Positron-Electron Differences in Energy Loss and Multiple Scattering*

F. ROHRLICH† AND B. C. CARLSON
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey

(Received August 24, 1953)

The Bhabha cross section for positron-electron scattering is used to derive the average rate of collision loss by positrons passing through matter, which differs from the corresponding expression for negative electrons by a few percent. The spin and exchange terms of the Bhabha and Møller cross sections produce small corrections also in the Landau distribution of energy losses by positive and negative electrons in thin foils, the most probable energy loss being less affected than the shape of the distribution. Positron-electron differences in the average cosine of the multiple-scattering angle, and in the penetration depth at which the particles have lost their memory of initial direction, are calculated from the appropriate elastic cross sections; the difference in rate of energy loss is included in the calculation. Numerical examples for Al and Pb are given.

I. INTRODUCTION

THE theoretical scattering cross section of positrons differs from that of electrons for scattering by a nuclear Coulomb field, as well as for scattering by another electron. In the first case the difference has been confirmed by the single-scattering experiments of Lipkin and White,[1] and in the second case by those of Howe and MacKenzie and of Ashkin and Woodward.[2] Although the fundamental differences have been known for a long time, there seems to be little published literature concerning the theoretical evaluation of their effect on the passage of electrons and positrons through matter.[3] Several recent experiments indicate the usefulness of such investigations.[4] Differences in stopping power, energy straggling, multiple scattering, and range need to be discussed, and it seems worth while at least to estimate the magnitude of these differences and to inquire under what conditions they need to be taken into account. The present paper is intended as a step in this direction, for the energy range from 50 kev to a few Mev.

We shall consider in Sec. II the average collision loss and in Sec. III the energy straggling on transmission through a thin foil. These two effects involve only the difference between electron-electron and positron-electron scattering. In Sec. IV we shall discuss the longitudinal distribution caused by multiple scattering in an infinite medium. This involves the positron-electron difference in both elastic and inelastic scattering.

II. AVERAGE ENERGY LOSS

The well-known Bethe-Bloch formula[5] for the average energy loss by collisions is derived for electrons under the assumption that above a certain fractional energy transfer, ϵ_1, the atomic electrons can be regarded as free, so that Møller's cross section[6] for scattering of free electrons by free electrons at rest in Born approximation is applicable:

$$\left(\frac{d\sigma}{d\epsilon}\right)^- = \frac{\chi}{T}\left[\frac{1}{\epsilon^2}+\frac{1}{(1-\epsilon)^2}+\left(\frac{\gamma-1}{\gamma}\right)^2 - \frac{2\gamma-1}{\gamma^2}\cdot\frac{1}{\epsilon(1-\epsilon)}\right], \quad (1)$$

where

$$\chi=\frac{2\pi e^4}{mv^2}=\frac{2\pi r_0^2 mc^2}{\beta^2}, \quad r_0=\frac{e^2}{mc^2}. \quad (2)$$

The incident total energy is $E=\gamma mc^2$, the kinetic energy is $T=(\gamma-1)mc^2$, and ϵ is the energy transfer in units of T. Since the outgoing electron of higher energy is by definition the primary electron, the maximum energy transfer is $\epsilon_m=\frac{1}{2}$. Therefore, the average energy loss per atom of Z electrons due to hard collisions is

$$ZT\int_{\epsilon_1}^{\frac{1}{2}} \epsilon\left(\frac{d\sigma}{d\epsilon}\right)^- d\epsilon = Z\chi\left[\ln\frac{1}{4\epsilon_1}+1 - \frac{2\gamma-1}{\gamma^2}\ln 2+\frac{1}{8}\left(\frac{\gamma-1}{\gamma}\right)^2\right]. \quad (3)$$

For low energy transfers ($\epsilon<\epsilon_1$) an explicit summation over the various excitation probabilities of the atom must be carried out. One finds[5]

$$ZT\int_0^{\epsilon_1} \epsilon\sigma(\epsilon,\gamma)d\epsilon = Z\chi\left[\ln\frac{2T^2\epsilon_1(\gamma+1)}{I^2}-\beta^2\right], \quad (4)$$

* This work was supported in part by the U. S. Atomic Energy Commission and the Higgins Scientific Trust Fund.

† Now at Department of Physics, State University of Iowa, Iowa City, Iowa.

[1] H. J. Lipkin and M. G. White, Phys. Rev. **79**, 892 (1950); **85**, 517 (1952).

[2] H. A. Howe and K. R. MacKenzie, Phys. Rev. **90**, 678 (1953); A. Ashkin and W. M. Woodward, Phys. Rev. **87**, 236 (1952).

[3] Notable exceptions include W. Miller, Phys. Rev. **82**, 452 (1951) and Chang, Cook, and Primakoff, Phys. Rev. **90**, 544 (1953). We are indebted to Professor Primakoff for sending us a copy of this paper before its publication.

