The complete set of special values of the 12j symbol when any one of its twelve arguments is zero is as follows:

$$\begin{Bmatrix} a0ap \\ cdfq \\ gdrs \end{Bmatrix} = \begin{Bmatrix} dd0a \\ fspg \\ cqar \end{Bmatrix} = \begin{Bmatrix} rpa0 \\ qsdc \\ gfad \end{Bmatrix} = \begin{Bmatrix} acgq \\ 0ddp \\ afrs \end{Bmatrix}$$

$$\begin{Bmatrix} rgqd \\ aa0f \\ pcsd \end{Bmatrix} = \begin{Bmatrix} rqgc \\ psf0 \\ adad \end{Bmatrix} = \begin{Bmatrix} dfcg \\ dsqa \\ 0par \end{Bmatrix} = \begin{Bmatrix} rapf \\ gacd \\ q0sd \end{Bmatrix}$$

$$= \frac{(-1)^{a+2c+d+p+q+r+s}}{\{[a][d]\}^{\frac{1}{2}}} \begin{Bmatrix} pcq \\ dsf \end{Bmatrix} \begin{Bmatrix} pcq \\ gra \end{Bmatrix}, \quad \text{(A8)}$$

together with

$$\begin{Bmatrix} 0bbp \\ cdfq \\ chrs \end{Bmatrix} = \begin{Bmatrix} dcfp \\ b0bq \\ hcsr \end{Bmatrix} = \begin{Bmatrix} rqcc \\ psfb \\ bh0d \end{Bmatrix} = \begin{Bmatrix} sqhb \\ prbc \\ fcd0 \end{Bmatrix}$$

$$= \{[b][c]\}^{-\frac{1}{2}} \begin{Bmatrix} rbp \\ cdf \\ qhs \end{Bmatrix}. \quad \text{(A9)}$$

A group of 16 symmetry relations of the 12j symbol, together with a convenient new notation, has been found by R. J. Ord-Smith and will be reported upon shortly.

The 12j symbol is being used in Southampton in nuclear structure calculations on light nuclei with interconfigurational mixing.

PHYSICAL REVIEW VOLUME 93, NUMBER 2 JANUARY 15, 1954

The Elastic Scattering of Protons by Alpha Particles*

S. Sack, L. C. Biedenharn, and G. Breit
Yale University, New Haven, Connecticut

(Received October 13, 1953)

The $p_{\frac{1}{2}}$ and $p_{\frac{3}{2}}$ phase shifts determined from analysis of the elastic scattering of protons by alpha particles are treated in terms of a potential interaction between the proton and alpha-particle core, with a spin-orbit interaction of the Thomas type. The results are sensitive to the well shape chosen and favor a Gauss well interaction. The limitations of this type of treatment are discussed.

I. INTRODUCTION

THE elastic scattering of nucleons by alpha particles has been extensively studied by many workers,[1–7] mainly on account of its possible importance for the determination[4,7] of the magnitude of the spin orbit interaction in nuclei. The origin of the latter is still obscure, and none of the work done so far, including that reported here, has any certainty of being more than an attempt to fit the data with formulas having only a slight foundation from a fundamental viewpoint. The possibly large importance of many-body forces in nuclei throws doubt on the applicability of considerations based on interactions between pairs of nucleons. The fact that the spin-orbit interaction is appreciably larger than expected from a simple application of the Thomas term theory or its extensions is an additional deterrent to elaborate calculation with wave functions based on two-body forces and spin-orbit forces which follow from relativistic considerations[8,9] in such cases.

From a purely phenomenological point of view, it is not clear that one-body treatments of nuclear structure are meaningless in such cases as the p-α interaction. There is much evidence that the alpha particle behaves approximately as a unit in nuclei, and the small nuclear radius as well as the large internal tightness of binding of the alpha particle lend plausibility to a view in which this particle produces a general field to which the incident proton is exposed. From this admittedly naive viewpoint it is of interest to determine the magnitude of the p-α interaction in terms of potential well parameters for wells of prescribed shape. The interaction is not taken to be the same for s, $p_{\frac{1}{2}}$, $p_{\frac{3}{2}}$, $\cdots$ terms, but it is supposed below that the potential wells for the $p_{\frac{1}{2}}$ and $p_{\frac{3}{2}}$ states are related to each other as though the difference between them were caused by a Thomas term with an adjustable constant of proportionality. Wells of various shapes are fitted to experimental data, and it is found that the data favor the Gauss potential well shape in preference to the long-tailed exponential well or the short-tailed square well. The wells just referred to are meant to be idealized wells in the absence of the Thomas term. The possibility of distinguishing between wells of different shapes rests on the experimental

* Assisted by the joint program of the U. S. Office of Naval Research and the U. S. Atomic Energy Commission and by the Office of Ordnance Research, U. S. Army.

