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Hamiltonian Mechanics of Fields

R. H. GOOD, JR.*
Department of Physics, University of California, Berkeley, California
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In relativistic field mechanics one ordinarily introduces the time derivative of a field component as its velocity and the partial derivative of the Lagrangian density with respect to the velocity as its canonically conjugate momentum. In order to treat the time and space equivalently, Born and Weyl once treated the four space-time derivatives of a field component as four velocities and introduced the four partial derivatives of the Lagrangian density with respect to the velocities as four momenta. In the present paper this idea is carried further in order to introduce the generalizations of the point-mechanics ideas of Hamiltonian equations, Lagrange brackets, Poisson brackets, and integrals of motion.

I. INTRODUCTION

THE electrodynamics of Schwinger,[1] Feynman,[2] and Dyson[3] has been very successful, especially in interpreting the Lamb-Retherford shift[4] and the anomalous magnetic moment of the electron.[5,6] In this electrodynamics the idea of covariance with respect to Lorentz transformations is specially emphasized, for the success of the theory depends on expressing the theory in a covariant fashion. Accordingly there is an incentive for constructing a field theory in which the time coordinate is treated entirely equivalently to the space coordinates so that the equations can be written in an even more obviously covariant form at every step of the development. The problem has been studied by Born[7] and by Weyl;[8] this paper is an attempt to extend some of their ideas.

If one follows Heisenberg and Pauli,[9] the mechanics of fields is ordinarily built up by comparison with the classical mechanics of point particles and rigid bodies. Starting from a given Lagrangian density which is a function of the field amplitudes and their four space-time derivatives, one uses the theory of functionals in such a way that the amplitudes of the field at the various points in space are analogous to different degrees of freedom. The partial derivatives of the amplitudes with respect to the time are called the velocities, and the partial derivatives of the Lagrangian density with respect to the velocities are called the momenta. In this way a momentum is defined for each component of the field, and a Hamiltonian mechanics for the field may be set up, still in parallel with classical mechanics. In contrast to this, Born and Weyl treat the four partial derivatives of a component of the field with respect to space and time as four independent velocities. Consequently for each component of the field they introduce four momenta, defined as the partial derivatives of the Lagrangian density with respect to the various velocities. The advantage of this point of view is that the momentum canonically conjugate to the field is a covariant quantity, for if the field is a tensor with independent components, the momentum is a tensor of one higher rank, and if the field variable is a four-component Dirac function, the momentum transforms like the gradient of the adjoint function. In what follows, the Born and Weyl point of view is used; the object of the paper is to see how the classical ideas of Hamiltonian equations, Lagrange brackets, Poisson brackets, and integrals of motion carry over into this different point of view.

As is shown in detail below, the starting point discussed above leads naturally into a generalization of Hamiltonian mechanics in which the four space-time coordinates together take the place that the time alone takes in Hamiltonian mechanics, and the number of components of the field takes the place of the number of degrees of freedom. Many of the classical ideas can be carried directly into this field mechanics; others have a limited parallel. In particular, one can set up a generalization of the Hamiltonian equations and study in detail the transformations which leave their form unchanged. The most general such canonical transformation is found to be a transformation in which the new coordinates used to describe the field are functions of the old coordinates, independent of the momenta. The generalized Lagrange and Poisson brackets are found to be vector operations which are in a limited sense reciprocal. For every infinitesimal canonical transformation which leaves the functional form of the Hamiltonian unchanged, there is a vector function which has zero divergence and which, if it does not explicitly depend on the spacetime, has a zero Poisson bracket with the Hamiltonian. This property is in close parallel with point mechanics, but the converse of the theorem, which is also true in point mechanics, does not hold in this field mechanics.

This work is perhaps interesting because it shows a generalization of classical mechanics which might not

* Now at Department of Physics, The Pennsylvania State University, State College, Pennsylvania.
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[7] M. Born, Proc. Roy. Soc. (London) **A143**, 410 (1934).
[8] H. Weyl, Phys. Rev. **46**, 505 (1934). [See also H. Weyl, Ann. Math. **36**, 607 (1935).]
[9] W. Heisenberg and W. Pauli, Z. Physik **56**, 1 (1929).

have been anticipated. There is also the possibility that it might lead to some new ideas about field quantization.