[4] Groetzinger, Humphrey, and Ribe, Phys. Rev. **85**, 78 (1952); H. H. Seliger, Phys. Rev. **88**, 408 (1952); reference 3.

[5] See, for example, H. A. Bethe, *Handbuch für Physik* (Julius Springer, Berlin, 1933), Vol. 24/2, p. 273.

[6] C. Møller, Ann. Physik **14**, 531 (1932).

where I is an average ionization energy. From (3) and (4) the average collision loss per unit path length s of electrons in a medium with N atoms per unit volume is

$$\left(-\frac{dE}{ds}\right)^{-}=NZ\chi\left[\ln\left(\frac{T^2}{I^2}\cdot\frac{\gamma+1}{2}\right)+f^{-}(\gamma)\right], \quad (5)$$

$$f^{-}(\gamma)=1-\beta^2-\frac{2\gamma-1}{\gamma^2}\ln 2+\frac{1}{8}\left(\frac{\gamma-1}{\gamma}\right)^2. \quad (6)$$

In the case of positrons, the derivation of the average energy loss proceeds along exactly the same lines, with only two basic differences: (a) the upper limit of the energy transfer is now $\epsilon_m=1$ rather than $\frac{1}{2}$, because electron and positron are distinguishable; (b) for collisions with large energy transfer, the relativistic electron-electron collision cross section has to be replaced by the corresponding positron-electron cross section, which was first derived by Bhabha,[7]

$$\left(\frac{d\sigma}{d\epsilon}\right)^{+}=\frac{\chi}{T}\left\{\frac{1}{\epsilon^2}-\frac{\gamma^2-1}{\gamma^2}\cdot\frac{1}{\epsilon}+\frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)^2 -\left(\frac{\gamma-1}{\gamma+1}\right)\cdot\left[\frac{\gamma+2}{\gamma}\cdot\frac{1}{\epsilon}-2\frac{\gamma^2-1}{\gamma^2} +\epsilon\left(\frac{\gamma-1}{\gamma}\right)^2\right]+\left(\frac{\gamma-1}{\gamma+1}\right)^2\cdot\left[\frac{1}{2}+\frac{1}{\gamma}+\frac{3}{2\gamma^2} -\left(\frac{\gamma-1}{\gamma}\right)^2\epsilon(1-\epsilon)\right]\right\}. \quad (7)$$

The difference between the Bhabha and Møller cross sections is indicated in Fig. 1. In this figure are plotted the ratios of these cross sections to their common leading term, $\chi/T\epsilon^2$. Three different incident energies are shown, $\gamma=1.1$ ($\sim$50 kev), $\gamma=\sqrt{2}$ (212 kev), and $\gamma=3$ ($\sim$1 Mev). In the intermediate case the initial slopes of the two cross sections are equal.

The contributions to the energy loss from small energy transfers are determined by the oscillator strengths of the atom and are, therefore, to a good approximation, independent of the charge of the incident particle. Thus, Eq. (4) will also be valid for positrons.

In this manner, one finds for the average collision loss per unit path length of positrons

$$\left(-\frac{dE}{ds}\right)^{+}=NZ\chi\left[\ln\left(\frac{T^2}{I^2}\cdot\frac{\gamma+1}{2}\right)+f^{+}(\gamma)\right], \quad (8)$$

$$f^{+}(\gamma)=2\ln 2-\frac{\beta^2}{12}\left[23+\frac{14}{\gamma+1} +\frac{10}{(\gamma+1)^2}+\frac{4}{(\gamma+1)^3}\right]. \quad (9)$$

[7] H. J. Bhabha, Proc. Roy. Soc. (London) **A154**, 195 (1936).

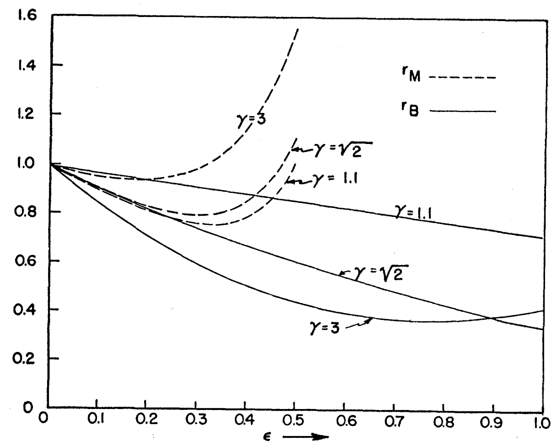

FIG. 1. The ratios r_M and r_B of the Møller and Bhabha scattering cross sections to classical relativistic Rutherford scattering as functions of the fractional energy transfer ϵ. Curves for three different kinetic energies $T=(\gamma-1)mc^2$ are shown. For $\gamma=\sqrt{2}$ the slopes of r_M and r_B at $\epsilon=0$ coincide.

The positron-electron difference in collision loss is determined by the functions $f^{\pm}(\gamma)$ given in Eqs. (6) and (9). These functions, which depend only on the incident energy and are independent of the atomic number, are plotted in Fig. 2. Collisions in which a positron loses more than half its energy make a roughly constant contribution of about 0.4 to f^{+}.