[1] Freier, Lampi, Sleator, and Williams, Phys. Rev. **75**, 1345 (1949).
[2] C. L. Critchfield and D. C. Dodder, Phys. Rev. **76**, 602 (1949).
[3] M. Heusinkveld and G. Freier, Phys. Rev. **85**, 80 (1952).
[4] R. K. Adair, Phys. Rev. **86**, 155 (1952).
[5] Kreger, Kerman, and Jentschke, Phys. Rev. **86**, 593 (1952).
[6] T. M. Putnam, Phys. Rev. **87**, 932 (1952).
[7] D. C. Dodder and J. L. Gammel, Phys. Rev. **88**, 520 (1952).
[8] G. Breit, Phys. Rev. **51**, 248 (1937).
[9] G. Breit, Phys. Rev. **53**, 153 (1938).

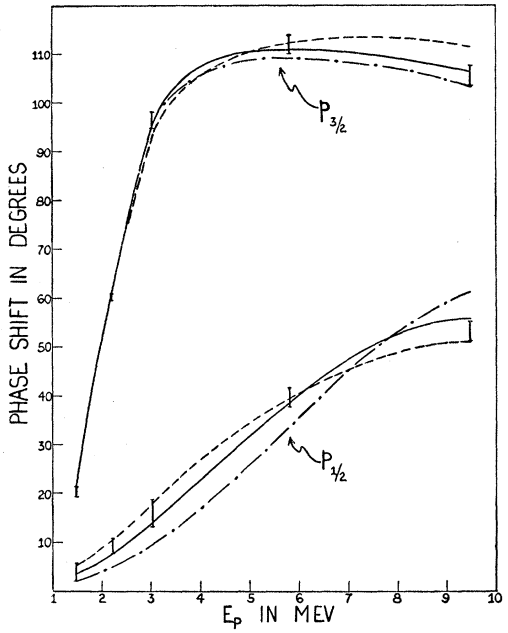

FIG. 1. Phase shifts for p-α scattering as a function of proton energy. The points represent the experimental phase shifts. The curves show the fits to the experimental phases for three well shapes. Legend: —·—·— square well; ——— Gauss well; - - - - - - - exponential well.

information regarding the $p_{\frac{1}{2}}$ and $p_{\frac{3}{2}}$ phase shifts at different energies and the influence of well shape on the phase-shift energy dependence.

As in p-p and n-p scattering, the influence of well shape is hard to distinguish from velocity dependence of potential well depth or range. The interpretation of the results presented here is thus subject to reservations regarding the absence of serious effects of such energy dependence. It is possible, therefore, that the significance of the preference of the experimental data for potentials with intermediate-length tails is not direct and that the best effective potentials will be useful primarily as a summary of the data.

Since two-body forces are still one of the most definite ways of dealing with nuclear structure, a discussion is given regarding the approximations needed in order to replace two-body spin-orbit forces by a spin-orbit force corresponding to a central potential. This discussion is explicit only regarding potentials of the Thomas-term type, but requires little modification for more general two-body potentials.

II. EXPERIMENTAL DATA AND PHYSICAL ASSUMPTIONS

The most extensive and quantitatively reliable measurements have been of the p-α differential cross section. Measurements for proton energies of 1 to 3 Mev[1] were analyzed by Critchfield and Dodder[2] giving two sets of phase shifts (s,$p_{\frac{1}{2}}$,$p_{\frac{3}{2}}$) allowed by the data, corresponding to inverted or normal $p_{\frac{1}{2}}-p_{\frac{3}{2}}$ doublets. Measurements of the polarization of the scattered protons[3] showed the correctness of the inverted doublet phase shifts. Measurements at 5.81 Mev[5] and 9.48 Mev[6] were analyzed by Dodder and Gammel,[7] yielding s, $p_{\frac{1}{2}}$, $p_{\frac{3}{2}}$, $d_{\frac{3}{2}}$, $d_{\frac{5}{2}}$ phase shifts, the p and d doublets being inverted. The phase shifts, with errors as estimated in the phase-shift analyses,[2,7] are reproduced in Fig. 1.

