where

$$\omega_r^2 = [\sum_l K_{00l} \cos(l\pi r/N)]/[\sum_l M_l(\cos l\pi r/N)]. \quad (20)$$

If $e^{-\sigma D}$ is so small that only nearest neighbors need to be retained, (20) may be written as

$$\omega_r \cong \omega_0 \left[1+\frac{1}{2}\left(\frac{K_{001}}{K_{000}}-\frac{M_1}{M_0}\right)\cos(\pi r/N)\right], \quad (21)$$

$\omega_0 = (K_{000}/M_0)^{\frac{1}{2}}$ being the harmonic oscillator frequency, and each level is seen to be split into N levels by the coupling.

On account of the interaction in all conceivable directions, extension to three dimensions of (19) and (20) is more complicated than previously, but presents no serious difficulties,[4] and there is no need for writing down the corresponding expressions here. In any case, for all σD, the coefficients K and M in each series of (20) or its equivalent fall off exponentially; each series converges for all r.

From the conservation laws,[5] the quantity

$$\mathcal{G}_k = (\partial\mathcal{L}/\partial\dot{\psi})(\partial\psi/\partial x_k)$$

should be interpreted as the k component of the momentum density of the field, and the momentum then becomes, with (1) and (13),

$$G = \sum_{ij} q_i q_j \int \theta(r-iD)\nabla\theta(r-jD)dV.$$

Since it is generally possible simultaneously to diagonalize two quadratic forms (such as the two appearing in the Hamiltonian) but not three, it follows that G will not commute with H.

I wish to thank Professor Nathan Rosen for patiently guiding the preparation of the thesis[4] from which this paper is taken, and for making countless suggestions. I am also grateful to Professor Wayne A. Bowers for discussions.

[4] H. B. Rosenstock, thesis, University of North Carolina, Chapel Hill, North Carolina, 1951 (unpublished).

[5] G. Wentzel, *Quantentheorie der Wellenfelder* (J. W. Edwards, Ann Arbor, 1946), Sec. 2.

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Nucleon Isobars in Intermediate Coupling

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The energies of excited states of nucleons are calculated for the symmetric pseudoscalar meson theory with a fixed, extended source. A version of the Tomonaga method, which is shown to be approximately correct in the limit of strong coupling, is used. The calculation is carried out for moderate coupling by using trial functions with 0, 1, 2, and 3 mesons. Numerical results are presented for a source of the Yukawa shape. For coupling stronger than a critical strength which varies with state and source size, isobars are found for angular momenta $\frac{1}{2}$, $\frac{3}{2}$ and for isotopic spins $\frac{1}{2}$, $\frac{3}{2}$. The $(\frac{3}{2},\frac{3}{2})$ state always lies lowest in energy. A very large source ($\sim\frac{1}{2}$ the meson Compton wavelength) yields a $(\frac{3}{2},\frac{3}{2})$ isobar at $\sim$350–400 Mev excitation, and a degenerate $(\frac{3}{2},\frac{1}{2})$ and $(\frac{1}{2},\frac{3}{2})$ pair at $\geqslant$500 Mev. Other isobars lie considerably higher. Smaller sources correspond to higher excitation energies in this range of coupling strength.

I. INTRODUCTION

THE existence of nucleon isobars was first predicted by strong coupling meson theory. Since that time, isobars have been used in phenomenological theories in attempts to explain meson-nucleon[1] and nucleon-nucleon[2,3] scattering, and photomeson production.[4] Until now, the isobars of symmetric pseudoscalar theory have been examined only in the strong coupling limit.[5] The only relevant low-lying isobar predicted in that limit is the one for which $I=\frac{3}{2}$, $J=\frac{3}{2}$. As a result, this excited state is the one which has been considered in scattering problems, although there was no guarantee that at some smaller coupling another state could not lie lower in energy. We have, therefore, calculated the positions of the low-lying isobars using the intermediate-coupling theory of Tomonaga.[6,7]

There is no attempt in this paper to calculate the widths of the isobaric levels. The results should, nonetheless, be useful in predicting the order and approximate spacing of the excited nucleon levels. The motivation is comparable to that of theories of nuclear structure which attempt to predict the same features of the levels of complex nuclei.

One of the results of these calculations is that nucleon isobars do exist for surprisingly low coupling. The

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[1] K. A. Brueckner, Phys. Rev. **86**, 106 (1952).
[2] J. Iwadare, Prog. Theoret. Phys. (Japan) **9**, 94 (1953).
[3] R. B. Raphael and J. Schwinger, Phys. Rev. **90**, 373 (1953).
[4] B. T. Feld, Phys. Rev. **89**, 330 (1953).
[5] W. Pauli and S. M. Dancoff, Phys. Rev. **62**, 85 (1942).
[6] S. Tomonaga, Prog. Theoret. Phys. (Japan) **2**, 6 (1947).
[7] R. Christian and T. D. Lee (to be published).

excitation energies for various source sizes are summarized in Figs. 2 through 5, and a discussion of these results is given in Secs. VI and VII. Some speculations on the relevance of these isobars to meson-nucleon scattering are presented in Sec. VIII.

II. THE TOMONAGA METHOD

We use pseudoscalar symmetric meson theory with a fixed, extended source. Inclusion of the repulsive core term which results from γ_5 coupling has been shown to be equivalent, in an energy calculation, to a change in the source shape and the coupling constant.[8] These changes are considered to have been included here. In particular, we choose a Yukawa source for the *modified* shape

$$\rho_{\mathrm{eff}}(r)=(M^2/4\pi r)\exp(-Mr) \qquad (1)$$

[with Fourier transform: $v(k)=M^2/(M^2+k^2)$], where M is the parameter which determines the size of the source.

