Some *p*- and *d*-Shell Nuclei in Intermediate Coupling

G. E. TAUBER* AND TA-YOU WU
National Research Council, Ottawa, Canada

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The theory of intermediate coupling has been applied to nuclei having either two particles or "holes" in the first p or d shell, such as He^6, Li^6, N^{14}, C^{14}, O^{18}, F^{18}, Ca^{38}, and A^{38}. A central nucleon-nucleon interaction of the form $(mP+nQ)V(r_{12})$, where P denotes the Majorana and Q the Bartlett operator, is assumed and calculations have been carried out for the exponential, Yukawa, and Gaussian types of potential $V(r_{12})$ with various "ranges." The coefficients m, n and the constants of the potential $V(r_{12})$ have been estimated from data for two free nucleons.

The secular equations have been solved and thus the degeneracy in the energy levels in the two extreme types of couplings has been removed. The energy levels of various spins for the various types of interactions and "ranges" have been plotted against the spin-orbit interaction parameter ζ. The range of ζ giving the correct spin for the ground state is found to depend on the type and "range" of $V(r_{12})$. The magnetic moments for these nuclei can also be obtained with the appropriate wave function for intermediate coupling. It is found that in the case of Li^6, a Yukawa potential having a "range" of about 1.0×10^{-13} cm accounts for both known excited states and gives a magnetic moment of $0.87\mu_N$. For N^{14}, the calculated magnetic moment is $0.36\mu_N$.

I. INTRODUCTION

THE empirical value found for the spin of the ground state of the Li^6 nucleus is one of the notable exceptions to the signal success of the nuclear shell model of Mayer *et al.*[1] According to it the nuclear configuration of Li^6 is $(1s)^2{}_P(1p)_P(1s)^2{}_N(1p)_N$,[2] which, in the extreme jj coupling has as the lowest state a group of degenerate levels with spin $I=3$, 2, 1, and 0. From symmetry considerations Feenberg[3] concludes that the spin of the lowest state for Li^6 is 3 while the observed value is 1.

To understand the observed spin of the ground states of Li^6, Inglis[4] has investigated the positions of the various levels of the configurations $(p)_P(p)_N$ for arbitrary couplings between the LS and jj limits. The purpose of the present paper is to extend the investigation of Inglis to see whether, by assuming nucleon-nucleon interactions of various forms and ranges and various spin-orbit interactions, it is possible to account for the two known excited states of Li^6,[5] as well as for $I=1$ of the ground state, and to calculate the magnetic moment of the ground state and also to apply the method to the configuration $(d)_P(d)_N$.

II. THE SECULAR EQUATION

The Hamiltonian of a many-nucleon nucleus may be assumed to be

$$H=\sum_i H(r_i)+\sum_{i\neq j} V_{ij}(r_{ij})+\sum_i H_i(\mathbf{l}_i\cdot\mathbf{s}_i), \quad (1)$$

where $H(r_i)$ is the one-particle Hamiltonian of a nucleus in a central field, V_{ij} the nucleon-nucleon interaction (between nucleons outside the complete shells) which is in addition to the part represented by the central field, and $H_i(\mathbf{l}_i\cdot\mathbf{s}_i)$ is the spin-orbit interaction. According as

$$V_{ij}\gg H_i \quad (2)$$

or

$$V_{ij}\ll H_i, \quad (3)$$

one has the limiting cases of LS coupling and jj coupling, respectively. Appropriate to these cases, one may conveniently use the LS representation and the JM representation. In an intermediate coupling scheme, one may express the wave function of the system in terms of either representation, and obtain the energy values from the corresponding secular equation.[6] If the coupling is known to be close to either limit, the eigenvalues and eigenfunctions may be calculated by the perturbation method. Thus if (2) is nearly satisfied, one may employ the LS representation in which the matrix of $\Sigma H(r_i)+\Sigma V_{ij}(r_{ij})$, where V_{ij} is central and does not contain tensor forces, is diagonal and $\Sigma H_i(l_i\cdot s_i)$ in general non-diagonal. Similarly, if the coupling is close to the jj extreme, i.e., (3) is nearly satisfied, the JM representation is used, where the matrix of $\Sigma H_i(l_i\cdot s_i)$ is diagonal and V_{ij} not.

For the configuration of a p proton and p neutron outside closed shells, the 36 states can be grouped in the (LS) limiting case into the levels

$$^{31}D_2,\ ^{33}P_{0,1,2},\ ^{31}S_0,\ ^{13}D_{1,2,3},\ ^{11}P_1,\ ^{13}S_1. \quad (4)$$

For two identical nucleons, the singlet isotopic spin states are excluded by the Pauli principle. The matrix elements of $l_i\cdot s_i$ in this representation are known[7] and the secular equations obtained. These, together with the other cases considered in this section, are given in Appendix I. In the (jj) limiting case, on the other hand,

