either $\frac{1}{2}$ or $\frac{3}{2}$. Therefore all the terms in d^3, which has been treated by Jahn[4] in the LS limit, have to appear in d^2d, too. Using the fractional parentage coefficients $(d^3TSL[\![d^2(T_1S_1L_1), d, TSL)$ obtained by Jahn, we can calculate the matrix elements of the spin-orbit interaction as follows:

$$(d^3TSL, TJM|\sum_{i=1}^{3}(s_i, l_i)|d^3T'S'L', T'JM)$$

$$= (-1)^{S+L'-J}(d^3TSL\|\sum_{i=1}^{3}(s_i, l_i)\|d^3T'S'L')\times W(SLS'L'; J1),$$

where

$$(d^3TSL\|\sum_{i=1}^{3}(s_i, l_i)\|d^3T'S'L')$$
$$=3\sum_{T_1S_1L_1}(d^3TSL[\![d^2(T_1S_1L_1)d, TSL)$$
$$\times(T_1S_1L_1, d_3, TSL\|(s_3, l_3)\|T_1S_1L_1, d_3, T'S'L')$$
$$\times(d^2(T_1S_1L_1)d, T'S'L']\!]d^3T'S'L')$$
$$=(9\sqrt{5})[(2S+1)(2S'+1)(2L+1)(2L'+1)]^{\frac{1}{2}}$$
$$\cdot\sum_{T_1S_1L_1}(-1)^{S_1+L_1-\frac{1}{2}-S'-L'}(d^3TSL[\![d^2(T_1S_1L_1)d, TSL)$$
$$\times W(\tfrac{1}{2}S\tfrac{1}{2}S'; S_11)W(2L2L'; L_11)$$
$$\times(d^2(T_1S_1L_1)d, T'S'L']\!]d^3T'S'L'),$$

and the final form of this matrix is given in Table I.

The matrix for $T=\frac{3}{2}$ is the same as that obtained by Racah and therefore is omitted. Multiplying it by the function $(-1)^{S+L'-J}W(SLS'L'; J1)$, we easily get the final matrices of spin-orbit interaction for any definite value of J. The results will be reported later with complete numerical results for F^{19}.

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[4] H. A. Jahn, Proc. Roy. Soc. (London) **201**, 516 (1950); **205**, 192 (1951).

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Characteristics of a Proposed Double-Mode Cyclotron*

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It is proposed to accelerate ions in a fixed-frequency cyclotron, either with the dee voltages 180° out-of-phase, or with the dee voltages in-phase. This is possible if the angular extent of the dee electrodes is made less than the customary 180°. Formulas have been derived for the energy gain per turn of ions with various charge-to-mass ratios as a function of the angle of the dees. The threshold dee voltage for acceleration to a given energy is somewhat larger than that for conventional two-dee cyclotrons. For appropriate dee angles it is shown that both protons and deuterons can be accelerated in the nonrelativistic energy region at substantially the same magnetic field and oscillator frequency. Suggested applications to acceleration of ions heavier than alpha particles are discussed.

INTRODUCTION AND SIMPLE THEORY

A CONVENTIONAL two-dee cyclotron accelerates ions with the dee voltages 180° out-of-phase. No use is made of the in-phase mode of oscillation of the dee system. It can be brought about by appropriate minor changes in the coupling of the oscillator to the dee system as, for example, by reversing one of the dee stem coupling loops. The frequency of the two modes of oscillation is approximately the same.[1]

In order to show that ions can be accelerated with the dees resonating in-phase, we consider a two-dee cyclotron as shown in Fig. 1. Each dee subtends an angle θ; the grounded dummy dees simply insure a proper accelerating field. We assume that the peak dee-to-ground potential, V_0, is the same for each dee. The angular frequency of the ions is Ω; the cyclotron oscillator frequency is ω. q is the charge of each ion. ϕ is the phase angle of the ion with respect to the oscillator voltage. It is defined as zero if the ion crosses the center of the dee as the dee voltage is zero and decreasing. The ion energy gain per turn is then given by

$$\Delta W=2qV_0\cos\left(\frac{\omega}{\Omega}\phi\right)\sin\left(\frac{\omega}{\Omega}\frac{\theta}{2}\right)\times\left[1+\cos\left(\frac{\omega}{\Omega}\pi+\psi\right)\right], \quad (1)$$

provided ω/Ω is nearly an integer. If ω/Ω were not an integer, the energy gain per turn will vary as $\cos[2m(\omega/\Omega)\pi]$, where m is the number of turns and acceleration would not result. ψ is zero or π corre-

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[1] J. Backus, Rev. Sci. Instr. **22**, 84 (1951); F. H. Schmidt and M. J. Jakobson, Rev. Sci. Instr. (to be published).

sponding to the in-phase or out-of-phase mode, respectively.