In this modified problem, only $l=1$ waves interact with the source. Including only these terms, the Hamiltonian is (with $\hbar=c=$ meson mass $=1$)

$$H=\sum_{j\alpha k}\omega\bar{a}_j{}^\alpha(k)a_j{}^\alpha(k)+g(12\pi R)^{-\frac{1}{2}} \times\sum_{j\alpha k}k^2v(k)\omega^{-\frac{1}{2}}\sigma_j\tau^\alpha[a_j{}^\alpha(k)+\bar{a}_j{}^\alpha(k)], \qquad (2)$$

where the superscripts $\alpha=x,y,z$ refer to Cartesian components in charge space, and the subscripts $j=x,y,z$ refer to Cartesian components in ordinary space. $g^2/4\pi$ is the analog of the fine structure constant, R is the radius of the sphere in which the radial functions are quantized, and $\omega=(1+k^2)^{\frac{1}{2}}$. The $l=1$ part of the meson field operators is given by

$$\varphi^\alpha=\sum_{jk}\left(\frac{3}{4\pi\omega R}\right)^{\frac{1}{2}}\frac{x_jF_k(r)}{r^2}[a_j{}^\alpha(k)+\bar{a}_j{}^\alpha(k)],$$

where $F_k(r)=krj_1(kr)$ of Schiff,[9] if the repulsive core is absent.

The Hamiltonian commutes with the operators I^2, I_z, J^2, and J_z where

$$I_z=\tfrac{1}{2}\tau^z-i\sum_{jk}[\bar{a}_j{}^x(k)a_j{}^y(k)-\bar{a}_j{}^y(k)a_j{}^x(k)] \equiv\tfrac{1}{2}\tau^z+T_z, \qquad (3)$$

$$J_z=\tfrac{1}{2}\sigma_z-i\sum_{\alpha k}[\bar{a}_x{}^\alpha(k)a_y{}^\alpha(k)-\bar{a}_y{}^\alpha(k)a_x{}^\alpha(k)] \equiv\tfrac{1}{2}\sigma_z+L_z. \qquad (4)$$

A transformation which interchanges σ and τ and which simultaneously interchanges upper and lower indices leaves (2) unchanged and interchanges I and J. Thus, all energies and phase shifts will be unchanged by an interchange of I and J.

We now solve the problem $H\Psi=E\Psi$ by means of a variational method in which the trial function is subjected to the following restrictions: (a) It shall be an exact eigenfunction of each of the operators I^2, I_z, J^2, and J_z. (b) All mesons shall be in one mode. This point is further discussed in Appendix A.

The minimization of the energy with a trial function of this type is equivalent to solving exactly the new problem $H_T\Psi=E\Psi$ with

$$H_T=\bar{\omega}\sum_{j\alpha}[\bar{A}_j{}^\alpha A_j{}^\alpha+\gamma\tau^\alpha\sigma_j(A_j{}^\alpha+\bar{A}_j{}^\alpha)]\equiv\bar{\omega}h_T. \qquad (5)$$

Here

$$A_j{}^\alpha\equiv\sum_k f(k)a_j{}^\alpha(k);\quad \sum_k|f(k)|^2=1;$$

and

$$\bar{\omega}\equiv\sum_k\omega|f(k)|^2;$$
$$\bar{\omega}\gamma\equiv g(12\pi R)^{-\frac{1}{2}}\sum_k k^2v(k)\omega^{-\frac{1}{2}}f(k).$$

As Christian and Lee[7] have pointed out, the form of $f(k)$ can be determined from the energy minimization

$$f(k)=\frac{N(\lambda)}{R^{\frac{1}{2}}}\frac{k^2v(k)}{\omega^{\frac{1}{2}}(\omega+\lambda)},$$

where λ is a real parameter to be determined by further energy minimization and $N(\lambda)$ is a normalizing constant.

If we set

$$E=\bar{\omega}(\lambda)U(\gamma^2), \qquad (6)$$

then the problem separates into two parts. The first is independent of source, involving only the solution of the eigenvalue equation

$$h_T\Psi=U(\gamma^2)\Psi \qquad (7)$$

for various values of the parameter γ. The second part, which does depend on source size and shape, is the minimization of the energy (6) with respect to λ for a fixed value of g. It is this feature of the separability of the problem which facilitates the calculation of results for a large number of source sizes.

III. MINIMIZATION OF THE ENERGY

The eigenvalue equation (7) is discussed in Secs. IV and V; here we develop the results of the energy minimization condition. It is convenient to define

$$L_n(\lambda)\equiv\frac{1}{\pi}\int_0^\infty\frac{k^4v^2(k)dk}{\omega(\omega+\lambda)^n}, \qquad (8)$$

from which follows the property $L_n'(\lambda)=-nL_{n+1}(\lambda)$. The expression for $L_1(\lambda)$ is given in Appendix B. We then have

$$\frac{1}{N^2}=L_2,\quad \bar{\omega}=\frac{L_1}{L_2}-\lambda,\quad \gamma^2=\left(\frac{g^2}{4\pi}\right)\frac{(L_1)^2}{3L_2\bar{\omega}^2}. \qquad (9)$$

From the foregoing it follows that

$$\frac{d\bar{\omega}}{d\lambda}=\frac{2L_1L_3}{(L_2)^2}-2,\quad \frac{d\gamma^2}{d\lambda}=\frac{\gamma^2}{\bar{\omega}}\left(\frac{\bar{\omega}L_2}{L_1}-2\right)\frac{d\bar{\omega}}{d\lambda}, \qquad (10)$$

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[8] C. H. Chang and B. A. Jacobsohn (to be published).

[9] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 77.

so that from (6) the condition of energy minimization becomes

$$\frac{\bar{\omega}L_2}{L_1}=2-\frac{U(\gamma^2)}{\gamma^2}\frac{d\gamma^2}{dU}. \tag{11}$$

The left side of (11) is plotted as a function of λ in Fig. 1 for several source sizes.