* National Research Laboratories Postdoctorate Fellow.
[1] M. G. Mayer, Phys. Rev. **74**, 235 (1948); **75**, 1969 (1949); **78**, 16 (1950); Haxel, Jensen, and Suess, Phys. Rev. **75**, 1766 (1949).
[2] P. F. A. Klinkenberg, Revs. Modern Phys. **24**, 63 (1952).
[3] E. Feenberg, Phys. Rev. **76**, 1275 (1949).
[4] D. R. Inglis, Phys. Rev. **87**, 915 (1952); Revs. Modern Phys. **25**, 390 (1953).
[5] F. Ajzenberg and T. Lauritsen, Revs. Modern Phys. **24**, 321 (1952).
[6] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1951), Chap. XI.
[7] See reference 6, Table I, p. 268.

one obtains the following levels in ascending order:

$$\begin{aligned} &(\tfrac{3}{2}\ \tfrac{3}{2})_{3,2,1,0}: && \zeta, \\ &(\tfrac{3}{2}\ \tfrac{1}{2})_{2,1},\quad (\tfrac{1}{2}\ \tfrac{3}{2})_{2,1}: && -\tfrac{1}{2}\zeta, \\ &(\tfrac{1}{2}\ \tfrac{1}{2})_{1,0}: && -2\zeta. \end{aligned} \tag{5}$$

Here the spin-orbit interaction parameter ζ is defined by[8]

$$\zeta = \hbar^2 \int_0^\infty R_{nl}^2(r)\xi(r)dr$$

and assumed to be equal for neutrons and protons. (It is negative for a particle and positive for a "hole.") The appropriate matrix elements of the nucleon-nucleon interaction in this representation are conveniently obtained by transforming the diagonal levels (4) into the JM representation.[9]

For the configuration of a d proton and d neutron outside a closed shell one obtains similarly the following levels in the LS limiting case:

$$\begin{aligned} &{}^{31}G_4,\ {}^{31}D_2,\ {}^{31}S_0,\ {}^{33}F_{2,3,4},\ {}^{33}P_{0,1,2}, \\ &{}^{13}G_{3,4,5},\ {}^{13}D_{1,2,3},\ {}^{13}S_1,\ {}^{11}F_3,\ {}^{11}P_1; \end{aligned} \tag{4a}$$

and in the jj limit again

$$\begin{aligned} &(5/2\ \ 5/2)_{5,4,3,2,1,0}: && 2\zeta, \\ &(5/2\ \ 3/2)_{4,3,2,1},\quad (3/2\ \ 5/2)_{4,3,2,1}: && -\tfrac{1}{2}\zeta, \\ &(3/2\ \ 3/2)_{3,2,1,0}: && -3\zeta. \end{aligned} \tag{5a}$$

III. CALCULATION OF ENERGY LEVELS IN THE LS LIMIT

A general nucleon-nucleon interaction of the form

$$V_{12} = (mP+nQ)V(|\mathbf{r}_1-\mathbf{r}_2|) \tag{6}$$

has been assumed, where P and Q are the Majorana and Bartlett operators, respectively, and V a central potential. For any two-particle problem the elements of the energy matrix can be readily calculated from trace invariance with the help of the wave functions of the individual two-particle states (Appendix III). Furthermore, as we are dealing only with central interaction, the potential can be expanded in a series of Legendre polynomials[10]

$$V(|\mathbf{r}_1-\mathbf{r}_2|) = \sum_{k=0}^{\infty} f_k(r_1, r_2)P_k(\cos\omega_{12}), \tag{7}$$

and the resulting interaction expressed in terms of Slater integrals defined by

$$F^k(n, n') = \int_0^\infty \int_0^\infty R_n^2(r_1)R_{n'}^2(r_2)f_k(r_1, r_2)dr_1dr_2. \tag{8}$$

The energies of the various states in the LS limit for the p-p and d-d configurations are given in Table I.

TABLE I.

State	$mP+nQ$
Energy states for two p nucleons in the LS limit	
${}^3D$	$(m+n)(F_0+F_2)$
${}^1D$	$(m-n)(F_0+F_2)$
${}^3P$	$(m-n)(-F_0+5F_2)$
${}^1P$	$(m+n)(-F_0+5F_2)$
${}^3S$	$(m+n)(F_0+10F_2)$
${}^1S$	$(m-n)(F_0+10F_2)$
Energy states for two d nucleons in the LS limit	
${}^3G$	$(m+n)(F_0+4F_2+F_4)$
${}^1G$	$(m-n)(F_0+4F_2+F_4)$
${}^3F$	$(m-n)(-F_0+8F_2+9F_4)$
${}^1F$	$(m+n)(-F_0+8F_2+9F_4)$
${}^3D$	$(m+n)(F_0+3F_2+36F_4)$
${}^1D$	$(m-n)(F_0-3F_2+36F_4)$
${}^3P$	$(m-n)(-F_0-7F_2+84F_4)$
${}^1P$	$(m+n)(-F_0-7F_2+84F_4)$
${}^3S$	$(m+n)(F_0+14F_2+126F_4)$
${}^1S$	$(m-n)(F_0+14F_2+126F_4)$

