathbf{x}) = \int_{-\infty}^{\infty} d\mathbf{x}' \phi(t', \mathbf{x}') p(t', \mathbf{x}'; t, \mathbf{x}). \tag{18a}$$

This equation expresses that the property $\phi(t, \mathbf{x})$ in the region $d\mathbf{x}$ and at the instant t, must have originated from somewhere in the whole region $\mathbf{x}'$ at an earlier instant t'. Evidently,

$$\int_{-\infty}^{\infty} d\mathbf{x} \phi(t, \mathbf{x}) = \int_{-\infty}^{\infty} d\mathbf{x}' \phi(t', \mathbf{x}').$$

In problems of moments it is known that the probability function p can be determined by the moments $\langle l^m \rangle$. Therefore, a relation for ϕ can be formed in terms of the moments from (18a). For this purpose we shall develop the integrand of (18a) into series. Before going to the development into series, it can be remarked that, in a Gaussian function, the parameters $t', \mathbf{x}'; t, \mathbf{x}$ figure exclusively in the form of the differences $t-t'$, $\mathbf{x}-\mathbf{x}'$. In functions of general type, however, t', $\mathbf{x}'$ themselves (or in another representation t, $\mathbf{x}$) must also be present. Let

$$t-t' = \tau; \quad \mathbf{x}-\mathbf{x}' = \mathbf{l};$$

then we can write the dispersion function $p(t', \mathbf{x}'; t, \mathbf{x})$ as a function P of $(t-\tau, \mathbf{x}-\mathbf{l}; \tau, \mathbf{l})$:

$$p(t', \mathbf{x}'; t, \mathbf{x}) = P(t-\tau, \mathbf{x}-\mathbf{l}; \tau, \mathbf{l}).$$

This second mode of writing is advantageous when we want to express that a dispersion function varies more slowly with t, $\mathbf{x}$ than with τ, $\mathbf{l}$. In particular this will be the case when τ is small. A development into a Taylor series with respect to $\mathbf{x}$ means a development of $p(t', \mathbf{x}'; t, \mathbf{x})$ into a series proceeding simultaneously with respect to $\mathbf{x}'$ and $\mathbf{x}$, with equal increments of both variables. The development of the integrand of (18a) into a Taylor series is then

$$\begin{aligned} \phi(t', \mathbf{x}') p(t', \mathbf{x}'; t, \mathbf{x}) &= \phi(t-\tau, \mathbf{x}-\mathbf{l}) P(t-\tau, \mathbf{x}-\mathbf{l}; \tau, \mathbf{l}) \\ &= \phi(t-\tau, \mathbf{x}) P(t-\tau, \mathbf{x}; \tau, \mathbf{l}) - l_\alpha \frac{\partial}{\partial x_\alpha}(\phi P) \\ &\quad + \tfrac{1}{2} l_\alpha l_\beta \frac{\partial^2}{\partial x_\alpha \partial x_\beta}(\phi P). \end{aligned}$$

[8] C. M. Tchen, "Mean Value and Correlation Problems Connected with the Motion of Small Particles Suspended in a Turbulent Fluid," Mededeelingen No. 51, (1947), Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool, Delft, Netherlands.

[9] C. M. Tchen, J. Research Natl. Bur. Standards **46**, 480 (1951).

Here repeating indices denote summations. Substituting the value above into (18a) and applying formulas (14a), (16a), and (17a), we obtain

$$\phi(t,\mathbf{x})=\phi(t',\mathbf{x})-\frac{\partial}{\partial x_\alpha}[\langle l_\alpha(t',\mathbf{x})\rangle\phi]+\frac{\partial^2}{\partial x_\alpha\partial x_\beta}[\tfrac{1}{2}\langle l_\alpha l_\beta\rangle\phi].$$

If we apply the Kolmogoroff assumptions that, for indefinitely decreasing τ, the ratios $\langle\mathbf{l}\rangle/\tau$ and $\langle l^2\rangle/2\tau$ assume constant values, and divide by τ, we can write the partial differential equation:

$$\frac{\partial\phi(t,\mathbf{x})}{\partial t}=-\frac{\partial}{\partial x_\alpha}\left[\frac{\langle l_\alpha\rangle}{\tau}(t,\mathbf{x})\phi\right]+\frac{\partial^2}{\partial x_\alpha\partial x_\beta}\left[\frac{\langle l_\alpha l_\beta\rangle}{2\tau}\phi\right].\qquad(19a)$$

In this form, Eq. (19a) is called the Fokker-Planck equation.