Once (7) has been solved for a particular state, then (11) gives the best value of λ for a given value of γ^2. The corresponding values of E and g^2 are found from (6) and (9).

In the calculation, only those solutions with $\lambda\geqslant-1$ are considered. This follows from the requirement that the meson field vanish at large distances, so that $f(k)$ must not have a pole on the real k axis. Except in the case of the ground state, this results in a minimum value of γ^2 (at $\lambda=-1$) below which the isobar ceases to exist. This critical value of γ^2 decreases with increasing source size.

IV. THE STRONG COUPLING LIMIT

Before proceeding to the intermediate coupling calculations, we wish to compare the Tomonaga predictions in the limit $\gamma^2\gg1$ with those of strong coupling theory.

Part of the Hamiltonian treated by Pauli and Dancoff[5] is similar to that in (5). From their paper we find for the low-lying levels:

$$U=-3\gamma^2+C_1+\frac{C_2}{\gamma^2}+\frac{J(J+1)-\frac{3}{4}}{8\gamma^2},\quad I=J. \tag{12}$$

C_1 and C_2 are constants independent of I and J. We ignore them since they do not contribute to the excitation energy. They would have to be retained, however, in order to calculate the energy to the point of disappearance of the isobar ($\lambda=-1$) for a very small source.

Inserting (12) into (11) we find

$$\lambda\simeq-\left(\frac{L_1}{L_2}\right)_0\frac{J(J+1)-\frac{3}{4}}{12\gamma^4}.$$

The leading term in (6) for $\gamma^2\rightarrow\infty$ gives

$$E=-L_1(0)g^2/4\pi$$

and the Yukawa shape for the meson field, in exact agreement with the Pauli-Dancoff result.[5] Disagreement occurs in the excitation energies, where our result must be divided by a source-size dependent factor $K=[L_1L_3/(L_2)^2]_0$ in order to agree with strong-coupling theory. Some typical values of K are $K=1.23$ for $M=0$, $K=1.59$ for $M=2$, $K=1.94$ for $M=7$, and $K=2.47$ for $M=\infty$ (point source). We see, then, that this version of the Tomonaga method, which agrees exactly with the leading term of weak-coupling theory, agrees qualitatively in the limit of strong coupling.

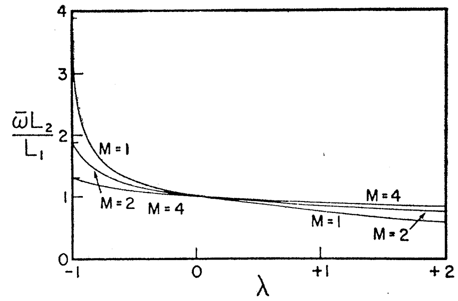

FIG. 1. $\bar{\omega}L_2/L_1$ plotted as a function of λ for various source sizes [see Eq. (11)].

V. MODERATE COUPLING

We now turn to the problem of solving the eigenvalue problem (7) with $\gamma^2\lesssim1$. One should notice particularly that it is not g^2 but γ^2 which determines the coupling strength. They are related by the source-dependent factor of (9). In this region, one may expect to get a good approximation to $U(\gamma^2)$ by using a trial function which contains only a small number of mesons less than or equal to some given number N. The validity of this limitation is discussed in Sec. VI. Since this region probably covers the case of physical interest, it is the only one we have investigated.

The trial function is now of the form

$$\Psi=\sum_{n=0}^{N}\sum_{L=J\pm\frac{1}{2}}\sum_{T=I\pm\frac{1}{2}}\sum_{j}{}^{(n)}{}_ja_L{}^T\,{}^{(n)}{}_j\Psi_{JL}{}^{IT}, \tag{13}$$

where the ${}^{(n)}{}_j\Psi_{JL}{}^{IT}$ are orthonormal eigenfunctions with the indicated quantum numbers, containing just n mesons. The subscript j enumerates the several possible states that may exist with the same quantum numbers and the same n. Table I lists the number of meson states which exist for various n when all mesons are put into one spatial mode. The table was constructed by making use of the properties of the representations of the symmetric group. Only those meson states are included which can be combined with nucleon spins to give I, $J=\frac{1}{2}$ and $\frac{3}{2}$. Since the number of states doubles in going from 3 to 4 mesons, we have cut off the calculation at $N=3$. The wave functions ${}^{(n)}{}_j\Psi_{JL}{}^{IT}$ are formed from the spin and isotopic spin functions of the nucleon combined with the meson functions which are given in Appendix C.

TABLE I. The number of states with given T and L for various values of n (number of mesons).

(T,L)	$n=0$	1	2	3	4	5	6
(0,0)	1		1	1	2	1	3
(0,1) or (1,0)							
(1,1)		1	1	2	2	4	4
(0,2) or (2,0)			1		2	1	3
(1,2) or (2,1)				1	1	2	2
(2,2)			1	1	3	3	7

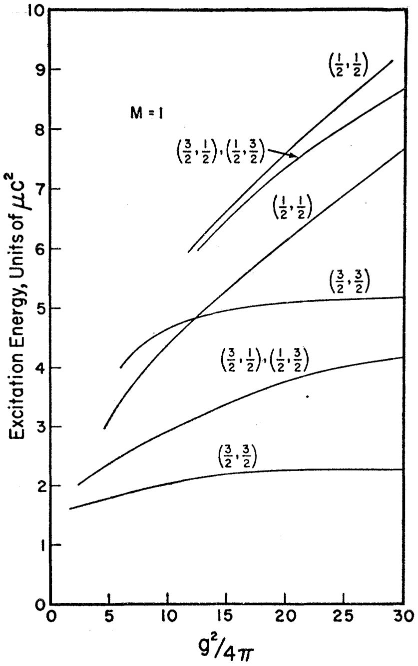

FIG. 2. Excitation energy as a function of coupling for $M=1$.