Talmi[11] has shown that it is possible to use harmonic-oscillator wave functions and carry out the integration for various types of potentials. In terms of his integrals I_l,

$$I_l = N_l^2 \int_0^\infty \exp(-\nu r^2)r^{2l+2}V(r)dr, \tag{9}$$

the F's are given as follows:

p-p nucleon:[12]

$$\begin{aligned} F_0 &= [5(I_0+I_2)+2I_1]/12, \quad F_0=F^0, \\ F_2 &= [(I_0+I_2)-2I_1]/12, \quad F_2=F^2/25; \end{aligned} \tag{10}$$

d-d nucleon:

$$\begin{aligned} 15F_0 &= (63/16)(I_0+I_4)+(7/4)(I_1+I_3)+(29/8)I_2, \\ &\qquad F_0=F^0; \\ 21F_2 &= (9/16)(I_0+I_4)-(1/2)(I_1+I_3)-(1/8)I_2, \\ &\qquad F_2=F^2/49; \\ 35F_4 &= (3/16)(I_0+I_4)-(3/4)(I_1+I_3)+(9/8)I_2, \\ &\qquad F_4=F^4/441. \end{aligned} \tag{11}$$

Here ν is the constant in the central, simple harmonic potential $V(r)=\nu^2\hbar^2r^2/2m$ and is related to the extension $1/(\nu)^{\frac{1}{2}}$ of the harmonic oscillator wave function which, from results of mirror nuclei, has been taken to be $1/(\nu)^{\frac{1}{2}}=2.4\times10^{-13}$ cm by Talmi.[11]

In order to make any explicit calculations it is necessary to specify the potential. We shall consider the

[8] See reference 6, p. 122.

[9] See reference 6, Table I, p. 294.

[10] J. C. Slater, Phys. Rev. **34**, 1293 (1929).

[11] I. Talmi, Helv. Phys. Acta **25**, 185 (1952).

[12] The F_0 and F_2 are related to the K, L integrals of E. Feenberg and M. Phillips [Phys. Rev. **51**, 597 (1937)] through $L=F_0+F_2$, $K=3F_2$.

following types of potentials:

$$\text{(i) Exponential} \quad V(r)=V_0{}^E e^{-r/r_0},$$

$$\text{(ii) Yukawa} \quad V(r)=V_0{}^Y e^{-r/r_0}(r_0/r), \tag{12}$$

$$\text{(iii) Gaussian} \quad V(r)=V_0{}^G \exp(-r^2/r_0^2),$$

where V_0 is the depth and r_0 the "range" of the potential. The explicit expressions for the integrals I_n for the cases (ii) and (iii) have been given by Talmi,[11] while the corresponding expressions for the case (i) are given in Appendix II.

Concerning the nucleon-nucleon potential (6), in the absence of sufficient knowledge, we shall assume that it is comparable with that between two nucleons unbound in a nucleus. On this assumption, the constants m and n and the depth of the potential V_0 have been obtained from the data on slow neutron scattering at low energies, namely the effective range $=1.72\times10^{-13}$, 2.7×10^{-13} cm, for the triplet and singlet state, respectively.[13] For a ^3S state the interaction (6) simply becomes

TABLE II. Constants of potential V in (6) and (10).

Type of potential and values of r_0 (cm)	$(m-n)V_0$ (Mev)	$(m+n)V_0$ (Mev)
	(i) Exponential	
$r_0=1.0\times10^{-13}$	-57.63	-109.54
$r_0=1.4\times10^{-13}$	-29.28	-57.95
$r_0=2.0\times10^{-13}$	-14.24	-28.62
	(ii) Yukawa	
$r_0=1.0\times10^{-13}$	-65.69	-117.20
$r_0=1.4\times10^{-13}$	-32.27	-59.67
$r_0=2.0\times10^{-13}$	-15.34	-29.24
	(iii) Gaussian	
$r_0=1.0\times10^{-13}$	-111.89	-138.0
$r_0=1.4\times10^{-13}$	-55.82	-74.8
$r_0=2.0\times10^{-13}$	-26.73	-42.8

$$(m+n)V, \tag{13}$$

while for the singlet ^1S, it is

$$(m-n)V. \tag{13a}$$

Only these two combinations enter into the expressions for the energy levels. Applying the method of Wu and Foley[14] to the potentials (10), the values of $(m+n)V$ and $(m-n)V$ for the various types of potentials and "ranges" can be obtained. These are given in Table II.

With these values the Talmi integrals I_l for the various interactions and ranges are calculated, and the energies of the various states (Table I) occurring in the p-p and d-d configurations can be obtained by inserting the appropriate expressions for the integrals F's.

[13] E. E. Salpeter, Phys. Rev. **82**, 60 (1951).

[14] T.-Y. Wu and H. M. Foley, Phys. Rev. **75**, 1681 (1949).