II. THE HAMILTONIAN EQUATIONS

The usual way to discuss a field with amplitudes $q_i(x_\alpha)$,[10] given the Lagrangian density function

$$L(q_i,\, dq_j/dx_\alpha,\, x_\beta),$$

is to introduce a Lagrangian by

$$\tilde{L}(t)=\int Ld^3x, \tag{1}$$

the space integration corresponding to a sum over the degrees of freedom. Then the variational principle,

$$\delta\int \tilde{L}dt=0, \tag{2}$$

or equivalently

$$\delta\int Ld^4x=0, \tag{3}$$

(where the field amplitudes are to be varied arbitrarily within the integration region but kept fixed on the boundary) gives the equations of motion, which are written in either of the forms

$$\frac{d}{dt}\frac{\delta\tilde{L}}{\delta(dq_i/dt)}-\frac{\delta\tilde{L}}{\delta q_i}=0, \tag{4}$$

or

$$\frac{d}{dx_\alpha}\frac{\partial L}{\partial(dq_i/dx_\alpha)}-\frac{\partial L}{\partial q_i}=0. \tag{5}$$

In Eq. (4), the δ indicates the functional or variational derivative. Following the correspondence with point mechanics, which is clearly visible in Eqs. (2) and (4), one ordinarily introduces momenta by

$$p_i=\frac{\delta L}{\delta(dq_i/dt)}=\frac{\partial L}{\partial(dq_i/dt)}, \tag{6}$$

and proceeds to build up a field Hamiltonian mechanics which differs from the point mechanics only in having integrals replacing certain sums over degrees of freedom.

In adopting the point of view of Born and Weyl, one remarks that in point mechanics the problem is to find certain functions of the time $q_i(t)$, whereas in field mechanics the problem is to find certain functions of space and time $q_i(x_\alpha)$. Consequently, given

$$L(q_i,\, dq_j/dx_\alpha,\, x_\beta),$$

one postulates Eq. (3) directly. This approach of course produces the same equations of motion, Eq. (5), but it leads to the introduction of four momenta for each field amplitude:[11]

$$p_{i\alpha}=\frac{\partial L}{\partial(dq_i/dx_\alpha)}. \tag{7}$$

To conform with this point of view, in what follows, L will be called the Lagrangian and other quantities which arise will be named the same as their point-mechanical analogues. It is easily verified that if, under a Lorentz transformation of the coordinates,

$$x_\alpha'=a_{\alpha\beta}x_\beta,\quad a_{\alpha\beta}a_{\alpha\gamma}=\delta_{\beta\gamma}, \tag{8}$$

the field components transform according to

$$q_i'(x')=b_{ij}q_j(x) \tag{9}$$

and

$$q_j(x)=(b^{-1})_{ji}q_i'(x'), \tag{10}$$

then, assuming the Lagrangian is a scalar and the components of the field are independent,

$$p_{i\alpha}'(x')=(b^{-1})_{ji}a_{\alpha\beta}p_{j\beta}(x). \tag{11}$$

From this one sees that, if the field is a tensor, the momentum is a tensor of one higher rank. Also if the $q_i(x_\alpha)$ form a four-component Dirac function, then the adjoint function transforms according to[12]

$$q_i^{A\prime}(x')=(b^{-1})_{ji}q_j^A(x), \tag{12}$$

(time-reversal will not be considered), and the momentum transforms like the gradient of the adjoint of $q_i(x_\alpha)$.

The Hamiltonian equations are found from the Lagrangian equations, Eq. (5), and the definition of the momenta, Eq. (7), with the use of the techniques of partial differentiation. One must assume that Eq. (7) can be solved for the velocities in terms of the momenta and coordinates (this cannot be done with first-order field equations), and then the general procedure is to eliminate the velocities in favor of the momenta. One finds that

$$\begin{aligned}\left.\frac{\partial L}{\partial p_{i\alpha}}\right|_{q,p}&=\left.\frac{\partial L}{\partial(dq_j/dx_\beta)}\right|_{q,dq/dx}\left.\frac{\partial(dq_j/dx_\beta)}{\partial p_{i\alpha}}\right|_{q,p}\\&=p_{j\beta}\left.\frac{\partial(dq_j/dx_\beta)}{\partial p_{i\alpha}}\right|_{q,p}\\&=-\frac{dq_i}{dx_\alpha}+\frac{\partial}{\partial p_{i\alpha}}\left.\left(p_{j\beta}\frac{dq_j}{dx_\beta}\right)\right|_{q,p},\end{aligned} \tag{13}$$

$$\begin{aligned}\left.\frac{\partial L}{\partial q_i}\right|_{q,p}&=\left.\frac{\partial L}{\partial q_i}\right|_{q,dq/dx}+\left.\frac{\partial L}{\partial(dq_j/dx_\beta)}\right|_{q,dq/dx}\left.\frac{\partial(dq_j/dx_\beta)}{\partial q_i}\right|_{q,p}\\&=\frac{dp_{i\alpha}}{dx_\alpha}+p_{j\beta}\left.\frac{\partial(dq_j/dx_\beta)}{\partial q_i}\right|_{q,p}\\&=\frac{dp_{i\alpha}}{dx_\alpha}+\frac{\partial}{\partial q_i}\left.\left(p_{j\beta}\frac{dq_j}{dx_\beta}\right)\right|_{q,p}.\end{aligned} \tag{14}$$