The calculations from this point on involve many details which are omitted here. A summary of the most important formulas involved in the solution of (7) is given in Appendix D. The solutions for $U(\gamma^2)$ are combined with graphs of the relevant source-dependent quantities appearing in (6), (9), and (11) to give the energy of each state as a function of g^2. Figures 2 through 5 show the results of the calculations of excitation energies for various source sizes. In Fig. 6 we have plotted the absolute energies of the ground state for comparison with the magnitudes of the excitation energies.

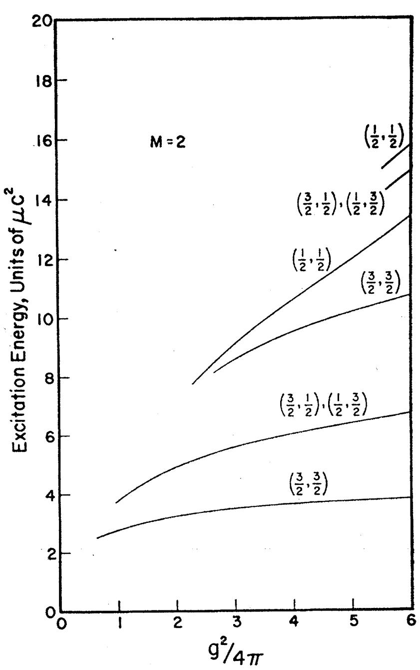

FIG. 3. Excitation energy as a function of coupling for $M=2$.

VI. EFFECT OF NEGLECTING THE FOUR-MESON AMPLITUDES

The errors introduced by cutting off the trial functions at three mesons are here shown to be small in the coupling region of interest. This may be seen from two aspects.

Consider, first, the probabilities P_n for finding n mesons. These are plotted as functions of γ^2 for the ground state in Fig. 7, for the lowest $(\frac{3}{2},\frac{3}{2})$ isobar in Fig. 8, and for the lowest $(\frac{3}{2},\frac{1}{2})$ isobar in Fig. 9. The probabilities do not approach the proper strong coupling limit of zero, since they were calculated from the amplitudes of Appendix D. Nevertheless, it is reasonable to assume that the curves are qualitatively correct in the region for which $P_3(\gamma^2)$ is less than one-half of its asymptotic value. From this, we are led to infer that P_4, which we have neglected, is actually quite small for $\gamma^2 \leqslant \frac{3}{4}$.

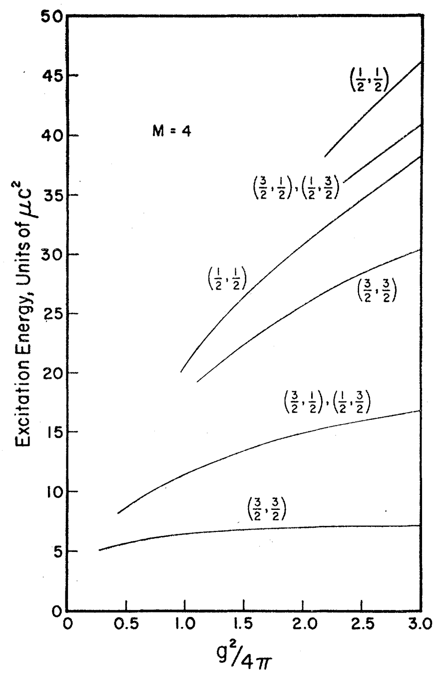

FIG. 4. Excitation energy as a function of coupling for $M=4$.

The association of these probability curves with Figs. 3, 5, and 6 may be accomplished through Table II which gives the relation between γ^2 and $g^2/4\pi$ for several states and source sizes. The first entry in the table for each state is the critical coupling below which the isobar disappears. Table II shows quite clearly the smallness of γ^2 for a large source ($M=2$) and the beginning of the breakdown of the approximation for a small source ($M=7$).

The second justification for the neglect of P_4 is presented in Figs. 10 and 11, which refer only to the case $M=2$. Figure 10 displays the ground-state energy with the trial functions cut off successively at $N=1$, $N=2$, and $N=3$ mesons; Fig. 11 shows the decrease in excitation energy as the calculation is improved one step.[10] The latter figure especially shows the satisfactory empirical convergence of the excitation energy for a large source.

In summary, the above evidence indicates that the excitation energies of the two lowest isobars for large

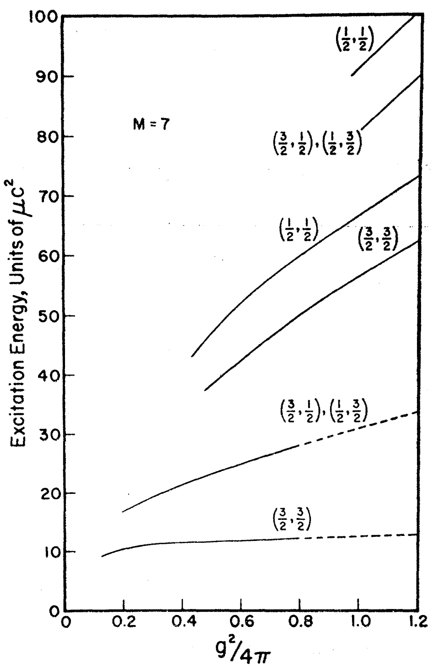

FIG. 5. Excitation energy as a function of coupling for $M=7$.

sources ($M \lesssim 2$) would not be changed significantly by the inclusion of more mesons in the trial function (13). The accuracy of our results decreases for higher isobars with a given source size, or for smaller sources with a given isobar. However, even in a rather extreme case, our results should have at least a rough qualitative significance; for example, for $M=7$ (see Fig. 5) the first $(\frac{1}{2},\frac{1}{2})$ isobar begins at $\gamma^2=1.4$, and the second $(\frac{3}{2},\frac{3}{2})$ begins at $\gamma^2=1.54$.