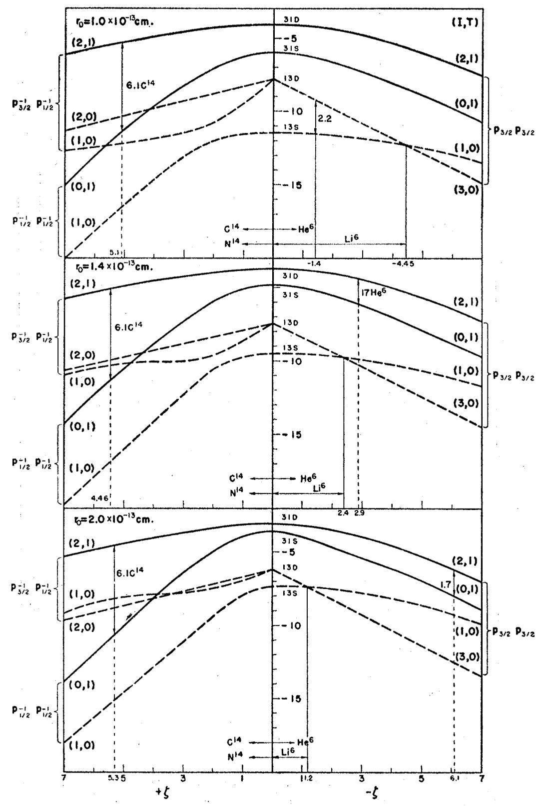

FIG. 1. Energy levels for two p nucleons with exponential potential and various "ranges" indicated on top of each section. Only low-lying energy levels have been included. Curves in full correspond to levels with $T=1$, while dashed curves give levels with $T=0$. Asymptotic behavior is indicated in the margin. Scale: 1 Mev per division for energy levels and spin-orbit interaction. Horizontal arrows indicate range of values of ζ for which the spin of the ground state agrees with the observed spin. The vertical arrows indicate the value of the parameter ζ for which the known excited states are obtained. In all these figures, $31G$ stands for ^{31}G, etc.

IV. ENERGY LEVELS IN INTERMEDIATE COUPLING

To obtain the energies for any intermediate coupling between **LS** and **jj**, the secular equations (given in Appendix I) are solved with the values of the F integrals already obtained, and the energy levels are obtained as a function of the spin-orbit parameter ζ. For three values for the range parameter r_0 in each of the three potentials in (12), the equations have been solved numerically and the energies thus calculated are plotted in Figs. 1–6 as functions of the spin-orbit parameter and the various potentials and ranges considered. In these figures, the curves in dashes represent those states antisymmetric in isotopic spin and hence excluded by the Pauli principle for two identical nucleons.

For a negative value of the spin-orbit parameter ζ one obtains all members of the polyad[15] having two nucleons outside closed shells, while for positive values of ζ the

[15] This expression is due to Inglis[4] and denotes all nuclei having a specified value of A, e.g. Py^6 includes Li^6, He^6, and Be^6 if it existed.

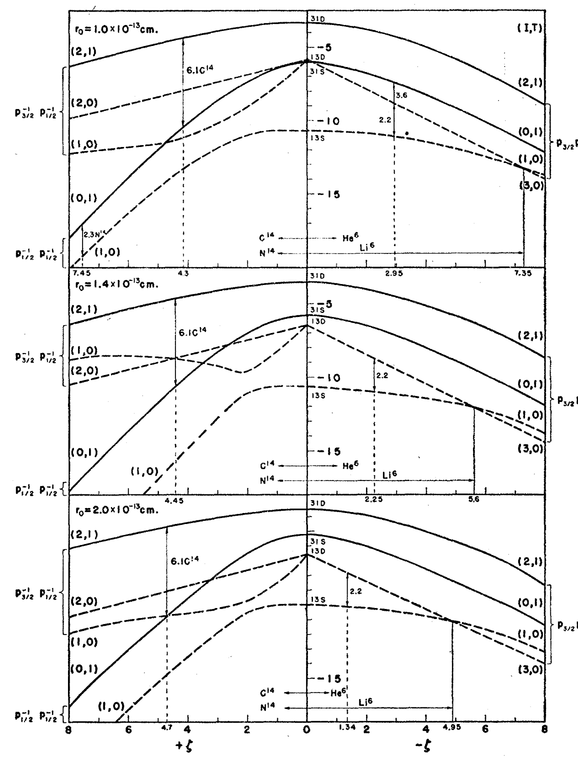

FIG. 2. Energy levels for two p nucleons with Yukawa potential and various "ranges" indicated on top of each section. For notation and scale see Fig. 1.

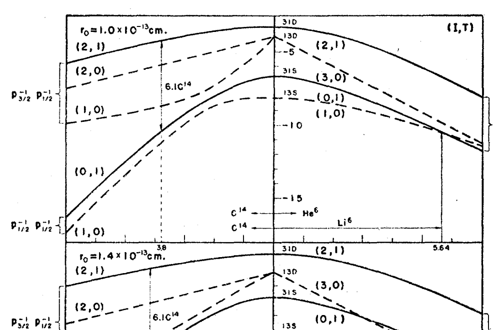

FIG. 3. Energy levels for two p nucleons with Gaussian potential and various "ranges" indicated on top of each section. For notation and scale see Fig. 1.

corresponding polyad with two "holes" is obtained. Thus the curves for $\zeta<0$ in Figs. 1–3 are applicable to He^6, Li^6, and Be^6, having the configuration $(p_N)^2$, p_Np_P, $(p_P)^2$, respectively, while the portions $\zeta>0$ of the curves are applicable to O^{14}, N^{14}, C^{14}, having the configuration $(p_N)^{-2}$, $(p_N)^{-1}(p_P)^{-1}$, $(p_P)^{-2}$, respectively. The range of values of ζ for which the spin of the ground state agrees with the observed spin is indicated by a horizontal line in the figures. Wherever possible the known excited states are also marked in the figures. In nearly all these cases it is possible to find a value for the spin-orbit parameter ζ depending on the potential and "range." For this range the spin of the ground state agrees with