The term $\sin(\omega\theta/2\Omega)$ expresses the dependence of the energy gain per turn on the angle of the dees. $\mathrm{Cos}(\omega\phi/\Omega)$ expresses the dependence on the phase of the ion with respect to the accelerating voltage. ϕ can vary from $-(\Omega/\omega)(\pi/2)$ to $+(\Omega/\omega)(\pi/2)$ and back again. This, as we shall see later, leads to the same total limiting phase variations as in conventional cyclotrons. The first cos term indicates that the ions are more drastically bunched the higher the order of the harmonic, and the number of ion bunches per orbit increases as the order of the harmonic. The term in square brackets expresses the dependence of the energy gain per turn upon the relative phase of the two similar dees. For integral values of ω/Ω the square-bracket term bccomes two if the in-phase mode is chosen for ω/Ω even and the out-of-phase mode chosen for ω/Ω odd. The maximum energy gain per turn then becomes

$$\Delta W = 4qV_0 \left| \sin\left(\frac{\omega}{\Omega}\frac{\theta}{2}\right)\right|. \tag{2}$$

Hence, those ions for which ω/Ω is odd are accelerated in the out-of-phase mode and those ions for which ω/Ω is even are accelerated in the in-phase mode. Equation (2) shows that all ions for which the ratio of oscillator to ion angular frequency, ω/Ω, is an integer will have an energy gain per turn of $4qV_0$ modulated by $|\sin(\omega\theta/2\Omega)|$.

As an example, consider a double-mode cyclotron with two similar 120° dees. With the dee resonating in phase, ions for which $\omega/\Omega=2, 4, 8, 10, \cdots$ would be accelerated

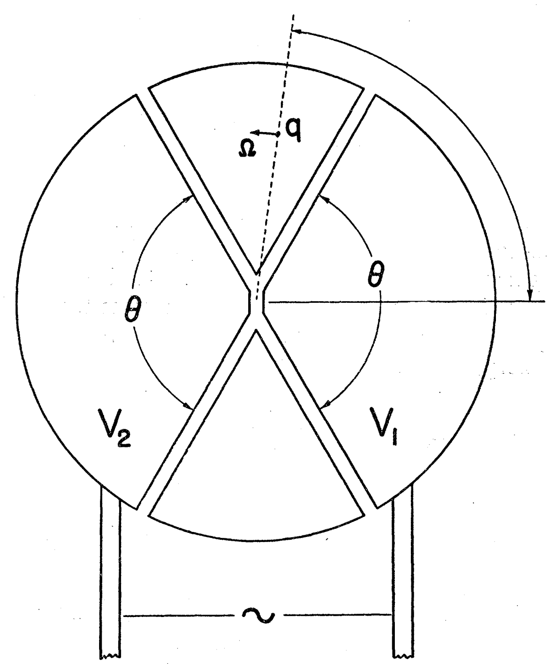

FIG. 1. A schematic of the dee electrodes for a double-mode cyclotron.

with a maximum energy gain per turn of $2\sqrt{3}qV_0$, as contrasted with $4qV_0$ for a conventional two-dee cyclotron. With the dees resonating 180° out of phase, ions for which $\omega/\Omega=1, 5, 7, 11, \cdots$ would be accelerated with the same energy gain per turn. Ions for which $\omega/\Omega=3, 6, 9, \cdots$ would not be accelerated.

If 90° dees were used instead of 120° dees, then ions for which $\omega/\Omega=2, 6, 10, \cdots$ would be accelerated in the in-phase mode with a maximum energy gain per turn of $4qV_0$, and ions for which $\omega/\Omega=1, 3, 5, 7, \cdots$ are accelerated in the out-of-phase mode with a maximum energy gain per turn of $(4/\sqrt{2})qV_0$. In this case, ions for which $\omega/\Omega=4, 8, 12, \cdots$ would not be accelerated. Note that a double-mode cyclotron is unique in that ions for which $\omega/\Omega=1$ or 2, (such as protons and deuterons) can be accelerated in the same machine without changing appreciably the oscillator frequency or magnetic field.

The above calculations have been made assuming a step-function change in potential at the edge of the dees. The fringing electrical field would not change the essential properties of the acceleration. Only the case where both dees have the same angle θ has been considered. If the dees were unequal in angle and less than 180°, electrical forces would make the ion orbits precess and thus tend to produce instability.