None of the foregoing remarks should be interpreted as an estimate of the accuracy of the Tomonaga approximation, the justification of which remains solely its agreement with the limits of weak and strong coupling theory.

[10] In each case, the excitation energy was calculated by subtracting the ground-state energy with $N-1$ from the isobar energy with N.

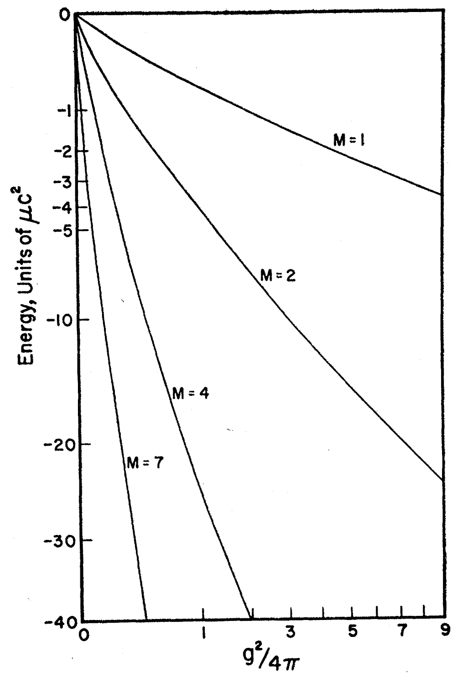

FIG. 6. Energy of the ground state as a function of coupling for various source sizes.

VII. DISCUSSION OF ISOBAR ENERGIES

From the results shown in Figs. 2 through 5, we observe the following features:

(a) No stable isobars ($E_{ex}<1$) exist for this range of coupling and source size.

(b) In all cases, the first excited nucleon state is $(\frac{3}{2},\frac{3}{2})$ and the second is the pair $(\frac{3}{2},\frac{1}{2})$, $(\frac{1}{2},\frac{3}{2})$. The crossing of two higher levels in Fig. 2 illustrates why it was not *a priori* obvious that the two lower levels would never cross.

(c) It is believed that the initial rise in excitation energy with increasing coupling is a real effect (see Sec. VI), although the lowest state must eventually drop.

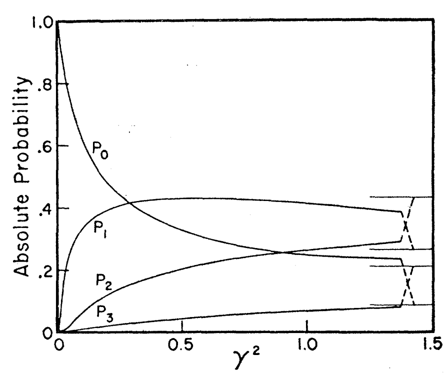

FIG. 7. The probabilities P_n of finding n mesons in the ground state plotted as a function of γ^2.

(d) The special emphasis given to the $(\frac{3}{2},\frac{3}{2})$ isobar by strong coupling theory is lacking in the intermediate region. The only exceptional behavior of this state, as compared with the $(\frac{3}{2},\frac{1}{2})$ isobar, is the flatness of the excitation-energy curve when plotted against g^2. The energy is thus a sensitive function of source size only.

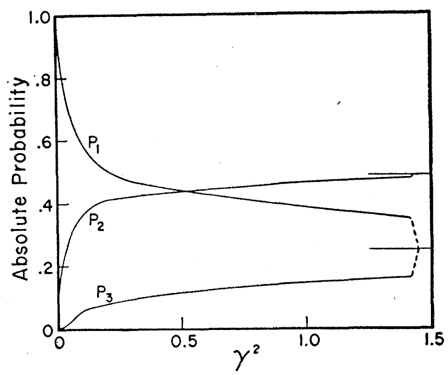

FIG. 8. The probabilities P_n of finding n mesons in the first $(\frac{3}{2},\frac{3}{2})$ isobar plotted as a function of γ^2.

VIII. IMPLICATIONS IN MESON-NUCLEON SCATTERING

Since the concept of isobars is directly related to meson-nucleon scattering, it is of interest to speculate on the consequences of these results when applied qualitatively to a discussion of phase shifts. (A calculation of the effect of the isobaric states on the scattering is under way.)

(a) From the consideration of a simple analogy, the scattering of a particle by a square potential barrier, we extrapolate the following comments on the width of

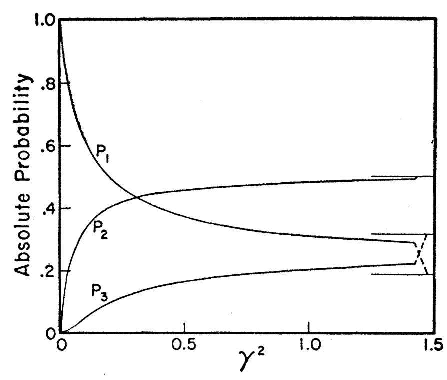

FIG. 9. The probabilities P_n of finding n mesons in the first $(\frac{3}{2},\frac{1}{2})$ isobar plotted as a function of γ^2.

the isobaric levels. They should be quite narrow for couplings much larger than critical; as the coupling decreases, their breadths should increase. It is probable that the disappearance of the isobar in Figs. 2 through 5 marks the attempt of the trial function to represent the disappearance of the true resonance in the scattering as the coupling is decreased. The value of the critical coupling, however, is markedly affected by changes in the trial function (see Fig. 11), moving apparently in the direction of weaker coupling as the trial function is improved.