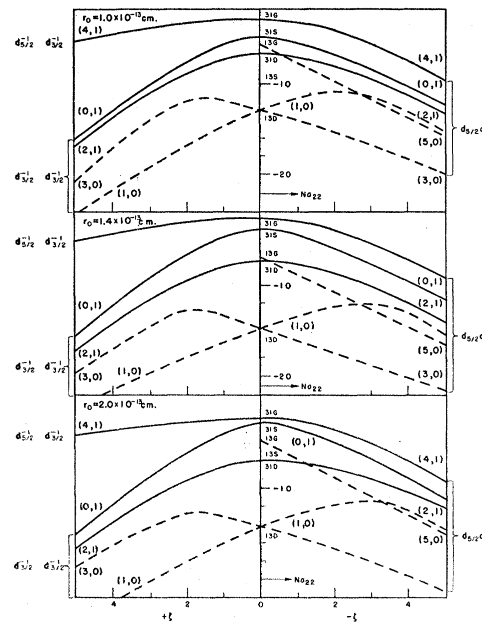

FIG. 4. Energy levels for two d nucleons with exponential potential and various "ranges" indicated on top of each section. Only low-lying energy levels have been included. Curves in full correspond to levels with $T=1$, while dashed curves give levels with $T=0$. Asymptotic behavior is indicated in the margin. Scale: 2 Mev per division for energy levels and 1 Mev per division for spin-orbit interaction. Horizontal arrows indicate range of values of ζ for which the spin of the ground state agrees with the observed spin. The vertical arrows indicate the values of the parameter ζ for which the known excited states are obtained.

that observed. It is interesting to note that in the case of the Yukawa potential (Fig. 2) values of $\zeta=-3$ and $r_0=1.0\times10^{-13}$ cm give both known excited states of Li^6 [3.58 Mev ($I=0$) and 2.2 Mev ($I=3$)[16] above ground][5] as well as the spin of the lowest state. (This is also approximately true with the Gaussian potential with $r_0=2\times10^{-13}$ cm. See Fig. 3.) This may indicate that this interaction and range seem to be favored over the others considered. It is also possible to obtain the

[16] According to the latest experimental results reported at the Second Conference of Medium Energy Nuclear Physics at the University of Pittsburgh, 1953, it seems to be established that the observed value of I for this level is indeed $I=3$ as predicted by the shell model.

energy levels of another member of the Py^6, *viz.*, He^6, but unfortunately not with the same interaction, as seen from Figs. 1 and 2.

In the case of nuclei with two holes in the p shell, namely, N^{14} and C^{14}, the spin of the ground state is correctly given for all values of ζ. For N^{14}, only the Yukawa potential with $r_0 = 1.0 \times 10^{-13}$ cm can fit the first excited state at 2.3 Mev with $I=0$ and $T=1$ with a value $\zeta \sim 7.45$ Mev. While no great significance should be attached to this matching of one excited state alone, it is nevertheless of interest to note that the same potential also accounts for three states of Li^6 but with a smaller value of $|\zeta|$. That N^{14} corresponds to a larger spin-orbit coupling has already been pointed out by Inglis.[4] Further discussion of the coupling in N^{14} will be given in Sec. V below.

In the case of C^{14}, it has been found not possible to fit the observed level at 4.1 Mev with $I=0$, since with all the potentials and range parameter r_0 considered, the other state $I=0$ of the p^2 configuration always lies more than 4 Mev above the ground state $I=0$.[17] It is possible, however, to account for the level observed at 6.11 Mev if its spin is 2, with a value of about 5 Mev in all the potentials considered.

In Figs. 4–6 the portions $\zeta<0$ of the curves are applicable to $O^{18}(d_N{}^2)F^{18}(d_Nd_P)Ne^{18}(d_P{}^2)$, while the $\zeta>0$ portions are applicable to $Ca^{38}(d_N{}^{-2})$, $K^{38}(d_N{}^{-1}d_P{}^{-1})$, $A^{38}(d_P{}^{-2})$.

It is seen that only for the Yukawa potential with $r_0 = 1.0 \times 10^{-13}$ cm and the Gaussian potential with $r_0 = 1.0$ and 1.4×10^{-13} cm is it possible to obtain a range of values of ζ for which the ground state has spin $I=1$ as is required for F^{18}. It is interesting to note that in these cases the ranges which give the correct spin for I for the ground state for both F^{18} and O^{18} overlap, so that it is possible to account for both these cases by assuming an intermediate coupling with about the same value for ζ. This shows that the observed value $I=0$ for O^{18} does not necessarily have to be taken to suggest an extreme j-j coupling. Moreover, the Gaussian potential with $r_0 = 1.0 \times 10^{-13}$ cm seems to be preferred, for it is possible also to match the excited state (1.05 Mev and $I=0$).[18] We have also included Na^{22}, which has a ground state spin $I=3$ and, according to the older picture,[19] the configuration $(1s)^2 \cdot (1p)^6 (1d)(2s)^2$ for the protons and a similar configuration for the neutrons.[2] In the case of A^{38} it is possible to obtain both the ground-state spin $I=0$ and excited state $I=2$, but not the second excited state $I=3$ (Fig. 6). The parity of that state is not known, however, and would be due to a different configuration if it is odd.