The property ϕ must moreover satisfy the equation of continuity:

$$\frac{\partial\phi}{\partial t}+\frac{\partial}{\partial x_\alpha}(\phi u_\alpha)=0.\qquad(20a)$$

Equations (19a) and (20a) form a system of two fundamental equations of the transport processes and will serve as a basis for determining the phase correlation (13), as required in the Heisenberg transfer function.

Some remarks can be made about the range of application of ϕ. First, if we apply to the case $\phi=1$, Eq. (20a) degenerates to the equation of conservation of mass in an incompressible flow. Secondly, the case of $\phi=\rho$ transforms Eq. (20a) into an equation of conservation of mass in a compressible flow. Finally, if $\phi=\rho u$, we have an equation of conservation of momentum, by disregarding the effect of pressure and viscosity which are of molecular origin. Now for the purpose of studying the transfer of energy across the spectrum (transfer function), the pressure and the viscosity can be indeed dropped for the following reasons. The former has the role of equipartition of energy among the components and does not play a role in the energy of the three components as a whole. The viscosity plays only a role in the molecular dissipation function which is well known, and not in the transfer function. Therefore, we cease to investigate a more comprehensive type of continuity equation, and conclude that (19a) and (20a) can be legitimately applied to the cases $\phi=1$ and $\phi=\rho u$.

It is to be remembered that the dispersion, as characterized by the function p, does not, by definition, depend separately on the specific nature of ϕ (whether it represents a mass or a momentum), and, therefore, we shall have a unique turbulent viscosity,

$$\nu_{\alpha\beta}^*=\langle l_\alpha l_\beta\rangle/2\tau,$$

common, at least in spectral structure, both in the transport of mass and in the transport of momentum. We shall assume this to be the case considered here, and must leave open the question whether there exist other types of dispersion functions which do not possess this property.

$\nu_{\alpha\beta}^*$ has the dimension ul, and may be expressed in terms of F. Therefore it may be considered as related to ϕ, for $\phi=\rho u$. Thus Eq. (19a) is essentially a nonlinear partial differential equation, suitable for the study of phase correlation.

Further, we shall assume

$$\nu_{\alpha\beta}^*=\nu^*\delta_{\alpha\beta}.$$

With the aid of the property that ν^* is common to both cases of transport $\phi=1$ and $\phi=\rho u$, a simplified form can be written for (19a). By putting $\phi=1$ and integrating, we have

$$\langle l_\alpha\rangle/\tau=\partial\nu^*/\partial x_\alpha;\qquad(21a)$$

the constant of integration may be rendered immaterial by a proper shift of coordinates. Substitution of (21a) transforms (19a) into

$$\frac{\partial\phi}{\partial t}=\frac{\partial}{\partial x_\alpha}\left(\nu^*\frac{\partial\phi}{\partial x_\alpha}\right).\qquad(22a)$$

Equation (21a) represents a current of diffusion due to the inhomogeneity of the turbulent field. Consider the case of a turbulent field where the diffusion increases with increasing values of x. Elements diffusing out of $d\mathbf{x}$ will take ever greater movements when they are displaced in the positive direction. Hence they will have greater chances to be dispersed farther away than those elements displaced in the opposite direction, and at the end of a small interval of time there will result a mean displacement in the positive direction, proportional to the gradient of the turbulent viscosity.

IV. CHARACTERISTIC FUNCTIONS

In order to study the phase correlation resulting from the transport of momentum between various modes, it is advantageous to pass from the probability function to the characteristic function. The characteristic function, denoted by a bar, is defined as the Fourier transform of the probability function. Thus the characteristic function $\bar{p}$ is

$$\bar{p}(t',\mathbf{x}';t,\mathbf{k})=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}d\mathbf{x}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}p(t',\mathbf{x}';t,\mathbf{x}),$$

and, conversely,

$$p(t',\mathbf{x}';t,\mathbf{x})=\int_{-\infty}^{\infty}d\mathbf{k}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\bar{p}(t',\mathbf{x}';t,\mathbf{k}).$$