(b) A resonance in $(\frac{3}{2},\frac{3}{2})$ scattering has been suggested[1] to be at about 300 Mev. Figures 3 and 4 indicate possible agreement with the large source sizes ($M=2$ to 4) indicated by Chew.[11]

(c) For a large source, the $(\frac{3}{2},\frac{1}{2})$ isobar lies at one- to two-hundred Mev above the $(\frac{3}{2},\frac{3}{2})$ level. If the former level is broad, then this is consistent with experiments[12,13] which suggest that the phase shift may change sign at 90 Mev. The low-energy phase shift is predicted by Chew[11] to be negative, but if there is an isobar, $\tan\delta$ will approach $+\infty$ as the meson energy approaches resonance from below. The phase shift should, therefore, change from negative to positive below the resonance energy, although a broad resonance is needed to produce this effect as low as ~100 Mev.

TABLE II. The relation between g^2 and γ^2 for various states and source sizes. $N=3$.

State	$M=2$ $g^2/4\pi$	γ^2	$M=7$ $g^2/4\pi$	γ^2
$(\frac{1}{2},\frac{1}{2})$	0	0	0	0
ground	0.61	0.066	0.039	0.068
	1.53	0.140	0.097	0.138
	4.19	0.297	0.147	0.188
	7.3	0.46	0.267	0.287
$(\frac{3}{2},\frac{3}{2})$	0.644	0.135	0.123	0.396
first isobar	1.18	0.225	0.180	0.470
	1.50	0.270	0.254	0.562
	2.39	0.379	0.337	0.635
	3.11	0.453	0.433	0.750
	3.83	0.519	0.508	0.818
	6.86	0.743	0.843	1.08
$(\frac{3}{2},\frac{1}{2})$	0.975	0.205	0.197	0.617
first isobar	1.55	0.307	0.279	0.728
	2.37	0.427	0.343	0.812
	3.72	0.590	0.453	0.943
	5.99	0.812	0.562	1.06

We would like to thank Dr. John S. Blair for many stimulating and helpful discussions.

APPENDIX A. THE SINGLE MODE RESTRICTION

In his original application of the variational method to charged scalar mesons, Tomonaga[6] introduced two independent non-orthogonal modes, one for positive and one for negative mesons. In this way, he was able to achieve exact agreement in excitation energy with the strong coupling theory. He was able to make this two-mode assumption because his concern was only with charge symmetry. In our case, the requirement that the trial functions possess symmetry properties

[11] G. F. Chew, Phys. Rev. **89**, 591 (1953).
[12] Anderson, Fermi, Martin, and Nagle, Phys. Rev. **91**, 155 (1953).
[13] Bodansky, Sachs, and Steinberger, Phys. Rev. **90**, 997 (1953).

with respect to continuous rotations in charge and ordinary space makes it impossible to put the different kinds of mesons in different modes.

At the expense of considerable extra computational labor, it would be possible to relax the single mode restriction in a way that is illustrated by the following example. A $(\frac{1}{2},\frac{1}{2})$ wave function can always be written

$$\Psi_{\frac{1}{2}}{}^{\frac{1}{2}}=\Psi_{\frac{1}{2},0}{}^{\frac{1}{2},0}+\Psi_{\frac{1}{2},1}{}^{\frac{1}{2},1}+\Psi_{\frac{1}{2},0}{}^{\frac{1}{2},1}+\Psi_{\frac{1}{2},1}{}^{\frac{1}{2},0}. \qquad \text{(A.1)}$$

One can choose a trial function in which the mesons in the first state are all in one mode, those of the second state are in another non-orthogonal mode, and those in the last two states are in combinations of the two. We have not chosen such trial functions, since their use would be at the sacrifice of the simple splitting of the problem into the source independent part, (7), and source dependent part, (11).

APPENDIX B. THE INTEGRALS L_n

The integrals defined in (8) can be found from the relation $L_n'(\lambda)=-nL_{n+1}(\lambda)$ and the following expression:

$$L_1(\lambda)=\frac{M^4}{\pi(M^2+\lambda^2-1)^2}\left\{(\lambda^2-1)^{\frac{1}{2}}\ln[\lambda+(\lambda^2-1)^{\frac{1}{2}}]-\frac{M^2\lambda(M^2+\lambda^2-1)}{2(M^2-1)}+\frac{M\pi}{4}(M^2+3\lambda^2-3)+\frac{M\lambda(2M^2\lambda^2-3M^2-3\lambda^2+3)}{4(M^2-1)^{\frac{3}{2}}}\ln\frac{M-(M^2-1)^{\frac{1}{2}}}{M+(M^2-1)^{\frac{1}{2}}}\right\}, \qquad \text{(B.1)}$$

for $\lambda>1$. In the case that $|\lambda|<1$, the first term in the bracket is replaced by $(1-\lambda^2)^{\frac{1}{2}}\arccos\lambda$. (The value of $\arccos\lambda$ varies between 0 and π.) We have used a Yukawa source for which $v(k)=M^2/(M^2+k^2)$.

At $\lambda=0$, the first three integrals are:

$$L_1(0)=\frac{M^4(M+2)}{4(M+1)^2},$$

$$L_2(0)=\frac{M^4}{2\pi(M^2-1)^2}\left[M^2+2-\frac{3M}{2(M^2-1)^{\frac{1}{2}}}\times\ln\frac{M+(M^2-1)^{\frac{1}{2}}}{M-(M^2-1)^{\frac{1}{2}}}\right],$$

$$L_3(0)=\frac{M^4}{4(M+1)^3}. \qquad \text{(B.2)}$$

APPENDIX C. WAVE FUNCTIONS

Since the amplitudes ${}^{(n)}{}_ja_L{}^T$ of Eq. (13) are useful in scattering calculations, we will give the expressions from which they may be found in Appendix D; first, it is necessary to give explicitly the phases of the functions that they multiply. We have adhered to the Condon and Shortley[14] conventions for adding nucleon- and meson-angular momenta, and for the relative phases of functions of different M_T and M_L. For each T and L, however, we must give one function to determine phases completely.