[10] Greek indices will be used to correspond to the four space-time coordinates $x_1, x_2, x_3, x_4=ict$. Roman indices will correspond to the components of the field so that $i=1$ for a scalar field, $i=1$ to 4 for a vector field, etc. Indices appearing twice in a product are to be summed. The ∂ symbol will be used for partial differentiation when the field components, the velocities or momenta, and the space-time coordinates are taken as independent variables. The symbol d/dx_α will indicate the gradient operator when only the space-time coordinates are taken to be independent.

[11] Momenta corresponding to the space derivatives of the field components have also been considered by Satosi Watanabe, Progr. Theoret. Phys. Japan 2, 71 (1947).

[12] W. Pauli, Revs. Modern Phys. 13, 220 (1941).

Accordingly the Hamiltonian will be defined by

$$H(q,p,x)=p_{j\beta}dq_j/dx_\beta-L, \tag{15}$$

so that the equations of motion are

$$dq_i/dx_\alpha=\partial H/\partial p_{i\alpha}, \tag{16}$$

$$-dp_{i\alpha}/dx_\alpha=\partial H/\partial q_i. \tag{17}$$

Evidently the Hamiltonian is a scalar. These equations reduce to the equations of point mechanics when the Greek indices range over only one value. The above argument shows that Eqs. (16) and (17) are necessary for Eqs. (5) and (7); it is easily verified that they are also sufficient. Equation (16) above is Weyl's Eq. (10), and Eq. (17) above is equivalent to Weyl's Eq. (9).[13]

III. CANONICAL TRANSFORMATIONS

By a canonical transformation is meant a transformation

$$\begin{aligned} q_i' &= q_i'(p,q,x), \\ p_{i\alpha}' &= p_{i\alpha}'(p,q,x), \end{aligned} \tag{18}$$

such that, for some new Hamiltonian function $H'(q',p',x)$,

$$dq_i'/dx_\alpha=\partial H'/\partial p_{i\alpha}', \tag{19}$$

$$-dp_{i\alpha}'/dx_\alpha=\partial H'/\partial q_i'. \tag{20}$$

In this field mechanics the only canonical transformations are point transformations of the type

$$q_i'=q_i'(q,x), \tag{21}$$

$$\frac{\partial q_j'}{\partial q_i}p_{j\alpha}'+\frac{\partial f_\alpha}{\partial q_i}=p_{i\alpha}, \tag{22}$$

$$H'=H+\frac{\partial q_i'}{\partial x_\alpha}p_{i\alpha}'+\frac{\partial f_\alpha}{\partial x_\alpha}, \tag{23}$$

where q_i' and f_α are functions of the q_i and x_α alone. [Presumably Eq. (22) will be solvable for the primed momenta in terms of the unprimed coordinates and momenta.] To see this, one observes that the existence of a canonical transformation implies that the variational principle,

$$\delta\int\left(p_{i\alpha}\frac{dq_i}{dx_\alpha}-H\right)d^4x=0, \tag{24}$$

transforms into

$$\delta\int\left(p_{i\alpha}'\frac{dq_i'}{dx_\alpha}-H'\right)d^4x=0. \tag{25}$$

The difference between the integrands will then be the divergence of some vector, say E_α:

$$p_{i\alpha}\frac{dq_i}{dx_\alpha}-H-p_{i\alpha}'\frac{dq_i'}{dx_\alpha}+H'=\frac{dE_\alpha}{dx_\alpha}. \tag{26}$$

[13] Reference 8, p. 507.