The percentage difference in energy loss is shown in Fig. 3. One sees that, almost independent of atomic number, positrons lose energy more rapidly than electrons below about 345 kev, but less rapidly above that energy. The effect increases percentagewise with average ionization potential, but is only about $\frac{1}{6}$ to $\frac{1}{3}$ larger in lead than in aluminum. A positron-electron difference of several percent may seem surprising until one remembers that the hard collisions, although relatively rare, are weighted heavily in the average energy loss. The question now arises whether this difference is detectable in experiments concerned with the penetration of electrons and positrons through matter. The answer to this question depends on the problem of

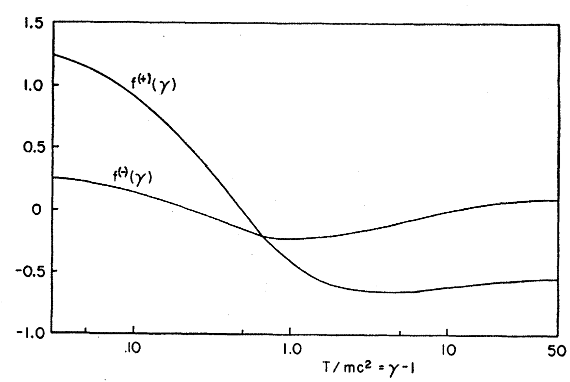

FIG. 2. The functions $f^{(+)}(\gamma)$ and $f^{(-)}(\gamma)$, which occur in the average energy loss formulas for positrons and electrons, as functions of the kinetic energy T of the incident particle in units of mc^2.

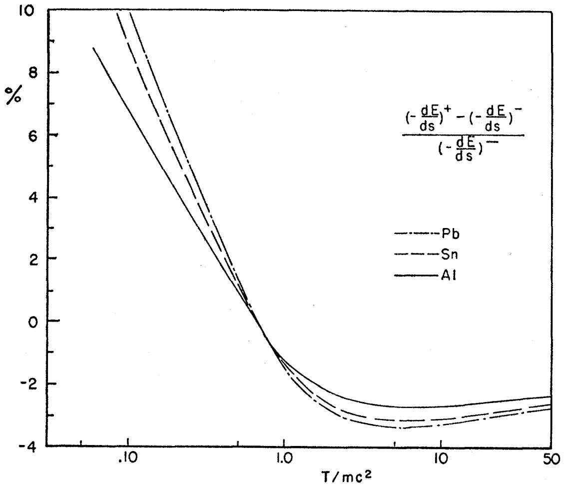

FIG. 3. Percentage positron-electron difference in average energy loss for Pb, Sn, and Al as a function of the kinetic energy T in units of mc^2.

multiple scattering, to be discussed in Sec. IV, but we turn first to a case in which the distribution of losses, rather than the average loss, is of interest.

III. ENERGY STRAGGLING

If a beam of monoenergetic electrons passes through a foil, the electrons emerge with a distribution of energies because of the statistical nature of the collision-loss process. If the foil is thin compared with the electron range, the distribution has a characteristic shape and, by suitable choice of units, can be compared with a universal curve derived theoretically by Williams[8] and Landau.[9] The appropriate unit of energy is $\zeta T = NZ\chi z$; it serves to define a dividing line between "soft" and "hard" collisions, according to whether the fractional energy transfer ϵ in the collision is smaller or greater than ζ. The definition is so chosen that on the average an electron suffers one hard collision in traversing the foil.

ζ depends on the electron energy through $\chi = 2\pi e^4/mv^2$ and on the characteristics of the foil through the number NZ of electrons per unit volume and the foil thickness z. (More exactly, z should be regarded as the path length s. Unless the foil is very thin, multiple scattering will produce a distribution of path lengths and thereby an extra contribution to the energy straggling which is not included in the Landau curve. This effect has been discussed by Williams[8] and Yang.[10]) In experiments on straggling, ζ is typically about 0.005; ζT is ordinarily large compared with the average ionization potential ($I = \text{Ry}\cdot Z$); and in most of the collisions with energy transfers comparable to ζ, the atomic binding effects have already assumed a minor role.

[8] E. J. Williams, Proc. Roy. Soc. (London) **A125**, 420 (1929).
[9] L. Landau, J. Phys. (U.S.S.R.) **8**, 201 (1944).
[10] C. N. Yang, Phys. Rev. **84**, 599 (1951). See also Goldwasser, Mills, and Hanson, Phys. Rev. **88**, 1137 (1952).

Both hard and soft collisions with energy transfers of the order of ζ will be called intermediate collisions.

The peak of the straggling distribution occurs at the most probable energy loss, Δ_p, which is determined, as Williams pointed out, by the very soft and intermediate collisions; very hard collisions ($\epsilon \gg \zeta$) are too rare to affect it. For this reason Landau introduced only a negligible error in Δ_p by extending an integration of the free-collision cross section to infinite energy transfers; for this reason, also, it is clear that the difference of a factor two between the maximum energy transfers in positron-electron and electron-electron collisions has no effect on the most probable energy loss.