$n=1$:

The nine operators $\bar{A}_{0,+,-}{}^{0,+,-}$ are determined uniquely in terms of $\bar{A}_j{}^\alpha$ by defining ${}^{(1)}\Phi_{1,M}{}^{1,M'}\equiv\bar{A}_M{}^{M'}\Phi_{\text{vac}}$ and applying the operators $L_{+,-}$, L_z, $T_{+,-}$, and T_z [see (3) and (4)] to the equation ${}^{(1)}\Phi_{1,0}{}^{1,0}\equiv\bar{A}_z{}^z\Phi_{\text{vac}}$.

$n=2$:

$${}^{(2)}\Phi_{0,0}{}^{0,0}=(\sqrt{2}/3)[\bar{A}_+{}^+\bar{A}_-{}^-+\bar{A}_+{}^-\bar{A}_-{}^+-\bar{A}_0{}^+\bar{A}_0{}^- -\bar{A}_+{}^0\bar{A}_-{}^0+\tfrac{1}{2}(\bar{A}_0{}^0)^2]\Phi_{\text{vac}},$$

$${}^{(2)}\Phi_{1,1}{}^{1,1}=(1/\sqrt{2})[\bar{A}_+{}^+\bar{A}_0{}^0-\bar{A}_0{}^+\bar{A}_+{}^0]\Phi_{\text{vac}},$$

$${}^{(2)}\Phi_{0,0}{}^{2,2}=6^{-\frac{1}{2}}[2\bar{A}_+{}^+\bar{A}_-{}^+-(\bar{A}_0{}^+)^2]\Phi_{\text{vac}},$$

$${}^{(2)}\Phi_{2,2}{}^{2,2}=(1/\sqrt{2})(\bar{A}_+{}^+)^2\Phi_{\text{vac}}.$$

$n=3$:

$${}^{(3)}\Phi_{0,0}{}^{0,0}=6^{-\frac{1}{2}}\,\text{Det}\,|\bar{A}_{+,0,-}{}^{+,0,-}|\,\Phi_{\text{vac}},$$

$${}_\alpha{}^{(3)}\Phi_{1,1}{}^{1,1}=3(11)^{-\frac{1}{2}}\bar{A}_+{}^+\;{}^{(2)}\Phi_{0,0}{}^{0,0},$$

$${}_\beta{}^{(3)}\Phi_{1,1}{}^{1,1}=(\sqrt{2}/3)\;{}_\alpha{}^{(3)}\Phi_{1,1}{}^{1,1}-[(11)^{\frac{1}{2}}/10][\bar{A}_+{}^+\;{}^{(2)}\Phi_{0,0}{}^{2,0}-\sqrt{3}\bar{A}_+{}^0\;{}^{(2)}\Phi_{0,0}{}^{2,1}+6^{\frac{1}{2}}\bar{A}_+{}^-\;{}^{(2)}\Phi_{0,0}{}^{2,2}],$$

$${}^{(3)}\Phi_{1,1}{}^{2,2}=(1/\sqrt{2})\bar{A}_+{}^+\;{}^{(2)}\Phi_{0,0}{}^{2,1}-\bar{A}_+{}^0\;{}^{(2)}\Phi_{0,0}{}^{2,2},$$

$${}^{(3)}\Phi_{2,2}{}^{2,2}=(\sqrt{2}/\sqrt{3})\bar{A}_+{}^+\;{}^{(2)}\Phi_{1,1}{}^{1,1}.$$

[14] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (The Cambridge University Press, London, 1951), Chap. III.

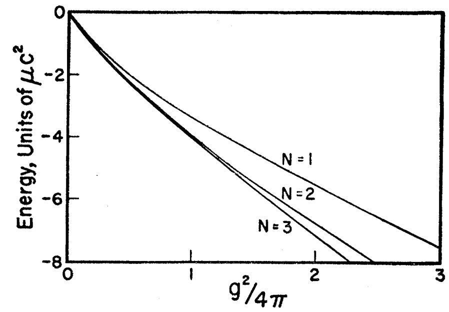

FIG. 10. Subsequent approximations to the ground-state energy for source size $M=2$. N refers to the largest number of mesons included in the calculation.

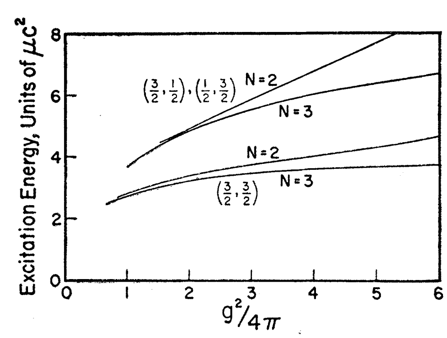

FIG. 11. Subsequent approximations to the excitation energies of the first two isobars for source size $M=2$.

The remaining functions can be found by interchanging upper and lower indices throughout any equation.

APPENDIX D. CALCULATIONS OF AMPLITUDES AND ENERGIES

The solution of (7) with the trial function (13) requires the use of sum rules and matrix elements given by Condon and Shortley.[14,15]

In each case we present the expression $\langle h_T \rangle$ which must be minimized, subject always to the restriction that $\sum |{}^{(n)}{}_j a_L{}^T|^2 = 1$. The result of this minimization is a set of equations for the amplitudes and an eigenvalue equation. In the following, we shall use $X \equiv U-1$, $Y \equiv U-2$, $Z \equiv U-3$.