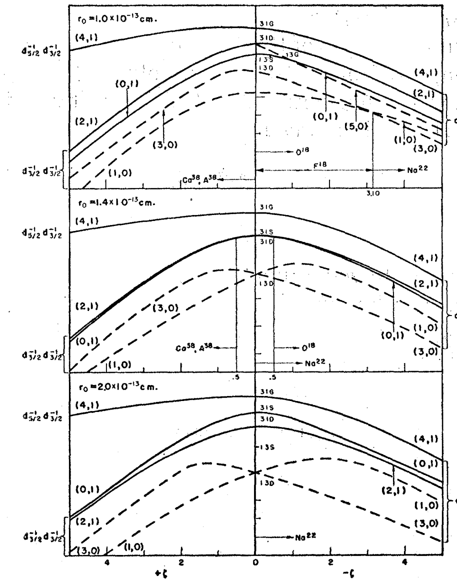

FIG. 5. Energy levels for two d nucleons with Yukawa potential and various "ranges" indicated on top of each section. For notation and scale see Fig. 4.

V. MAGNETIC MOMENT OF Li^6, N^{14}

The magnetic moment is defined as the expectation value of the operator

$$\mu = \sum_{i=P,N} (m_l{}^i g_l{}^i + m_s{}^i g_s{}^i)\mu_N, \qquad (14)$$

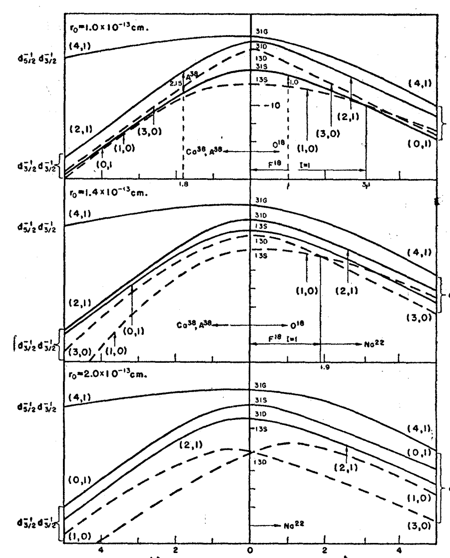

FIG. 6. Energy levels for two d nucleons with Gaussian potential and various "ranges" indicated on top of each section. For notation and scale see Fig. 4.

[17] It is possible that the low observed position of the level at 4.1 Mev may be due to the result of strong configuration interaction, or may arise from another configuration altogether, since the parity of that state is not definitely known.

[18] The value $I=0$ has been obtained by matching it against the ground state of O^{18} in the figure of reference 5.

[19] This assumption arose from early thoughts regarding F^{19} but is doubtful in view of results concerning O^{17} and Na^{23} [see A. R. Edmonds and B. H. Flowers, Proc. Roy. Soc. (London) **A215**, 128 (1952), who have considered d particles with a Gaussian interaction]. According to present belief the configuration of Na^{22} has three protons and three neutrons in the d shell.

where m_l^i and m_s^i are the orbital and spin angular momentum operators of the nucleon, respectively, ($i=P$ for proton, $i=N$ for neutron in our notation) and g_l^i and g_s^i are the "gyromagnetic ratios" of the orbital and spin, respectively, and are given by

$$g_l^P=1, \quad g_l^N=0, \qquad g_s^P=5.58, \quad g_s^N=-3.82. \tag{15}$$

The appropriate wave function is found from the solution of the secular equation. For the lowest state of spin 1 from two p nucleons, it is a linear combination of the (zero-order) wave functions corresponding to the states ${}^3S_1$, ${}^3D_1$, and ${}^1P_1$ and can be written as

$$\Psi(1,1)=a\psi({}^1P_1)+b\psi({}^3D_1)+c\psi({}^3S_1), \tag{16}$$

where the coefficients are obtained from the appropriate solution of the secular equation for $J=1$, and satisfy the requirement $a^2+b^2+c^2=1$ as each individual wave function is already normalized. The (zero-order) wave functions are obtained by properly combining the states corresponding to the various values of M_l and M_s into multiplets. The multiplets and wave functions are given in Appendix III.

Applying the operator μ (14) to the wave function, one obtains for the magnetic moment

$$\langle\mu\rangle=(0.5a^2+0.31b^2+0.88c^2)\mu_N. \tag{17}$$

Thus, for Li^6, the value of $\zeta=-3$ for the Yukawa potential with $r_0=1.0\times10^{-13}$ cm (which was used in order to fit both excited levels) gives then for the magnetic moment the value $\mu=0.866\mu_N$, in better agreement with the observed value $0.822\mu_N$ than the value $0.63\mu_N$ calculated for the state $I=1$ in pure jj coupling.[20] It might also be pointed out that the magnetic moment is not very sensitive to the exact value of ζ and would yield a similar result for values of ζ in the immediate neighborhood of $\zeta=-3$. Similarily for N^{14}, the value of $\zeta=7.45$ for the Yukawa potential with $r_0=1.0\times10^{-13}$ cm gives then for the magnetic moment the value $\mu=0.355\mu_N$, which is close to the value 0.37 calculated in pure jj coupling,[5] while the observed value is $0.40\mu_N{}^2$.