Also introduce

$$\bar{\phi}(t,\mathbf{k})=\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}d\mathbf{x}e^{-i\mathbf{k}\cdot\mathbf{x}}\phi(t,\mathbf{x}).$$

Corresponding to (14a)–(22a) the following formulas can be obtained either by the Fourier resolutions of

(14a)–(22a) or by direct reasonings:

$$(2\pi)^3\bar{p}(t', \mathbf{x}'; t, 0)=1, \quad (14b)$$

$$(2\pi)^3 \lim_{t=t'}\bar{p}(t', \mathbf{x}'; t, \mathbf{k})=1, \quad (15b)$$

$$(2\pi)^3 \lim_{k=0}\frac{\partial}{\partial k_\alpha}\bar{p}(t', \mathbf{x}'; t, \mathbf{k})=-i\langle l_\alpha\rangle, \quad 16b)$$

$$(2\pi)^3 \lim_{k=0}\frac{\partial^2}{\partial k_\alpha\partial k_\beta}\bar{p}(t', \mathbf{x}'; t, \mathbf{k})=-\langle l_\alpha l_\beta\rangle, \quad (17b)$$

$$\bar{\phi}(t, \mathbf{k})=(2\pi)^3\bar{\phi}(t', \mathbf{k})\bar{p}(t', \mathbf{x}'; t, \mathbf{k}), \quad (18b)$$

$$\frac{\partial\bar{\phi}}{\partial t}=\frac{1}{\tau}\int_{-\infty}^{\infty}d\mathbf{n}ik_\alpha\langle\bar{l}_\alpha\rangle(\mathbf{n})\bar{\phi}(\mathbf{k}-\mathbf{n}) - \int_{-\infty}^{\infty}d\mathbf{n}k_\alpha k_\beta\bar{\nu}_{\alpha\beta}^*(\mathbf{n})\bar{\phi}(\mathbf{k}-\mathbf{n}), \quad (19b)$$

$$\frac{\partial\bar{\phi}}{\partial t}=ik_\alpha\int_{-\infty}^{\infty}d\mathbf{n}a_\alpha(\mathbf{k}-\mathbf{n})\bar{\phi}(\mathbf{n}), \quad (20b)$$

$$\langle\bar{l}_\alpha\rangle/\tau=-ik_\alpha\bar{\nu}^*, \quad (21b)$$

$$\frac{\partial\bar{\phi}}{\partial t}=k_\alpha\int_{-\infty}^{\infty}d\mathbf{n}(k_\alpha-n_\alpha)\bar{\phi}(\mathbf{k}-\mathbf{n})\bar{\nu}^*(\mathbf{n}). \quad (22b)$$

Thus by comparing the right-hand sides of (20b) and (22b) we have the following relation of phase correlation:

$$\int_{-\infty}^{\infty}d\mathbf{n}a_\alpha(\mathbf{k}-\mathbf{n})\bar{\phi}(\mathbf{n})=-i\int_{-\infty}^{\infty}d\mathbf{n}n_\alpha\bar{\phi}(\mathbf{n})\bar{\nu}^*(\mathbf{k}-\mathbf{n}), \quad (23)$$

which, by putting $\phi=\rho u_i$ as mentioned before, can be written as follows:

$$\int_{-\infty}^{\infty}d\mathbf{n}a_j(\mathbf{n}-\mathbf{k}')a_i(-\mathbf{n}) =i\int_{-\infty}^{\infty}d\mathbf{n}n_ja_i(-\mathbf{n})\bar{\nu}^*(\mathbf{n}-\mathbf{k}'),$$
$$\int_{-\infty}^{\infty}d\mathbf{n}a_j(\mathbf{k}'-\mathbf{n})a_i(\mathbf{n}) =-i\int_{-\infty}^{\infty}d\mathbf{n}n_ja_i(\mathbf{n})\bar{\nu}^*(\mathbf{k}'-\mathbf{n}). \quad (24)$$

Equations (23) and (24) are called relations of phase correlation. Substituting (24) into (12), we obtain the following transfer function:

$$W_k=\frac{1}{2}\frac{8\pi^3}{V}\int_{0\leq k'\leq k}d\mathbf{k}'\int_{k\leq n\leq\infty}d\mathbf{n}k_j'n_j \times[a_i(\mathbf{k}')a_i(-\mathbf{n})\bar{\nu}^*(\mathbf{n}-\mathbf{k}') +a_i(-\mathbf{k}')a_i(\mathbf{n})\bar{\nu}^*(\mathbf{k}'-\mathbf{n})]. \quad (25)$$

This general expression will serve as the foundations of the Heisenberg and Obukhoff theories.