Williams showed further that the straggling distribution can be resolved into two parts: a Gaussian distribution caused by the soft collisions, centered about an energy loss close to the most probable loss and with rms deviation ζT; and a long tail due to the hard collisions. However, it is only the intermediate collisions ($\epsilon \sim \zeta$) which contribute significantly either to the Gaussian or to the tail. The very soft collisions ($\epsilon \ll \zeta$) are so frequent that the statistical fluctuation in the resulting energy loss is negligible; the very hard collisions ($\epsilon \gg \zeta$) are extremely rare and in any case influence only that remote part of the straggling tail which is too small to be ordinarily observable. In summary, the shape of the straggling distribution is essentially determined by the intermediate collisions alone, whereas the displacement of its peak from the incident energy is determined by those in conjunction with all softer collisions.

Any positron-electron difference in most probable energy loss or in the shape of the Landau curve must, therefore, result from a difference in cross section for fractional energy transfers of the order of 0.005, and is more likely to show up in the shape of the curve than in Δ_p. Since the spin and exchange terms of both the Møller and Bhabha cross sections are of minor importance at low fractional energy transfers, the difference will necessarily be small.

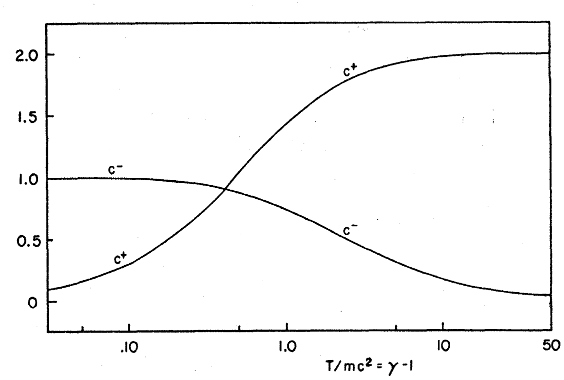

FIG. 4. The negative slopes c^+ and c^- at $\epsilon=0$ of the curves r_B and r_M, respectively (see Fig. 1), as functions of the kinetic energy T in units of mc^2.

In order to obtain a more quantitative estimate, we follow Landau's analysis but replace the Rutherford cross section by improved approximations to the Møller and Bhabha cross sections for low fractional energy transfers ϵ. An expansion of these cross sections in powers of ϵ leads (see Eqs. (1) and (7)) to

$$\left(\frac{d\sigma}{d\epsilon}\right)^{\pm}=\frac{\chi}{T}\cdot\frac{1}{\epsilon^2}\cdot(1-c^{\pm}\epsilon+\cdots), \tag{10}$$

$$c^{+}=\beta^2[2-(\gamma+1)^{-2}],$$

$$c^{-}=(2\gamma-1)/\gamma^2.$$

The functions c^+ and c^- are plotted in Fig. 4. It is analytically convenient, as well as a slightly better approximation, to use

$$\left(\frac{d\sigma}{d\epsilon}\right)^{\pm}=\frac{\chi}{T}\cdot\frac{1}{\epsilon^2}\cdot\exp(-c^{\pm}\epsilon). \tag{11}$$

This choice makes the analysis closely parallel to Landau's, the integration again being extended to infinite energy transfers, since the very hard collisions do not affect the observable straggling. The final inversion of the Laplace transformation can be reduced by change of variable to the integral which Landau evaluated numerically. The results given below reduce to Landau's results when $\pm$ signs are omitted and c is taken to be zero.

The probability (normalized to unity) of an energy loss in the range $d\Delta$ is

$$f^{\pm}(x,\Delta)d\Delta=\exp[-\alpha^{\pm}(\lambda+\ln\alpha^{\pm})]\varphi(\lambda)d\lambda, \tag{12}$$

where $\varphi(\lambda)$ is the universal Landau curve, $\alpha^{\pm}=c^{\pm}\zeta$, and

$$\lambda=\frac{\Delta}{\zeta T}-\left[\ln\frac{\zeta T^2 2(\gamma+1)}{I^2}-\beta^2+0.42\right]. \tag{13}$$

The peak of the distribution occurs at

$$\lambda_p{}^{\pm}=-0.05-\nu\alpha^{\pm}, \tag{14}$$

where

$$\frac{1}{\nu}=\left(-\frac{1}{\varphi}\frac{d^2\varphi}{d\lambda^2}\right)_{\lambda=-0.05},$$

and the corresponding most probable energy loss is

$$\Delta_p{}^{\pm}=\zeta T\left[\ln\frac{\zeta T^2 2(\gamma+1)}{I^2}-\beta^2+0.37-\nu\alpha^{\pm}\right]. \tag{15}$$

By numerical integration, ν is found to be 2.8. Since the logarithm is of the order of 10 to 20, the negative correction due to the last term is indeed very small. If $\zeta=0.007$ and $c=1$, it serves only to reduce 0.37 to 0.35, a correction of Δ_p of about 0.2 percent.