VI. CONCLUSION

Although the above calculations are of a somewhat exploratory nature and are based on various assumptions of nuclear interactions, they nevertheless show definitely that the order of energy levels with various I depend not only on the coupling (LS—intermediate—jj), but also on the nucleon-nucleon interaction (exchange properties, general form, and "range"). Thus, to find the correct coupling to account for the spin and the magnetic moment of the ground state of a nucleus, it is necessary to study the situation for various types of nucleon-nucleon interactions. The present work also seems to suggest that a nucleus having only two nucleons outside closed shells is much closer to LS coupling than to jj coupling. This is definitely demonstrated in the case of Li^6, where the assumption of jj coupling leads to wrong results for the spin of the ground state and gives not as good agreement with the experimental value of the magnetic moment as LS coupling.

At present work is being carried out to investigate nuclei with half-closed shells (such as B^{10}) in order to obtain more information concerning the relation between the coupling scheme and the number of nucleons outside closed shells.

We wish to express our thanks to Miss E. Motard for carrying out the numerical solutions of the secular equations involved. One of us (G.E.T.) also wishes to express his appreciation to the National Research Council of Canada for a postdoctorate fellowship which made this work possible.

APPENDIX I. SECULAR EQUATIONS

(1) Two p Nucleons

(a) States antisymmetric in the space and spin of the two nucleons: $T=1$.

$$I=2:\quad \begin{vmatrix} {}^1D-E & \frac{1}{2}\sqrt{2}\zeta \\ \frac{1}{2}\sqrt{2}\zeta & {}^3P+\frac{1}{2}\zeta-E \end{vmatrix}=0,$$

$$I=1:\quad {}^3P+\tfrac{1}{2}\zeta-E=0,$$

$$I=0:\quad \begin{vmatrix} {}^3P-\zeta-E & -\sqrt{2}\zeta \\ -\sqrt{2}\zeta & {}^1S-E \end{vmatrix}=0.$$

(b) States symmetric in the space and spin of the two nucleons: $T=0$.

$$I=3:\quad {}^3D+\zeta-E=0,$$

$$I=2:\quad {}^3D-\tfrac{1}{2}\zeta-E=0,$$

$$I=1:\quad \begin{vmatrix} {}^3D-\frac{3}{2}\zeta-E & -(5/6)^{\frac{1}{2}}\zeta & 0 \\ -(5/6)^{\frac{1}{2}}\zeta & {}^1P-E & (\frac{2}{3})^{\frac{1}{2}}\zeta \\ 0 & (\frac{2}{3})^{\frac{1}{2}}\zeta & {}^3S-E \end{vmatrix}=0,$$

where ${}^1D$ stands for the integral in Table I, etc.

(2) Two d Nucleons

(a) States antisymmetric in the space and spin of the two nucleons: $T=1$.

$$I=4:\quad \begin{vmatrix} {}^1G-E & \zeta \\ \zeta & {}^3F+\frac{3}{2}\zeta-E \end{vmatrix}=0,$$

$$I=3:\quad {}^3F-\tfrac{1}{2}\zeta-E=0,$$

$$I=2:\quad \begin{vmatrix} {}^3F-2\zeta-E & -(12/5)^{\frac{1}{2}}\zeta & 0 \\ -(12/5)^{\frac{1}{2}}\zeta & {}^1D-E & (21/10)^{\frac{1}{2}}\zeta \\ 0 & (21/10)^{\frac{1}{2}}\zeta & {}^3P-\frac{1}{2}\zeta-E \end{vmatrix}=0,$$

[20] I. Talmi, Phys. Rev. 83, 1248 (1951).

$I=1$: $^3P-\frac{1}{2}\zeta-E=0$,

$$I=0:\quad \begin{vmatrix} ^3P-\zeta-E & -\zeta\sqrt{6} \\ -\zeta\sqrt{6} & ^1S-E \end{vmatrix}=0.$$

(b) States symmetric in the space and spin of the two nucleons: $T=0$.

$I=5$: $^3G+2\zeta-E=0$,

$I=4$: $^3G-\frac{1}{2}\zeta-E=0$,

$$I=3:\quad \begin{vmatrix} ^3G-(5/2)\zeta-E & -(9/7)^{\frac{1}{2}}\zeta & 0 \\ -(9/7)^{\frac{1}{2}}\zeta & ^1F-E & (12/7)^{\frac{1}{2}}\zeta \\ 0 & (12/7)^{\frac{1}{2}}\zeta & ^3D+\zeta-E \end{vmatrix}=0,$$

$I=2$: $^3D-\frac{1}{2}\zeta-E=0$,

$$I=1:\quad \begin{vmatrix} ^3D-\frac{3}{2}\zeta-E & -(7/2)^{\frac{1}{2}}\zeta & 0 \\ -(7/2)^{\frac{1}{2}}\zeta & ^1P-E & \sqrt{2}\zeta \\ 0 & \sqrt{2}\zeta & ^3S-E \end{vmatrix}=0,$$

where the ^1G, ^3F, etc., are given in Table I.