V. ON THE HEISENBERG AND OBUKHOFF FORMULAS

Since we are concerned with the transfer function, the size of the eddies which we must be dealing with are mainly eddies of the inertial range. However, in order to discuss the mechanism of transfer, it is necessary to talk about the interactions between big and small eddies. It must be understood, then, that they are still eddies essentially close to that range. Let us consider some special cases and derive the Heisenberg and Obukhoff formulas from the general expressions of the transfer function. This will give us at the same time an opportunity of delimiting the circumstances under which the two theories are valid.

In its general form (25), the transfer function contains a gradient, of modes $\mathbf{k}'$ and $\mathbf{n}$, and a turbulent viscosity, of mode $(\mathbf{n}-\mathbf{k}')$. It lies in the nature of the phenomena of diffusion that a transfer of momentum from $\mathbf{k}'(0\leq k'\leq k)$ to $\mathbf{n}(k\leq n\leq\infty)$ occurs when the small eddies $\mathbf{n}$ in the role of turbulent viscosity displace themselves in a gradient field of larger eddies. As one of the special cases which we consider, it may happen that the functions of the two kinds of eddies, the one playing the role of turbulent viscosity and the other playing the role of gradient, are very distinct, so that for the motion of gradient-forming eddies, the smaller eddies may be, by approximation, abstracted and replaced by a uniform turbulent viscosity of the field; it is in the same way that the molecular motion has been usually abstracted in the hydrodynamical equation of motion and replaced by a uniform molecular viscosity ν. For such a case, we can write

$$\bar{\nu}^*(\mathbf{n}-\mathbf{k}')=|\nu^*|\delta(\mathbf{n}-\mathbf{k}'). \quad (26)$$

In this way, the mode of the turbulent viscosity becomes independent of the mode of gradients during the momentum transfer, so that the relations (24), becoming

$$\int_{-\infty}^{\infty}d\mathbf{n}a_j(\mathbf{n}-\mathbf{k}')a_i(-\mathbf{n})=ik_j'a_i(-\mathbf{k}')\nu^*$$

and

$$\int_{-\infty}^{\infty}d\mathbf{n}a_j(\mathbf{k}'-\mathbf{n})a_i(\mathbf{n})=-ik_j'a_i(\mathbf{k}')\nu^*,$$

prove the relation (13) for the part k, ∞ of the spectrum of ν^*.

Further, either by substituting (13) into (12) or by substituting (26) into (25), we obtain the Heisenberg formula (1a).

It may be assumed that such an independence of modes participating in the momentum transfer occurs more likely with smaller eddies than with bigger eddies. Therefore, the Heisenberg theory may extend more legitimately to small eddies. It is with them that the two gradients in the transfer function (the one originates from the transfer of momentum, and is associated with ν^* to form a shear stress, and the other entails from the transport of the shear stress to create the transfer of energy) are not of smooth pattern and may be intermingled.

In the second place, consider the other extreme case where the transport of momentum occurs in a vorticity field of smooth pattern, i.e.,

$$\mathbf{k}' \ll \mathbf{n}. \tag{27}$$

In this way $\bar{\nu}^*$ and a_i having the same argument n become inseparable. Substituting (27) into (12), we obtain

$$W_k = -\frac{1}{2}\int_{0\le k'\le k} d\mathbf{k}' ik_j' a_i(\mathbf{k}') \times \frac{8\pi^3}{V}\int_{k\le n\le\infty} d\mathbf{n} a_i(-\mathbf{n})a_j(\mathbf{n}) + \frac{1}{2}\int_{0\le k'\le k} d\mathbf{k}' ik_j' a_i(-\mathbf{k}') \times \frac{8\pi^3}{V}\int_{k\le n\le\infty} d\mathbf{n} a_i(\mathbf{n})a_j(-\mathbf{n}). \tag{28}$$

The right-hand member consists of a complex double integral and its conjugate. Each double integral reveals the product of a vorticity by an energy of shear origin. As the vorticity is expressed by

$$\left[2\int_0^k dk' k'^2 F(k')\right]^{\frac{1}{2}}$$

and the energy by

$$\int_k^\infty dnF(n),$$

Eq. (28) is transformed into the Obukhoff formula (2).