The exponential factor in Eq. (12) which corrects the shape of the Landau curve is larger than 1 for small λ and is shown for several values of α in Fig. 5. Its effect is always to make the tail smaller in comparison with the peak of the distribution, the change being greatest for high-energy positrons and low-energy electrons. As an example, the values of $\varphi(\lambda)$ at $\lambda=8$ and $\lambda=0$ are in the ratio one to ten. If $\zeta=0.005$ and $\gamma=7$ (about 3 Mev), the ratio is decreased by 8 percent for positrons, but by only 1 percent for electrons.

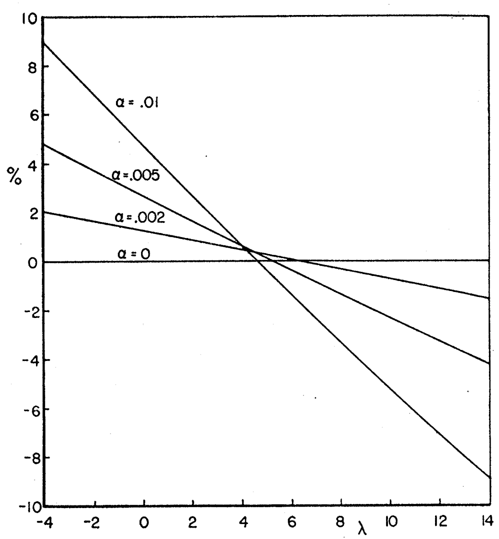

FIG. 5. The percentage correction to the Landau curve for various values of $\alpha=c\zeta$, plotted as a function of the Landau parameter λ [Eq. (13)].

The shape correction also produces a small change in the full width at half-height, $\Gamma=3.98\zeta T$. From the graph in Landau's paper, the slope of $\varphi(\lambda)$ is found to be approximately 0.12 and -0.030 at the half-height points. From these slopes, it can easily be shown that the half-height point near $\lambda=2.5$ shifts by $\Delta\lambda=-7.6\alpha$, while the point near $\lambda=-1.5$ shifts by $\Delta\lambda=-1.0\alpha$. The resultant change in $\Gamma/\zeta T$ is -6.6α, or $-(165\alpha)$ percent. Thus, the corrected full width at half-height is

$$\Gamma^{\pm}=\zeta T(3.98-6.6\alpha^{\pm}). \tag{16}$$

For the numerical example of the last paragraph, the width at half-height is decreased by 1.6 percent for positrons and 0.2 percent for electrons.

IV. MULTIPLE SCATTERING

The exact cross section (neglecting radiative corrections) for the elastic scattering of electrons and positrons by the Coulomb field of a charge Ze was given by Mott[11] in the form of a series in Legendre polynomials. An expansion of this cross section in powers of $\alpha Z(\alpha=1/137)$ is equivalent to the solution in iterated Born approximation. This expansion is useful for light elements where the second Born approximation

[11] N. F. Mott, Proc. Roy. Soc. (London) **A124**, 426 (1929); **A135**, 429 (1932).

is sufficient. One finds[12] for positrons and electrons:

$$\sigma_{\pm}(\theta,\gamma)=\left(\frac{d\sigma}{d\Omega}\right)^{\pm}=\tfrac{1}{4}(r_0Z/\beta^2\gamma)^2(1/\sin^4\tfrac{1}{2}\theta)$$
$$\times[1-\beta^2\sin^2\tfrac{1}{2}\theta\mp\alpha Z\pi\beta\sin\tfrac{1}{2}\theta(1-\sin\tfrac{1}{2}\theta)]. \quad (17)$$

The ratio $(\sigma_+-\sigma_-)/\sigma_-$ vanishes at $\theta=0°$ and 180° and is negative for all other angles. It has a minimum of

$$-2\left[(2/\pi\alpha Z)\cdot\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{1}{2}}+1\right]^{-1}$$

at $\sin(\theta/2)=\gamma/(\gamma+1)$, i.e., at an angle which varies from 60° to 180° as the energy increases from 0 to ∞.

For heavy elements the power series in αZ converges too slowly, and one must resort to a numerical summation of the Mott solution. This was done for Hg at various energies by Bartlett and Watson[13] for electrons, and by Massey[14] for positrons. Except at small angles the positron-electron differences are large and amount to as much as a factor of approximately three at suitable energies and angles. The electron cross section always exceeds the positron cross section.