The s phase shifts are negative and increase slowly in magnitude with energy, suggesting a repulsive interaction between the proton and alpha particle. The s phase shifts have been fitted qualitatively by Adair[4] and Dodder and Gammel[7] with a hard-sphere interaction having a range of 2.6×10^{-13} cm.

The $p_{\frac{1}{2}}$ and $p_{\frac{3}{2}}$ phase shifts show markedly different energy dependences. The phase shifts are positive, increasing with energy. The $p_{\frac{3}{2}}$ phase shifts increase rapidly with energy, passing through 90° at 2.8-Mev proton energy; the $p_{\frac{1}{2}}$ phase shifts increase more slowly, remaining less than 90° throughout the experimental energy range.

The energy dependence of the p phase shifts suggests a deeper effective attractive potential for $p_{\frac{3}{2}}$ than for $p_{\frac{1}{2}}$. The difference between the $p_{\frac{1}{2}}$ and $p_{\frac{3}{2}}$ potentials can then be interpreted as an effective spin-orbit interaction between nucleon and alpha particle.

Previous discussions of the p states[4,7] have been in terms of resonance models. Adair[4] and Dodder and Gammel[7] have treated the p states in terms of the "reduced width" and "characteristic energy" of the Wigner[10] formalism. These formulations give results similar to treatment in terms of effective potentials for the $p_{\frac{1}{2}}$ and $p_{\frac{3}{2}}$ states, insofar as their applicability depends on the simple energy dependence of the logarithmic derivative of the wave function of relative motion.

However, a description in terms of resonance model parameters has few advantages in the present case, since there are no marked resonance levels. Since the Wigner formalism involves some arbitrariness in the dependence of levels on the choice of the nuclear radius, and since it gives no direct connection between the interactions that apply to the two p states, the more elementary approach of one-body effective potentials was preferred as a temporary expedient. The limitations of this treatment have been mentioned in the introduction and a more precise understanding of such limitations could possibly be obtained by an application of the dispersion theory formalism.

The quantity of interest, for the approach used here as well as the Wigner formalism, is the logarithmic derivative of the wave function of relative motion at moderate internuclear distances. This is readily calculated from the experimental phase shifts and tables of Coulomb functions.[11,12] In the notation of reference 11,

[10] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947).

[11] Bloch, Hull, Broyles, Bouricius, Freeman, and Breit, Revs. Modern Phys. **23**, 147 (1951).

[12] *Tables of Coulomb Functions I*, National Bureau of Standards Applied Mathematics Series No. 17 (U. S. Government Printing Office, Washington, D. C., 1952).

the homogeneous logarithmic derivative $Y(r)$ is given by

$$Y(r)=rd\mathfrak{F}/\mathfrak{F}dr=(\rho dF/Fd\rho)-1/[(F^2/\rho)\cot K-FG/\rho].$$

The difference in the $p_{\frac{1}{2}}$ and $p_{\frac{3}{2}}$ state interactions is clearly illustrated in the energy dependence of the logarithmic derivatives.[13] The $p_{\frac{3}{2}}$ logarithmic derivative is more negative and decreases more rapidly with energy than that for $p_{\frac{1}{2}}$. These properties are expressed from the standpoint of the Wigner model as follows:[7] (1) the characteristic energy for the $p_{\frac{1}{2}}$ state is higher than that for $p_{\frac{3}{2}}$; (2) the $p_{\frac{1}{2}}$ state has a larger reduced width than the $p_{\frac{3}{2}}$ state. From the standpoint of the present work: (1) the difference in the logarithmic derivatives, $Y_{\frac{3}{2}}<Y_{\frac{1}{2}}$, indicates a deeper potential well for the $p_{\frac{3}{2}}$ state; (2) the different energy dependences, $(dY_{\frac{3}{2}}/dE)<(dY_{\frac{1}{2}}/dE)$, serve to fix qualitatively the dependence on distance of the difference between the $p_{\frac{1}{2}}$ and $p_{\frac{3}{2}}$ state potentials. The values of $Y_{\frac{1}{2}}$ and $Y_{\frac{3}{2}}$ fix the sign and magnitude of the effective spin-orbit interaction while the values of dY_J/dE determine qualitatively its shape.