Usual opinions are more in favor of the Heisenberg theory, because it extends to the viscous range of the spectrum (although not to $k\to\infty$, where the Brownian motion may come to interfere with turbulence) by giving a reasonable spectral law $F\sim k^{-7}$, while the Obukhoff theory does not. The above considerations of transport processes have given to those theories some foundations, and shown that they are both reasonable. The Heisenberg theory may extend more to large k, while the Obukhoff theory more to small k. As the Obukhoff theory is shown not applicable to large k, obviously it is not expected to deliver the k^{-7} law, as did the Heisenberg theory in the viscous range.

The relation of phase correlation (13) which serves as a basis of the Heisenberg theory, may be roughly interpreted by means of the Burgers'[5] model of turbulence under the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}.$$

It is a simplified form of the hydrodynamical equation (3), and can also be considered as describing in a simplified way the propagation of a series of plane shock waves, introduced one after another into a gas. For the description of homogeneous turbulence, we may best prescribe a system of impulsive forces acting simultaneously for a short instant in a series of arbitrary chosen points. In the beginning the u curve will picture the pattern of the initial impulsive forces. But later on it changes more and more its profile, and original details are gradually eliminated. There is a tendency to form steep fronts at negative slopes. After a finite lapse of time they will approach to a vertical position, while the positive slopes will gradually decrease, so that we obtain a series of rectangular triangles, forming the so-called saw-tooth profiles. The vertical fronts, once generated, remain vertical, and propagate with variable velocities of advance, so that the consecutive vertical fronts may overtake each other. At the intermingling of two fronts the element of smaller scale n submerges into the element of larger scale k' with a transfer of momentum, and the two elements are combined to form a single front, moving from now on like the front of the survived element k'. The transfer of momentum is a result of the intermingling of the fronts which are the seats of vorticity. It is, therefore, not strange that in formula (13), the transport is characterized by the merging of two elements k' and n, and as a result, the surviving larger element k' appears in the form of a vorticity $ik_j'a_i(-\mathbf{k}')$. It is to be noticed that the role of the smaller submerged element in this transport process cannot be completely forgotten: the transport, moreover, depends upon how actively the submerged element n can diffuse, and is, therefore, proportional to the turbulent viscosity formed by the smaller element.

VI. APPLICATION OF THE PHASE CORRELATION AND OF THE TRANSPORT PROCESSES TO SHEAR FLOW

By means to the dimensional reasonings on which the Heisenberg theory is based, Parker[10] has extended the treatment to shear flow, and considered the problems of the critical Reynolds numbers. We shall not go into the study of the origin of turbulence here, but we shall restrict ourselves to the fully developed turbulent state, and investigate the spectrum in shear flow. The energy spectrum in shear flow has been studied earlier on the basis of the Boussinesq-Prandtl concept

[10] Eugene N. Parker, Phys. Rev. **90**, 221 (1953); see also Eugene N. Parker, "The Concept of Physical Subsets and Application to Hydrodynamic Theory," Technical Memorandum No. 988, March 1953, Mechelson Laboratory, Inyokern, China Lake, California (unpublished).

of turbulent shear.[11] In the following, the shear spectrum as well as the energy spectrum will be derived directly from the relations (23) and (24) of phase correlation. The method followed will be first to derive the production function and the Boussinesq relation between the shear spectrum F_{ij} and the energy spectrum on the basis of the phase correlation. Then the equation of equilibrium between the functions of dissipation, production, and transfer will serve to determine F and F_{ij}.

We suppose that in a domain of size $\mathbf{X}$, there are superposed two flows: a main flow U_i (independent of time), and a secondary flow u_i (varying in time and space), respectively with Fourier components A_i and a_i defined by formulas of the types (5a) and (5b). Beside the dissipation function and the transfer function as studied in Secs. 1–5, we have to investigate the production function ψ_k which results from the interaction between the two motions. Other functions, such as diffusion and convection, which take their origin from inhomogeneity, are of large scales, and are supposed negligible as compared with ψ_k and W_k.