APPENDIX II. TALMI INTEGRALS

I for the exponential potential (i) in (10):

$$I_0=2V_0[(1-\phi)(\mu^2+\tfrac{1}{2})\exp(\mu^2)-\mu/\sqrt{\pi}],$$

$$I_1=\frac{4}{3}V_0[(1-\phi)(\mu^4+3\mu^2+\tfrac{3}{4})\exp(\mu^2)-\mu(\mu^2+5/2)/\sqrt{\pi}],$$

$$I_2=\frac{8}{15}V_0\left[(1-\phi)\left(\mu^6+\frac{15}{2}\mu^4+\frac{45}{4}\mu^2+\frac{15}{8}\right)\exp(\mu^2)-\mu\left(\mu^4+7\mu^2+\frac{33}{4}\right)\Big/\sqrt{\pi}\right],$$

$$I_3=\frac{16}{105}V_0\left[(1-\phi)\left(\mu^8+14\mu^6+\frac{105}{2}\mu^4+\frac{105}{2}\mu^2+\frac{105}{16}\right)\times\exp(\mu)^2-\mu\left(\mu^6+\frac{27}{2}\mu^4+\frac{183}{4}\mu^2+\frac{279}{8}\right)\Big/\sqrt{\pi}\right],$$

$$I_4=\frac{32}{945}V_0\left[(1-\phi)\left(\mu^{10}+\frac{45}{2}\mu^8+\frac{315}{2}\mu^6+\frac{1575}{4}\mu^4+\frac{4725}{16}\mu^2+\frac{945}{32}\right)\exp(\mu)^2-\mu\left(\mu^8+22\mu^6+147\mu^4+330\mu^2+\frac{2895}{16}\right)\Big/\sqrt{\pi}\right],$$

where ϕ is the error function

$$\phi(\mu)=\frac{2}{\sqrt{\pi}}\int_0^\mu \exp(-t^2)dt,$$

and

$$\mu=1/2\nu^{\frac{1}{2}}r_0.$$

APPENDIX III. WAVE FUNCTIONS FOR TWO *p* AND *d* NUCLEONS

(1) *p* Nucleons

There are 36 wave functions corresponding to the various values of m_l and m_s which can be grouped together in states having a definite of M_L and M_S. These form various multiplets shown in (4). The corresponding wave functions are linear combinations of the above, having a definite value of L and S and symmetry. They can be found by direct diagonalization or by a method of Gray and Wills[21] using the operators $\mathfrak{L}_\pm=L_x\pm iL_y$ and $\mathfrak{S}_\pm=S_x\pm iS_y$. The wave functions required are those linear combinations which have $J=1$, $M=1$ and occur in (16) and are given below:[22]

$$\psi(^3S_1)=\frac{1}{\sqrt{3}}[(1^+,-1^+)+(-1^+,1^+)-(0^+,0^+)],$$

$$\psi(^1P_1)=\tfrac{1}{2}[(1^+,0^-)-(1^-,0^+)-(0^+,1^-)+(0^-,1^+)],$$

$$\psi(^3D_1)=\frac{1}{\sqrt{10}}\left\{-\frac{\sqrt{3}}{2}[(1^+,0^-)+(0^+,1^-)+(1^-,0^+)+(0^-,1^+)]+\frac{1}{\sqrt{6}}[(1^+,-1^+)+2(0^+,0^+)+(-1^+,1^+)]+\sqrt{6}(1^-,1^-)\right\}.$$

(2) *d* Nucleons

In this case there are 100 wave functions corresponding to the various values of m_l and m_s. The multiplets formed are those shown in (4a). The corresponding wave functions are again linear combinations of the above and are found by the method indicated. The final wave functions corresponding to $J=3$ and $M=3$ are given below:

$$\psi(^3D_3)=\frac{1}{\sqrt{14}}[2(2^+,0^-)-6(1^+,1^+)+2(0^+,2^+)],$$

$$\psi(^1F_3)=\tfrac{1}{2}[(2^+,1^-)-(2^-,1^+)-(1^+,2^-)+(1^-,2^+)],$$

$$\psi(^3G_3)=\tfrac{1}{6}\{2\sqrt{7}(2^-,2^-)-(\tfrac{1}{2}\sqrt{7})[(2^+,1^-)+(1^+,2^-)+(1^-,2^+)+(2^-,1^+)]+(1/2\sqrt{7})\times[\sqrt{6}(2^+,0^+)+4(1^+,1^+)+\sqrt{6}(0^+,2^+)]\}.$$

[21] N. M. Gray and L. A. Wills, Phys. Rev. **38**, 248 (1931).

[22] The usual notation for wave functions is used: The first figure in each set of parentheses indicates the m_l value of the proton, the second that of the neutron; + or − denotes the spin functions for spin values $+\frac{1}{2}$ or $-\frac{1}{2}$, respectively.