The effect of the spin-orbit interaction shape may be seen with the aid of a well-known formula relating the change in logarithmic derivative to the change in potential:

$$(Y_{\frac{3}{2}}-Y_{\frac{1}{2}})_{r=a}=(2\mu a/\hbar^2)\left[(1/\mathfrak{F}_{\frac{1}{2}}\mathfrak{F}_{\frac{3}{2}})\int_0^a \mathfrak{F}_{\frac{1}{2}}\mathfrak{F}_{\frac{3}{2}}\Delta V(r)dr\right].$$

Here, Y is the homogeneous logarithmic derivative defined earlier, and $\Delta V(r)$ represents the effective spin-orbit interaction $V_{\frac{3}{2}}-V_{\frac{1}{2}}$. If the spin-orbit interaction has a δ function distance dependence $\delta(r-a)$, $(Y_{\frac{3}{2}}-Y_{\frac{1}{2}})_{r=a}$ is independent of energy. For an inverted doublet spin-orbit interaction at distances $r<a$, the bracketed quantity increases in magnitude with the energy, reproducing qualitatively the behavior of the experimental logarithmic derivatives.

A simple verification of the above conclusions is obtained by fitting experimental data for $p_{\frac{3}{2}}$ and $p_{\frac{1}{2}}$ separately by means of square-well potentials. For $p_{\frac{3}{2}}$ the data require an attractive square well with a range of 3.15×10^{-13} cm. The $p_{\frac{1}{2}}$ phase shifts call for a shallower attractive square well with a range of about 3.6×10^{-13} cm. The effective spin-orbit interaction $V_{\frac{3}{2}}-V_{\frac{1}{2}}$ is seen to be attractive for $r<3.15\times10^{-13}$ cm and repulsive for $r>3.15\times10^{-13}$ cm.

In order to relate the analysis to the customary form of the Thomas term, it was supposed that the Hamiltonian has the form

$$H=T+V+(\alpha\hbar/4M^2c^2)[\mathbf{p}\times\nabla V]\cdot\boldsymbol{\sigma},$$

where T is the kinetic energy, V is a function of r only, $\mathbf{p}$ is the momentum operator, and ∇ in ∇V applies to V only. This Hamiltonian can be written

$$H=T+[1-(\alpha\lambda\!\!\!^{-2}/4)(\mathbf{L}\cdot\boldsymbol{\sigma})d/rdr]V(r), \qquad (1)$$

where $\lambda\!\!\!^{-}=\hbar/Mc$ and $\mathbf{L}$ is the orbital angular momentum operator in units $\hbar$. On account of the relatively large difference between the proton mass M and the reduced mass $\mu=4M/5$, there is a question as to whether one should use $\alpha=1$ or $\alpha=25/16$ to reproduce the expectation on the simple scalar equation view. Since the empirical α is appreciably greater than either of these values, this question does not appear to be very significant, there being no possibility of agreement with the scalar equation. In the interests of simplicity, the standardization used here is such as to neglect corrections for reduced mass. There is no reason for supposing such corrections to be simply a matter of changing M to the reduced mass. In fact, in a system composed of several particles, there is a separate Thomas-like term for each particle, so that the contributions of terms referring to the four nucleons in the alpha particle cancel in first approximation.

The employment of Eq. (1) does not strictly correspond to the form which follows from the assumption that Thomas-like terms represent the spin-orbit interactions between pairs of particles, as has been noted by Breit and Stehn.[14] In the approximation of representing the wave function by Slater determinants, the spin-orbit energy for the $p_{\frac{3}{2}}$ state is

$$(\psi,H'\psi)=2(u_2{}^{\pi}v_1,A_{12}{}^{z}u_2{}^{\pi}v_1)+2(u_2{}^{\nu}v_1,A_{12}{}^{z}u_2{}^{\nu}v_1)-(u_1{}^{\pi}v_2,(A_{12}{}^{z}+A_{21}{}^{z})u_2{}^{\pi}v_1), \qquad (2)$$

where the superscripts π,ν refer to proton and neutron wave functions, respectively, and the remainder of the notation is as follows:

u,v = orbital functions for s and p states, respectively; the p function is for magnetic quantum number 1;