In order to estimate the effect of the positron-electron difference in multiple scattering, we shall calculate the average penetration depth at which the original direction of the particle beam has been essentially lost. For this problem we would need the solution of the Boltzmann equation for a semi-infinite medium. Since this solution is rather complicated, and since back-scattering from the target will be unimportant, we shall work under the assumption of an infinite medium. The exact solution for this case was given by Lewis.[15] Let s be the path length,

$$s=\int_0^s ds'=\int_E^{E_0}\left|\frac{dE}{ds'}\right|^{-1}dE, \quad (18)$$

TABLE I. Numerical values of the constants that occur in Eqs. (27) and (28) for the average cosine of the multiple-scattering angle.

		a	b
Al	e^+	0.297	0.014
	e^-	0.305	0.034
Pb	e^+	0.311	0.057
	e^-	0.430	0.052

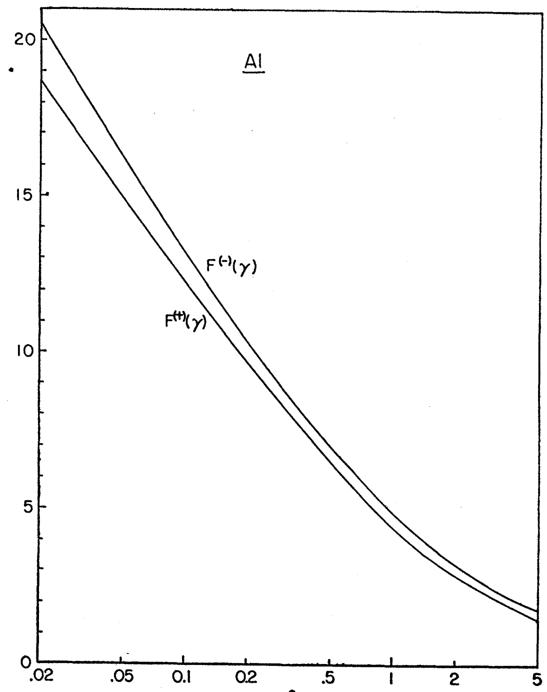

FIG. 6. The function $F(\gamma)=aZ\ln[(\gamma+1)/(\gamma-1)]+bZ/\beta$ for positrons and electrons in Al. The average cosine of the multiple scattering angle is $1/e$ for a kinetic energy $T=(\gamma-1)mc^2$, such that $F(\gamma)=F(\gamma_0)+1$.

and let z be the penetration depth into the medium. The longitudinal distribution is determined by $\langle z^nP_l(\cos\theta)\rangle_{\text{Av}}$ for all n and l. In particular,

$$\langle\cos\theta\rangle_{\text{Av}}=k_1, \quad (19)$$

$$\langle z\rangle_{\text{Av}}=\int_0^s k_1ds', \quad (20)$$

where

$$k_1(s)=\exp\left(-\int_0^s\kappa_1(s')ds'\right), \quad (21)$$

$$\kappa_1(s)=2\pi N\int_0^\pi\sigma(\theta,s)(1-\cos\theta)\sin\theta d\theta. \quad (22)$$

By Eq. (18) s is a monotonic function of E, so that the labels s and E are equivalent. $\sigma(\theta,E)$ is the elastic cross section per unit solid angle for particles of total energy E.

For the evaluation of κ_1 the elastic cross section is cut off at angles smaller than θ_0, which correspond to impact parameters exceeding $a_0/Z^{\frac{1}{3}}$ (approximately the Thomas-Fermi radius of the atom), i.e.,

$$\sin(\theta_0/2)=1/\xi_0, \quad \xi_0=2\beta\gamma/\alpha Z^{\frac{1}{3}}. \quad (23)$$

For light elements we find from (17), (22), and (23) that

$$\kappa_1^{\pm}(s)=\kappa_1^{\pm}(\gamma)=4\pi N(Zr_0/\beta^2\gamma)^2$$
$$\times[\ln\xi_0-\beta^2/2\mp\pi\alpha Z\beta(\tfrac{1}{2}-\xi_0^{-1})]. \quad (24)$$

---

[12] W. A. McKinley and H. Feshbach, Phys. Rev. **74**, 1759 (1948); R. H. Dalitz, Proc. Roy. Soc. (London) **A206**, 509 (1951).
[13] J. H. Bartlett and R. E. Watson, Proc. Am. Acad. Arts Sci. **74**, 53 (1940).
[14] H. S. W. Massey, Proc. Roy. Soc. (London) **A181**, 14 (1942). See also N. F. Mott and H. S. W. Massey, *Theory of Atomic Collisions* (Clarendon Press, Oxford, 1949), where the results of references 13 and 14 are given, but where our Eq. (17) is given incorrectly.
[15] H. W. Lewis, Phys. Rev. **78**, 526 (1950).

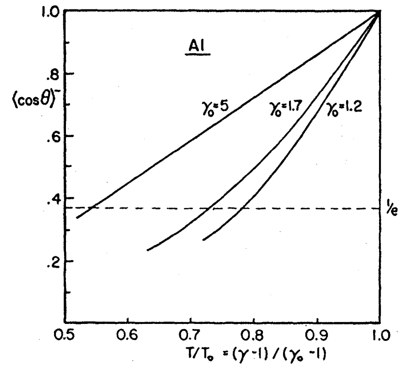

FIG. 7. The cosine of the multiple-scattering angle averaged over all (negative) electrons whose initial kinetic energy T_0 has dropped to T because of collision loss in Al. Curves for three different incident energies $T_0=(\gamma_0-1)mc^2$ are shown.