As we recall that the transfer function W_k was obtained by analyzing the inertia term

$$u_i u_j \partial u_i/\partial x_j$$

of the equation of energy, the production function ψ_k is given by the Fourier analysis of the shear term

$$u_i u_j \partial U_i/\partial x_j.$$

Thus by the same method used to derive W_k, we have

$$\psi_k = -\frac{1}{2}\frac{8\pi^3}{V}\int_{0\le k'\le k} d\mathbf{k}' ik_j' A_i(\mathbf{k}') \times\int_{k\le n\le\infty} d\mathbf{n}\, a_i(-\mathbf{n})a_j(\mathbf{n}-\mathbf{k}') + \frac{1}{2}\frac{8\pi^3}{V}\int_{0\le k'\le k} d\mathbf{k}' ik_j' A_i(-\mathbf{k}') \times\int_{k\le n\le\infty} d\mathbf{n}\, a_i(\mathbf{n})a_j(-\mathbf{n}+\mathbf{k}') \quad (29)$$

in analogy with (12), and

$$\psi_k = \frac{1}{2}\frac{8\pi^3}{V}\int_{0\le k'\le k} d\mathbf{k}'\int_{k\le n\le\infty} d\mathbf{n}\, k_j' n_j \times[A_i(\mathbf{k}')a_i(-\mathbf{n})\bar{\nu}^*(\mathbf{n}-\mathbf{k}') + A_i(-\mathbf{k}')a_i(\mathbf{n})\bar{\nu}^*(\mathbf{k}'-\mathbf{n})] \quad (30)$$

in analogy with (25).

Again we can apply to two extreme cases as described in Sec. 5.

[11] C. M. Tchen, J. Research Natl. Bur. Standards **50**, 51 (1953).

(1) First, we suppose a uniform ν^* in the transport processes. For this case we can use assumption (26), which substituted into (30) gives

$$\psi_k = \frac{8\pi^3}{V}\int_{0\le k'\le k} dk' 2\pi k'^4 \tfrac{1}{2}[A_i(\mathbf{k}')a_i(-\mathbf{k}') + A_i(-\mathbf{k}')a_i(\mathbf{k}')]\nu_k. \quad (31)$$

This supposes that there is a strong interaction between a_i and A_i. Therefore, A_i must possess a rough pattern with strong vorticity. The integral represents a mixed vorticity composed of two motions. If we suppose that the spectrum of A_i is approximately complete between 0 and k, due to its rapid convergence, (31) takes the following form:

$$\psi_k \sim \left[2\left(\frac{\partial U_i}{\partial x_j}\right)^2 \int_0^k dk' k'^2 F(k')\right]^{\frac{1}{2}} \nu_k, \text{ for large } U', \quad (32)$$

where

$$U'^2 = (\partial U_i/\partial x_j)^2.$$

For the determination of the energy spectrum F, consider the case of statistical equilibrium, and write the following equation of dissipation:

$$\epsilon = 2\nu\int_0^k dk' k'^2 F(k') + W_k + \psi_k, \quad (33)$$

where

$$\epsilon = 2\nu\int_0^\infty dk' k'^2 F(k').$$

In the inertial range, the viscous dissipation on the right-hand side of (33) is negligible, and so does W_k too, as compared with ψ_k, if

$$2\int_0^k dk' k'^2 F(k') \ll (\partial U_i/\partial x_j)^2.$$

This is the case of large U' and moderate k. Hence from (32) and (33) we have

$$\nu_k\left[U'^2 2\int_0^k dk' k'^2 F(k')\right]^{\frac{1}{2}} = \epsilon.$$

It follows:

$$F = (\epsilon/\kappa U')k^{-1}, \text{ for large } U'. \quad (34)$$

An estimate of the shear spectrum F_{ij} can be obtained by starting from (6), and writing the rate of change of the amplitude of the shear stress as

$$\frac{\partial}{\partial t}[a_i(\mathbf{k})a_j(-\mathbf{k}) + a_i(-\mathbf{k})a_j(\mathbf{k})].$$

The function of production of such a shear stress has an expression of structure similar to ψ_k, containing the triple product of A by two a's. If again the production term is the predominant one, one finds by similarity

$$2F_{ij} \sim k^{-1} \quad (35)$$

as the spectrum of the shear $u_i u_j$.