$A_{12}{}^{z}=(\hbar/4M^2c^2)[\mathbf{p}_1\times(\nabla_1V(r_{12}))]_z$, with the understanding that $\mathbf{p}_1=(\hbar/i)\nabla_1$ is an operator, while $(\nabla_1V(r_{12}))$ is a number;

$$(U_1V_2,O_{12}X_1Y_2)=\int U^*(\mathbf{r}_1)V^*(\mathbf{r}_2)O_{12}X(\mathbf{r}_1)Y(\mathbf{r}_2)d\mathbf{r}_1d\mathbf{r}_2;$$

where, for brevity, arguments are indicated by subscripts.

The first two terms in Eq. (2) represent direct interactions of the p particle with the s shells of neutrons and protons. The last term arises on account of the identity of the three protons and is an exchange effect. The first two terms can be represented as the effect of an equivalent central field; the last one cannot. It should be pointed out that the scattering problem can be stated in terms of a variational equation for the expectation value of the energy. The radial factors in u and v are varied. On varying v, there results an integro-differential equation for the radial factor of v. The first two terms in Eq. (2) contribute an ordinary potential, while the last contributes a part representable as an

[13] D. C. Dodder and J. L. Gammel, Phys. Rev. 88, 520 (1952), Fig. 1. Note that energy of relative motion is incorrectly given; the values in their Fig. 1 should be multiplied by 5/4.

[14] G. Breit and J. R. Stehn, Phys. Rev. 53, 459 (1938).

integral operator which is not expressible in terms of a local potential.

One finds

$$(u_2v_1,A_{12}{}^zu_2v_1)=-(\alpha\lambda\!\!\!^{-2}/4)$$
$$\times\int R_s{}^2(2)R_p{}^2(1)[V'(r_{12})/r_{12}]d\mathbf{r}_1d\mathbf{r}_2/(4\pi)^2, \quad (3)$$

where $R_s(r)$, $R_p(r)$ are the radial s and p functions, respectively, normalized so that

$$\int_0^\infty R_s{}^2(r)r^2dr=\int_0^\infty R_p{}^2r^2dr=1.$$

If, on the other hand, use is made of Eq. (1) for a single-particle model, and if V in this model is distinguished as U, one obtains for the spin-orbit energy in the $p_{\frac{3}{2}}$ state:

$$\langle(\mathbf{A}\boldsymbol{\sigma})\rangle=-(\alpha\lambda\!\!\!^{-2}/16\pi)\int R_p{}^2(1)[U'(r_1)/r_1]d\mathbf{r}_1. \quad (4)$$

If one compares this result with Eq. (3), agreement is obtained if one sets

$$\tilde{V}(r_1)=\int V(r_{12})R_s{}^2(2)d\mathbf{r}_2/(4\pi). \quad (5)$$

This equivalent one body $\tilde{V}$ is just the average potential at $\mathbf{r}_1$ caused by the s particles at $\mathbf{r}_2$. The possibility of interpreting the direct-interaction terms in Eq. (2) as due to an equivalent central potential is thus verified and holds more generally than in the present simple example. Collecting the results for different interactions of a p particle with an s shell for the case of Wigner forces as listed in reference 14, the expectation value is

$$(\psi,H'\psi)=-(\alpha\lambda\!\!\!^{-2}/16\pi^2)\int R_2{}^2Q_1{}^2(V'/r_{12})r_1{}^2d\mathbf{r}_1d\mathbf{r}_2$$
$$-(\alpha\lambda\!\!\!^{-2}/128\pi^2)\int R_1Q_2(V'/r_{12})$$
$$\times\{2(r_2{}^2-\mathbf{r}_1\cdot\mathbf{r}_2)R_2Q_1+[(R_2Q_1'/r_1)$$
$$-(R_2'Q_1/r_2)]r_1{}^2r_2{}^2\sin^2\theta\}d\mathbf{r}_1d\mathbf{r}_2, \quad (6)$$

where $R=R_s$, $R_p=rQ$, and the arguments r_1,r_2 of the radial functions are distinguished by subscripts. The radial functions for s protons and neutrons are assumed to be the same for simplicity although this point is not essential to the argument. If one varies Q, the first integral in Eq. (6) gives rise to a local interaction; the second is not expressible in such simple terms.