COMPARISON WITH A CONVENTIONAL CYCLOTRON

Some of the properties of an accelerator that must be considered from the standpoint of design and operation are (1) starting conditions for the ions, (2) orbit stability, (3) threshold voltage,[2] and (4) exciting power. It is instructive to compare a normal two-dee cyclotron with a double-mode cyclotron with 120° dees. Table I shows a comparison of these properties for the two accelerators.

It is not likely that difficulty would be experienced in starting ions in an accelerator with 120° dees. Starting conditions in the out-of-phase mode are similar to those in a conventional cyclotron. Starting conditions for the in-phase mode could be improved by inserting 60° grounded sectors between the 120° dees. This would provide larger potential gradients for the acceleration of the ions near the ion source.

The requirements for orbit stability are the same for either type accelerator. The magnetic induction must decrease with increasing radius to provide vertical focusing.

The threshold voltage is defined as the lowest dee voltage for which ions will be accelerated to the exit radius of the machine. As shown in Table I the threshold voltage will be slightly increased for a 120°-dee cyclotron accelerating ions *in the same harmonic* as a conventional cyclotron. This is due to the fact that the energy gain per turn is less. For the acceleration of deuterons in a double-mode cyclotron with 90° dees and $\omega=\Omega_{\text{proton}}$, the threshold voltage for corresponding energies is

[2] H. A. Bethe and M. E. Rose, Phys. Rev. 52, 1254 (1937).

twice that of a conventional cyclotron with $\omega=\Omega_{\text{deuteron}}$. The energy gain per turn is the same for either accelerator, but the deuterons are accelerated on a second harmonic for the 90°-dee cyclotron. The increase in threshold voltage is partially compensated for by the decrease in power required to excite the smaller angle dees. The exciting power, neglecting ion loading, is proportional to V_0^2/Z_s, where Z_s is the shunt impedance of the resonating dee electrodes. Since Z_s is doubled by decreasing the angular extent of the dees from 180° to 90°, an increase of $\sqrt{2}$ in dee voltage will result for a given power input. This indicates the same power is required to accelerate protons to a given radius in a 90°-dee cyclotron as in a 180°-dee cyclotron. The problem of maintaining higher dee voltages arises with the smaller angle dees since the threshold voltage is higher.

POSSIBLE APPLICATIONS

One of the possible uses of a double-mode cyclotron would be the acceleration of ions heavier than alpha particles. The principal difficulty encountered in producing large beams of such heavy ions is the necessity of stripping inner electrons from the atoms. The present method of obtaining ${}_6C^{12(6+)}$ ion beams relies on a sequence of acceleration that requires harmonic acceleration of large beams of low-energy ions, some of which are stripped and accelerated to higher energies.[3]

[3] J. Miller, Ph.D. thesis, University of California, 1952 (unpublished).

TABLE I. Comparison of normal two-dee cyclotron with 120° dee double-mode cyclotron.

Property	Normal two-dee cyclotron with 180° dees $\omega=\Omega_{\text{proton}}$	Double-mode cyclotron with 120° dees $\omega=\Omega_{\text{proton}}$ In-phase mode	Out-of-phase mode
Ions accelerated	ω/Ω odd	ω/Ω even except 6, 12, ⋯	ω/Ω odd except 3, 9, 15, ⋯
ΔW = energy gain	$4qV_0$	$2\sqrt{3}qV_0$	$2\sqrt{3}qV_0$
Starting conditions	Normal	Probably requires dummy dees	Normal
Orbit stability	Stable	Stable (1-dee cyclotron unstable)	Stable (1-dee cyclotron unstable)
Threshold voltage	V_0	$V_0/0.866$	$V_0/0.866$
Exciting power for threshold voltage	P	(8/9)P	(8/9)P

The principle advantage of a double-mode cyclotron for the purpose of heavy-ion acceleration would be that it permits the harmonic acceleration of a larger variety of ions. As an example, consider the acceleration of ${}_8O^{16}$. A 120° double-mode cyclotron could accelerate ${}_8O^{16(++)}$ $(\omega/\Omega=8)$. Some of these ions would be stripped and accelerated as ${}_8O^{16(8+)}$. The stripping could possibly be enhanced by a discharge maintained within one of the grounded dummy dees.

Another application would be to a medium-energy cyclotron for the purpose of accelerating protons and deuterons without changing the oscillator frequency or magnetic field. The exciting power required for a 90°-dee double-mode cyclotron would be the same as for a 180°-dee cyclotron of comparable size.