(2) Let us consider the case of a mean motion of smooth pattern, i.e., of small U',

$$2\int_0^k dk'k'^2F(k')\gg U'^2.$$

The assumption (27) is then valid, and after being substituted into (29), gives

$$\psi_k=\frac{\partial U_i}{\partial x_j}2\int_k^\infty dnF_{ij}(n). \tag{36}$$

Since U' is small, (36) indicates that W_k is predominant on the right-hand side of (33). Hence the spectrum of energy is that given by the Heisenberg and Obukhoff theories:

$$F=(8\epsilon/9\kappa)^{2/3}k^{-5/3}. \tag{37}$$

In order to determine F_{ij}, we start from the general relation of phase correlation (23), which may be written as follows:

$$\frac{8\pi^3}{V}\int d\mathbf{n}\langle a_j(\mathbf{n})\bar{\phi}(-\mathbf{n})\rangle=\frac{8\pi^3}{V}\int d\mathbf{n}in_j\langle\bar{\phi}(-\mathbf{n})\bar{\nu}^*(\mathbf{n})\rangle.$$

For $\phi=\rho(U_i+u_i)$, we have

$$\frac{8\pi^3}{V}\int d\mathbf{n}a_j(\mathbf{n})a_i(-\mathbf{n})=\frac{8\pi^3}{V}\int d\mathbf{n}in_jA_i(-\mathbf{n})\bar{\nu}^*(\mathbf{n}).$$

Further, by means of the relation

$$in_jA_i(-\mathbf{n})=-\frac{1}{8\pi^3}\int_{\mathbf{x}-\mathbf{X}}^{\mathbf{x}+\mathbf{X}}d\mathbf{x}'\frac{\partial U_i(\mathbf{x}')}{\partial x_j'}e^{i\mathbf{n}\cdot\mathbf{x}'}$$

as derived from the definition (5a), (38) can be transformed as follows:

$$\begin{aligned}\frac{8\pi^3}{V}\int_{k\leq n\leq\infty} d\mathbf{n}a_j(\mathbf{n})a_i(-\mathbf{n})&=-\frac{1}{V}\int_{k\leq n\leq\infty}d\mathbf{n}\int_{\mathbf{x}-\mathbf{X}}^{\mathbf{x}+\mathbf{X}}d\mathbf{x}'\frac{\partial U_i(\mathbf{x}')}{\partial x_j'}e^{i\mathbf{n}\cdot\mathbf{x}'}\bar{\nu}^*(\mathbf{n})\\&=-\nu_k\frac{\partial U_i}{\partial x_j},\end{aligned} \tag{38}$$

for a mean motion of smooth pattern. Hence

$$\begin{aligned}2\int_k^\infty dnF_{ij}(n)&=\frac{8\pi^3}{V}\int_{k\leq n\leq\infty}d\mathbf{n}\tfrac{1}{2}[a_i(\mathbf{n})a_j(-\mathbf{n})+a_i(-\mathbf{n})a_j(\mathbf{n})]\\&=-\nu_k\frac{1}{2}\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)\end{aligned} \tag{39}$$

for sufficiently large k.

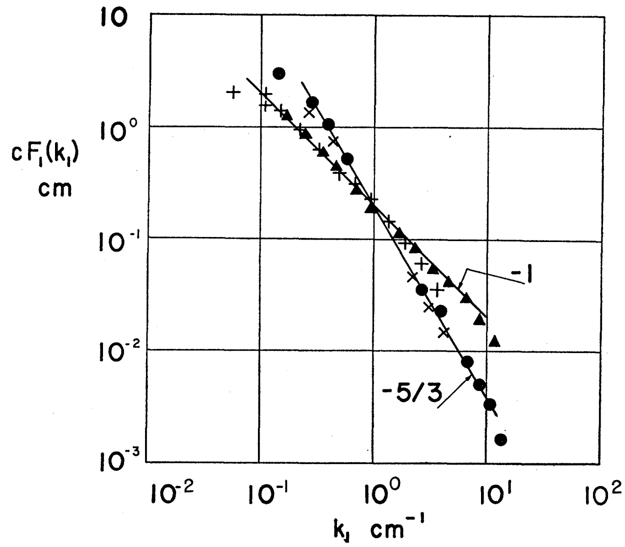

FIG. 1. Energy spectrum in a boundary layer and in a pipe (for data see Table I).