The potential energy $V_{12}+V_{23}\cdots$ also gives rise to exchange terms. An improved treatment including exchange effects with two-body forces would thus have to include a modification of the effects of the non-spin-orbit part of the potential. No attempt is made to calculate these effects in the present paper, partly because of the multiplicity of possibilities offered by combinations of exchange effects, and partly because of the unknown magnitude of many-body force contributions. The latter may conceivably result in making the equivalent one-body equation (1) a better approximation than a sum of potentials representing interactions between pairs of particles.

III. COMPARISON WITH EXPERIMENT

Three well shapes, giving a survey of possibilities for the spin-orbit interaction shape, were chosen. For each shape the values of three parameters, well depth, well range, and strength of spin-orbit interaction, were determined by minimizing the mean square deviation of the calculated p phases from experiment. Five experimental energies, $E_p=1.49$, 2.22, 3.04, 5.81, 9.48 Mev, were considered in the fitting procedure. These energies provided an adequate survey of the data without overemphasis of the low-energy data. The three shapes chosen were: (1) square well, giving a δ-function spin-orbit interaction located at the edge of the well, (2) Gauss well, giving a Gauss well with the same range parameter for the spin-orbit interaction, (3) exponential well, giving a Yukawa well shape for the spin-orbit interaction. These well shapes show a regular progression from concentrated potentials with externally located spin-orbit interaction to longer-tailed wells with spin-orbit interaction concentrated at small distances.

In all calculations the repulsive Coulomb field was taken as the field of a uniform volume distribution of charge with a radius of 2.31×10^{-13} cm. The fit for a given well shape does not depend sensitively on the assumed charge distribution. For the Gauss potential, the effect of replacing the uniform volume distribution by a point charge is compensated by increasing the depths of the $p_{\frac{1}{2}}$ and $p_{\frac{3}{2}}$ wells by 0.45 percent, with the range parameter left unchanged. The root-mean-square deviation of calculated phases from experiment is the same in both cases and the maximum difference in the "best fit" phase shifts for the two cases is 0.2°, an amount negligible in comparison with the errors in the experimental phase shifts.

(A) Square Well

The potential arising from a common p state potential, $V_0(r)=-D$, $(r\leq r_0)$, $V_0(r)=0$, $(r>r_0)$, is

$$V(r)=-D-(\beta/r_0)(\boldsymbol{\sigma}\mathbf{L})\delta(r-r_0), \quad (r\leq r_0);$$
$$V(r)=0, \quad (r>r_0),$$

where $\beta=(\alpha/4)(\hbar/Mc)^2$. The spin-orbit interaction produces an energy-independent splitting in the $p_{\frac{1}{2}}$ and $p_{\frac{3}{2}}$ state logarithmic derivatives, as is seen from the relation

$$(Y_{\frac{3}{2}}-Y_{\frac{1}{2}})_{r=a+}=-3(2\mu D/\hbar^2)\alpha\lambda\!\!\!^{-2}/4.$$

The parameters for the best fit are

$$r_0=3.21\times10^{-13}\text{ cm},\quad D=19.65\text{ Mev},\quad \alpha=48.8.$$

The root-mean-square deviation of calculated phases from experiment is 4.1°. The best fit is reproduced in Fig. 1. It is apparent that the calculated $p_{\frac{3}{2}}$ phases increase too slowly with energy, while the calculated $p_{\frac{1}{2}}$ phases increase too rapidly.

(B) Gauss Well

The interaction arising from a Gauss well, $V(r)=-A\exp[-(r/a)^2]$, is

$$V_{\frac{1}{2}}=-[1-(4\beta/a^2)]A\exp[-(r/a)^2],$$
$$V_{\frac{3}{2}}=-[1+(2\beta/a^2)]A\exp[-(r/a)^2],$$

where $\beta=(\alpha/4)(\hbar/Mc)^2$. The above set of potentials is equivalent to taking $V_J=-A_J\exp[-(r/a)^2]$, with

$$A_{\frac{1}{2}}=[1-(4\beta/a^2)]A,\quad A_{\frac{3}{2}}=[1+(2\beta/a^2)]A.$$

The parameters for the best fit are:

$$a=2.30\times10^{-13}\text{ cm},\quad A=47.32\text{ Mev},\quad \alpha=29.6,$$

or

$$A_{\frac{3}{2}}=53.17\text{ Mev},\quad A_{\frac{1}{2}}=35.61\text{ Mev}.$$

The phase shifts for the best fit are reproduced in Fig. 1. The root-mean-square deviation of calculated phases from experiment is 1.5°, slightly over $\frac{1}{3}$ the corresponding number for the square well.