(a) $I = J = \frac{1}{2}$

$$a = {}^{(0)}a_0{}^0, \quad b = {}^{(1)}a_1{}^1, \quad c = {}^{(2)}a_0{}^0, \quad d = {}^{(2)}a_1{}^1, \quad e = {}_\alpha{}^{(3)}a_1{}^1,$$
$$f = (83)^{-\frac{1}{2}}[(33)^{\frac{1}{2}}\,{}^{(3)}a_0{}^0 + (50)^{\frac{1}{2}}\,{}_\beta{}^{(3)}a_1{}^1]$$
$${}_\beta{}^{(3)}a_1{}^1 = (50/33)^{\frac{1}{2}}\,{}^{(3)}a_0{}^0$$
$$\langle h_T \rangle = |b|^2 + 2[|c|^2 + |d|^2] + 3[|e|^2 + |f|^2] + \gamma\{3\bar{b}a + \sqrt{2}[\bar{c} + 2\bar{d}]b + (11)^{\frac{1}{2}}\bar{e}c + (11)^{-\frac{1}{2}}[4\bar{e} + (83)^{\frac{1}{2}}\bar{f}]d + \text{comp. conj.}\}. \quad \text{(D.1)}$$

The eigenvalue equation is

$$UXY^2Z^2 - (20UX + 10UZ + 9YZ)YZ\gamma^2 + (83UX + 74UZ + 180YZ)\gamma^4 - 747\gamma^6 = 0. \quad \text{(D.2)}$$

As expected, there is only one root which goes to zero as $\gamma^2 \to 0$. This is the ground state for which Eq. (11) possesses solutions for $\lambda \geq 0$ with any value of γ^2. The next two roots belong to isobars, for they correspond to solutions of Eq. (11) with $\lambda > -1$ provided γ^2 is large enough. The three highest roots of (D.2) do not correspond to solutions of (11); i.e., the energy in these cases does not possess a minimum when λ is varied.

(b) $I = \frac{3}{2}$, $J = \frac{1}{2}$

$$a = {}^{(1)}a_1{}^1, \quad b = {}^{(2)}a_1{}^1, \quad c = {}^{(2)}a_0{}^2, \quad d = {}_\alpha{}^{(3)}a_1{}^1,$$
$$e = {}_\beta{}^{(3)}a_1{}^1, \quad f = {}^{(3)}a_1{}^2.$$

Rotate axes in d, e, f space, introducing three new orthonormal amplitudes g, h, k with

$$g = (88)^{-\frac{1}{2}}[(10)^{\frac{1}{2}}d - (45)^{\frac{1}{2}}e + (33)^{\frac{1}{2}}f],$$
$$h = -\left(\frac{5}{31}\right)^{\frac{1}{2}}\left[\frac{3}{10}\left(\frac{5}{11}\right)^{\frac{1}{2}}d + \frac{7}{4}\left(\frac{10}{11}\right)^{\frac{1}{2}}e + \frac{3}{2}\left(\frac{3}{2}\right)^{\frac{1}{2}}f\right].$$

One then finds $k=0$ and

$$\langle h_T \rangle = |a|^2 + 2[|b|^2 + |c|^2] + 3[|g|^2 + |h|^2] - \gamma\{\bar{a}[(2)^{\frac{1}{2}}b + (5)^{\frac{1}{2}}c] + \tfrac{1}{2}(5)^{\frac{1}{2}}\bar{b}[g - (31/5)^{\frac{1}{2}}h] + (8)^{\frac{1}{2}}\bar{c}g + \text{comp. conj.}\}, \quad \text{(D.3)}$$

$$XY^2Z^2 - (17X + 7Z)YZ\gamma^2 + (62X + 41Z)\gamma^4 = 0. \quad \text{(D.4)}$$

The two lowest roots correspond to solutions of (11) for sufficiently large values of γ^2.

(c) $I = J = \frac{3}{2}$

$$a = {}^{(1)}a_1{}^1, \quad b = {}^{(2)}a_1{}^1, \quad c = {}^{(2)}a_2{}^2, \quad d = {}_\alpha{}^{(3)}a_1{}^1,$$
$$e = {}_\beta{}^{(3)}a_1{}^1, \quad f = {}^{(3)}a_2{}^2, \quad \sqrt{2}g = {}^{(3)}a_1{}^2 = {}^{(3)}a_2{}^1.$$

Replace d, e, f, and g by the orthonormal amplitudes h, j, k, and l where

$$h = \left(\frac{2}{27}\right)^{\frac{1}{2}}\left[\frac{1}{(11)^{\frac{1}{2}}}d + \frac{5}{4}\left(\frac{2}{11}\right)^{\frac{1}{2}}e + \frac{5}{2}\left(\frac{3}{2}\right)^{\frac{1}{2}}f + \left(\frac{15}{4}\right)^{\frac{1}{2}}g\right],$$
$$j = -\frac{5}{3(61)^{\frac{1}{2}}}\left[\frac{13}{3}\left(\frac{3}{11}\right)^{\frac{1}{2}}d + \frac{7}{15}\left(\frac{6}{11}\right)^{\frac{1}{2}}e + \frac{7}{2}\sqrt{2}f - \frac{8}{5^{\frac{1}{2}}}g\right].$$

One then finds $k = l = 0$, and

$$\langle h_T \rangle = |a|^2 + 2[|b|^2 + |c|^2] + 3[|h|^2 + |j|^2] + \gamma\left\{\bar{a}[b + 5c]/\sqrt{2} + (27/2)^{\frac{1}{2}}\bar{b}h + \frac{5}{3(6)^{\frac{1}{2}}}\bar{c}h - \frac{2(61)^{\frac{1}{2}}}{3(3)^{\frac{1}{2}}}\bar{c}j + \text{comp. conj.}\right\}, \quad \text{(D.5)}$$

$$XY^2Z^2 - (23X + 13Z)YZ\gamma^2 + (122X + 161Z)\gamma^4 = 0. \quad \text{(D.6)}$$

Again the two lowest roots correspond to isobars.

[15] For more details refer to the doctoral thesis by F. H. Harlow at the University of Washington, 1953 (unpublished).