For heavy elements, numerical work is required to evaluate the integral in

$$\kappa_1^{\pm}(\gamma)=4\pi N(Zr_0/\beta^2\gamma)^2\left\{\ln\xi_0+\tfrac{1}{2}\int_{\theta_0}^{\pi}\times[r^{\pm}(\theta,\gamma)-1]\cot(\theta/2)d\theta\right\}, \quad (25)$$

where $r(\theta,\gamma)$ is the ratio of the scattering cross section to the relativistic Rutherford cross section for spinless particles. The ratios calculated for Hg by Bartlett and Watson[13] and by Massey[14] are valid for Pb with sufficient accuracy. Their results can be used directly in a numerical integration, or, alternatively, their curves can be fitted with polynomials in $\sin(\theta/2)$ and the integration carried out analytically. Polynomials of first and fourth degrees give good representations of r^+ and r^-, respectively, the coefficients being, of course, different for different energies.

When (24) or (25) is inserted in

$$-\ln\langle\cos\theta\rangle_{Av}^{\pm}=\int_0^s \kappa_1^{\pm}(s)ds=\int_{\gamma}^{\gamma_0}\frac{\kappa_1^{\pm}(\gamma)}{|d\gamma/ds|^{\pm}}d\gamma, \quad (26)$$

the integrand is found to be $Z/(\gamma^2-1)$ times a slowly varying function of γ which can be fitted quite satisfactorily by a function of the form $2a+b(\gamma^2-1)^{-\frac{1}{2}}$. The constants a and b differ for positrons and electrons but are of the same order of magnitude for small and large Z, as is shown in Table I. With this approximation, integration of (26) gives[16]

$$\langle\cos\theta\rangle_{Av}=k_1(\gamma_0,\gamma)=\frac{G(\gamma_0)}{G(\gamma)}, \quad (27)$$

$$G(\gamma)=\left(\frac{\gamma+1}{\gamma-1}\right)^{aZ}\cdot e^{bZ/\beta}=e^{F(\gamma)}. \quad (28)$$

[16] A similar approximation, with a single constant corresponding to a, was obtained from the spinless Rutherford cross section by C. H. Blanchard and U. Fano, Phys. Rev. **82**, 767 (1951). They do not distinguish between electrons and positrons.

$G(\gamma)$ is seen to increase very strongly with decreasing γ. The advantage of the particular form (27) is that it reduces the function k_1 of two variables to the function $G(\gamma)$ of a single variable. One can easily find $k_1(\gamma_0,\gamma)$ from a plot of $F(\gamma)=\ln G(\gamma)$, such as that shown in Fig. 6 for Al. $\langle\cos\theta\rangle_{Av}^{-}$ as a function of γ for several incident energies γ_0 is shown in Fig. 7 for Al. For heavy elements these curves would be much steeper.

The average total energy $E_d=\gamma_d mc^2$ at which the particles have lost their initial orientation is conveniently defined by $\langle\cos\theta\rangle_{Av}=1/e$. An approximate value of γ_d can easily be obtained from Fig. 6 for any given γ_0 by using $F(\gamma_d)=F(\gamma_0)+1$.

The average penetration depth corresponding to γ_d is

$$z_d=\int_{\gamma_d}^{\gamma_0} k_1(\gamma_0,\gamma)\left|\frac{d\gamma}{ds}\right|^{-1}d\gamma. \quad (29)$$

Typical values for $T_d^-=(\gamma_d^--1)mc^2$ and z_d^- (i.e., for negative electrons) can be found from Table II. z_d^- is given in units of the experimental range. A similar calculation for positrons permits the computation of the positron-electron difference in z_d and in γ_d. These results are also shown in Table II.

The intensity distribution of the electrons or positrons for penetration thicknesses larger than z_d has been studied by several authors[17] on the basis of the age equation of diffusion theory. This equation is obtained as an approximation of the Boltzmann transport equation and is found to be

$$\frac{\partial F}{\partial s}+\frac{1}{3\kappa_1}\nabla^2 F=0. \quad (30)$$

One finds for the intensity distribution due to a uniform plane source at $z=z_d$

$$I(z)=I(z_d)\left[1-\mathrm{erf}\left(\frac{z-z_d}{1.225r_{Av}}\right)\right], \quad (31)$$

TABLE II. Energy loss and penetration depth at which the average cosine of the multiple-scattering angle has dropped to $1/e$. $T_0=(\gamma_0-1)mc^2$ is the initial kinetic energy, $W^{\pm}=(\gamma_0-\gamma_d^{\pm})mc^2$ is the loss of kinetic energy, and $z_d^{\pm}$ is the corresponding average penetration depth. The experimental range R has been assumed to be the same for Pb as for Al when expressed in g/cm².