For the Gauss potential, the dV/rdr term has the same shape as V. The difference between the $p_{\frac{1}{2}}$ and $p_{\frac{3}{2}}$ potentials is, therefore, the same fraction of V at all r. This fraction has the large value 0.4, supporting the view that the spin-orbit interaction is not a relativistic correction, but a major effect.

(C) Exponential Well

The interaction resulting from a common p state potential, $V_0(r)=-B\exp(-2r/b)$, is

$$V(r)=-B[\exp(-2r/b)+(\sigma\mathbf{L})(4\beta/b^2)\exp(-2r/b)/(r/b)],$$

where $\beta=(\alpha/4)(\hbar/Mc)^2$. The spin-orbit interaction adds a potential having a Yukawa radial dependence, attractive for $p_{\frac{3}{2}}$ and repulsive for $p_{\frac{1}{2}}$. The parameters for the best fits are

$$b=1.924\times10^{-13}\text{ cm},\quad B=155.5\text{ Mev},\quad \alpha=20.0.$$

The spin-orbit splitting is here given by a Yukawa well, $\mathcal{C}\exp(-r/\hat{a})/(r/\hat{a})$, with $\mathcal{C}=110$ Mev and $\hat{a}=0.958\times10^{-13}$ cm. The calculated phases are reproduced in Fig. 1. The root-mean-square deviation of calculated phases from experiment is 2.5°. The exponential-well fits show too large a $p_{\frac{3}{2}}$ phase shift and too small a $p_{\frac{1}{2}}$ phase shift at the highest energies, a trend opposite to that shown by the square well fits. This may be attributed to the singular spin-orbit interaction at small distances, which is most effective at high energies for which the centrifugal barrier is least important.

IV. CONCLUSIONS

The best-fit phase shifts calculated for the three well shapes studied are plotted with the experimental p phase shifts in Fig. 1. The graphs indicate that: (1) an exponential well gives a splitting increasing too rapidly with energy; (2) a square well gives a splitting increasing too slowly with energy; (3) a Gauss well gives a good over-all fit to the experimental phase shifts.

The Gauss-well parameters which may be considered as giving a good representation of the p state proton-alpha interaction are

$$V=-[1-\beta(\sigma\mathbf{L})(d/rdr)]A\exp[-(r/a)^2]$$
$$a=2.30\times10^{-13}\text{ cm},\quad A=47.32\text{ Mev},\quad \beta=7.40(\hbar/Mc)^2.$$

These numbers support the general belief that the spin-orbit parameter required by experiment is much larger than the theory of the Thomas term would require, *viz.* $\beta=(1/4)(\hbar/Mc)^2$. The ratio of the spin-orbit interaction needed to account for experiment to that suggested by the Thomas term has the following values:

Square well	Gauss well	Exponential well
49	30	20

The conclusions as to well shape depend sensitively on the values taken for the experimental p phase shifts at the higher energies, as is evident in Fig. 1. The possible presence of small phase shifts for $L>2$, neglected in the phase-shift analysis,[7] might lead to changes in the p phases sufficient to change the preference for the Gauss well. An investigation of this point is outside the scope of the present work; however, the f phase shifts calculated from the same Gauss well that fitted the p phase-shift data are $\approx\frac{1}{2}°$ at 10 Mev, which indicates that the neglect of f phase shifts in the analysis of the experimental data is not unreasonable. On the other hand, there is no compelling reason for taking the potential to be the same in f and in p states, so that it is difficult to exclude f phase shifts of appreciably greater amounts than given by the above estimate. It is also difficult to exclude effects of changes in well shape for large r which can affect f wave phase shifts without producing serious effects on spin-orbit interaction.

For these reasons the phase-shift analysis used here, as well as its consequences as presented in this note, might need modification. In particular, the effect of phase shifts for $L>2$ on the s,p,d phase shifts cannot be claimed to be necessarily negligible in ascertaining effects of well shape on spin-orbit interaction,[8] as has been attempted here.