Formula (39) is known as the spectral equivalent of the Boussinesq-Prandtl formula, which, after differentiation, gives a relation between F_{ij} and F as follows:

$$\begin{aligned}F_{ij}(k)&=-\frac{1}{4}\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)\left(\frac{F(k)}{k^3}\right)^{\frac{1}{2}}\\&=-\frac{1}{4}\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)\left(\frac{8\kappa^2\epsilon}{9}\right)^{1/3}\cdot k^{-7/3}\end{aligned} \tag{40}$$

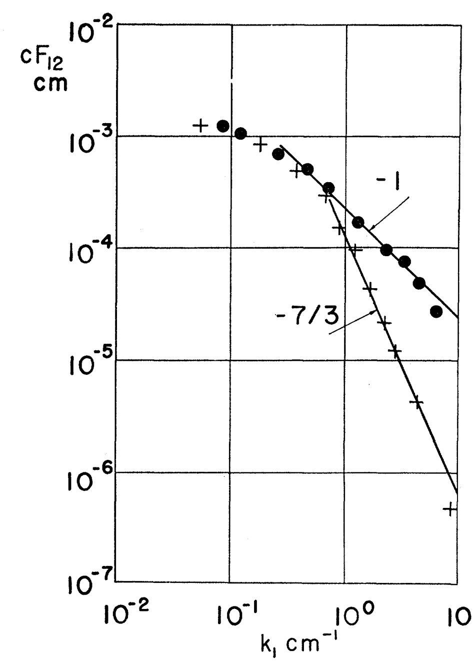

FIG. 2. Shear spectrum in a boundary layer (for data see Table II).

TABLE I. Data for the energy spectrum in a boundary layer and in a pipe.

Experimental points	Type of flow	δ; cm	U_m; cm/sec	Distance from wall; δ/x_2	Local mean velocity gradient	u' cm/sec	c
+	boundary layer	7.6	1524	0.05	large	119	1
▲	pipe	12.3	3048	0.008	large	256	3.87
×	boundary layer	7.6	1524	0.8	small	32	1
●	pipe	12.3	3048	0.69	small	113	2.51

TABLE II. Data for the shear spectrum in a boundary layer.

Experimental points	U_m cm/sec	Distance from wall; δ/x_2	Local mean velocity gradient	$c=\frac{\langle u_1u_2\rangle}{U_m^2}$
●	1524	0.05	large	0.0014
+	1524	0.58	small	0.00017

provided $\partial U_i/\partial x_j$ is small, and F is given by formula (37). The above derivation of (39) fixes the conditions of applicability of the Boussinesq-Prandtl formula.

We conclude that in the inertial (nonviscous) range, the spectral laws of energy are $F\sim k^{-5/3}$, k^{-1}, and the spectral laws of shear are $F\sim k^{-7/3}$, k^{-1}, respectively, for small and large U'.

From the experimental point of view, high values of U' can be found near the wall in flows of the boundary layer type. However, one must not go too close to the wall, where the turbulent Reynolds number drops, and the inertial range becomes absent. It is to be remarked that the spectra considered are three dimensional spectra. To date, no measurements of three dimensional spectra are available. However, some measurements of one dimensional spectra seem to give reasonable confirmation to the above power laws, if it can be assumed that the powers are conserved in the transformation between the one- and the three-dimensional spectra, especially at large k (compare Figs. 1 and 2).

It is to be remarked that for flows in a boundary layer and in a pipe, only one component of the mean flow with gradient plays a predominant role. Therefore, among the energy equations for the three components, one component only contains the production function, and hence may give the spectrum k^{-1}.

Figures 1 and 2 are based upon the measurements in a boundary layer and in a pipe, respectively, by Klebanoff[12] and Laufer.[13] In order to facilitate the comparison, the normalized spectra of energy and shear are plotted in Figs. 1 and 2. Thus the one-dimensional energy spectrum F_1 and the shear spectrum F_{12} are expressed in cm.

Let δ be the thickness of the boundary layer, and the radius of the pipe; U_m the maximum mean velocity; u' the root-mean-square of the velocity fluctuations in the x_1 direction; c a numerical constant required by normalization, and used to bring together the data. The essential data for those figures are found respectively in Tables I and II.

[12] P. S. Klebanoff, Natl. Advisory Comm. Aeronaut. Tech. Notes (to be published).
[13] John Laufer, Natl. Advisory Comm. Aeronaut. Tech. Notes, No. 2954 (1953).