	γ_0	T_0	W^-/T_0	z_d^-/R	$\frac{W^+-W^-}{W^-}$	$\frac{z_d^+-z_d^-}{z_d^-}$
		Mev	%	%	%	%
Al	1.2	0.102	22	34	7.0	1.0
	1.7	0.358	27	36	4.4	3.7
	3	1.02	36	39	2.9	4.9
	5	2.04	46	47	2.2	4.5
Pb	1.2	0.102	2.8	6.9	32	24
	1.7	0.358	3.6	7.1	34	35
	3	1.02	5.3	8.2	35	39
	5	2.04	7.9	10.1	34	39

[17] See, for example, Bethe, Rose, and Smith, Proc. Am. Phil. Soc. **78**, 573 (1938).

valid of course only for $z>z_d$. The energy loss is taken into account in r_{Av}, which is given by

$$r_{Av}^2=1.05\int_0^{\gamma_d}\frac{1}{\kappa_1}\left|\frac{d\gamma}{ds}\right|^{-1}d\gamma. \tag{32}$$

In this way one can calculate a theoretical estimate of the range,

$$R_{\rm th}=z_d+r_{Av}. \tag{33}$$

This theoretical range should be compared with the experimental range obtained by extrapolating the straight section of the plot of intensity against target thickness.[18] r_{Av} is the intersection of the tangent at $z=z_d$ of the distribution (31) with the z axis.

Equation (33) accounts for about 90 percent of the experimental range in Al, but only about half the experimental range in Pb. This disagreement is not very surprising. The transport mean free path $1/\kappa_1$ which enters Eq. (30) is easily found to be between 0.7 and 1.3 times the remaining range $(R_{\rm exp}-z_d)$ in Al, but only 0.11 to 0.15 times the remaining range in Pb. Compared to z_d the transport mean free path is up to three times larger in Al, but of the same order in Pb. It follows that Eq. (30) is not a good approximation to the actual situation and that a less naive approach, which takes better account of the strongly peaked cross section, must be adopted.

V. SUMMARY AND CONCLUSIONS

In passage through matter, the statistical behavior of positrons and electrons reflects, although not very strongly, the marked differences that distinguish them in single-scattering events. These are differences in (a), the cross section for inelastic scattering by the atomic electrons, (b), the maximum possible energy transfer in inelastic scattering, and (c), the elastic cross section. Single scatterings through large angles, for which these differences may amount to factors of 2 or 3, are relatively so rare that processes involving many collisions show much smaller differences.

The positron-electron difference in energy straggling in thin foils, which is affected only by (a), appears as a small difference in shape of the straggling distributions, with hardly any shift of the most probable energy loss. The Landau curve becomes slightly higher and narrower for positrons at energies above a few hundred kev, and for negative electrons at lower energies (Eqs. (15) and (16)).

The difference in stopping power, i.e., average rate of energy loss, receives contributions of the same order of magnitude from (b) as from (a). At energies below a few hundred kev, positrons lose energy a few percent more rapidly than electrons, and the reverse occurs at higher energies (see Fig. 3).

Although useful in calculations, the stopping power is not easy to measure directly; and in observable effects where it plays an important part, such as the range, the positron-electron difference in multiple scattering, resulting from (c), is larger than the difference in stopping power. This is particularly true in heavy elements, since the relative difference in multiple scattering is strongly Z-dependent, unlike the relative differences in stopping power and energy straggling. For example, in heavy elements positrons penetrate about 30 percent farther than electrons before losing their memory of initial direction, whereas the difference in light elements is only a few percent.

In addition to scattering, another process that plays an appreciable part in the passage of positrons through matter is annihilation in flight. The total probability of annihilation before coming to rest rises with initial energy to 10 or 15 percent for energies of a few Mev,[19] and is only weakly Z-dependent. The transmitted intensity of a positron beam as a function of target thickness will certainly be reduced by annihilation, but the extrapolated range may be fairly insensitive to a small progressive reduction in intensity. As yet there is no experimental evidence for a positron-electron difference in range,[20] but the available data for positrons seem to be restricted to ranges of β spectra in aluminum. However, the reduction in range due to annihilation, even for monoenergetic positrons, might be masked by the weaker multiple scattering of positrons as compared with electrons.

Estimates of z_d^+ are not affected by annihilation in flight, since z_d^+ is calculated as an average over positrons that have traversed equal path lengths and have, therefore, had equal opportunities to annihilate, independent of the directions of their paths. The disappearance of some of them will affect the intensity but not the shape of the distribution.

The investigations of this paper are naturally restricted to energies at which nearly all the energy loss is due to collisions. At higher energies radiation loss rapidly becomes the predominant effect.

[18] J. S. Marshall and A. G. Ward, Can. J. Research **15**, 39 (1937).

[19] H. A. Bethe, Proc. Roy. Soc. (London) **A150**, 129 (1935).

[20] L. Katz and A. S. Penfold, Revs. Modern Phys. **24**, 28 (1952).