One may regard this equation as an identity among the unprimed coordinates and primed momenta; in that case it may be written as

$$\begin{aligned} p_{i\alpha}\frac{dq_i}{dx_\alpha}-H+q_i'\frac{dp_{i\alpha}'}{dx_\alpha}+H' &= \frac{dE_\alpha}{dx_\alpha}+\frac{d}{dx_\alpha}(q_i'p_{i\alpha}') \\ &= \frac{d}{dx_\alpha}F_\alpha(q,p',x), \end{aligned} \tag{27}$$

where F_α is some definite vector function. From this it follows that

$$\partial F_\alpha/\partial p_{i\beta}'=\delta_{\alpha\beta}q_i', \tag{28}$$

$$\partial F_\alpha/\partial q_i=p_{i\alpha}, \tag{29}$$

$$\partial F_\alpha/\partial x_\alpha=H'-H. \tag{30}$$

Equation (28) implies that F_α depends on no other momenta than the $p_{i\alpha}'$ and that furthermore it depends only linearly on those, so that in general

$$F_\alpha=r_i(q,x)p_{i\alpha}'+f_\alpha(q,x). \tag{31}$$

One sees then that the r_i are only the new coordinates and that Eqs. (28), (29), and (30) reduce to Eqs. (21), (22), and (23). Thus for a canonical transformation the new coordinates can depend only on the old coordinates and the x_α. This is different from point mechanics where, in a canonical transformation, the new coordinates may also depend on the old momenta. As specializations of the above results, the generating vector $F_\alpha=q_ip_{i\alpha}'$ gives the identity transformation, and an infinitesimal canonical transformation with generating vector $G_\alpha(q,p,x)=s_i(q,x)p_{i\alpha}+g_\alpha(q,x)$ and smallness parameter ϵ is given by

$$F_\alpha=q_ip_{i\alpha}'+\epsilon G_\alpha(q,p',x)=q_ip_{i\alpha}'+\epsilon(s_ip_{i\alpha}'+g_\alpha), \tag{32}$$

$$\delta q_i=q_i'-q_i=-\frac{\epsilon}{4}\frac{\partial G_\alpha}{\partial p_{i\alpha}}(q,p,x)=\epsilon s_i, \tag{33}$$

$$\begin{aligned} \delta p_{i\alpha}=p_{i\alpha}'-p_{i\alpha} &= -\epsilon\frac{\partial G_\alpha}{\partial q_i}(q,p,x) \\ &= -\epsilon\left(\frac{\partial s_j}{\partial q_i}p_{j\alpha}+\frac{\partial g_\alpha}{\partial q_i}\right), \end{aligned} \tag{34}$$

$$\delta H=H'-H=\epsilon\frac{\partial G_\alpha}{\partial x_\alpha}(q,p,x)=\epsilon\left(\frac{\partial s_i}{\partial x_\alpha}p_{i\alpha}+\frac{\partial g_\alpha}{\partial x_\alpha}\right). \tag{35}$$

IV. LAGRANGE AND POISSON BRACKETS

When the notions of Lagrange and Poisson brackets are discussed, one must think of a set of independent functions of the coordinates and momenta which are not necessarily a new set of canonical variables. In case there are n components of the field, there are altogether $5n$ coordinates and momenta; let $u_\Gamma(q,p,x)$, $\Gamma=1, 2, \cdots 5n$, represent the independent functions of the coordinates and momenta. With respect to a certain set of coordinates and momenta, the Lagrange bracket between any two functions will be defined by

$$\{u_A, u_B\}_\alpha{}^{q,p}=\frac{\partial q_i}{\partial u_A}\frac{\partial p_{i\alpha}}{\partial u_B}-\frac{\partial q_i}{\partial u_B}\frac{\partial p_{i\alpha}}{\partial u_A}, \tag{36}$$

where the rest of the u's are to be held constant during each partial differentiation. One is led to this particular generalization of the point-mechanical definition by considering canonical transformations which are independent of the x_α so that the new Hamiltonian is equal to the old. In this case one may easily transform from the unprimed variables to the primed by multiplying Eq. (16) by $\partial p_{i\alpha}/\partial p_{j\beta}'$, Eq. (17) by $\partial q_i/\partial p_{j\beta}'$ and adding to obtain

$$\{q_k', p_{j\beta}'\}_\alpha{}^{q,p}\frac{dq_k'}{dx_\alpha}+\{p_{k\gamma}', p_{j\beta}'\}_\alpha{}^{q,p}\frac{dp_{k\gamma}'}{dx_\alpha}=\frac{\partial H'}{\partial p_{j\beta}'}, \tag{37}$$

and also by multiplying Eq. (16) by $\partial p_{i\alpha}/\partial q_j'$, Eq. (17) by $\partial q_i/\partial q_j'$ and adding to obtain

$$\{q_k', q_j'\}_\alpha{}^{q,p}\frac{dq_k'}{dx_\alpha}+\{p_{k\beta}', q_j'\}_\alpha{}^{q,p}\frac{dp_{k\beta}'}{dx_\alpha}=\frac{\partial H'}{\partial q_j'}. \tag{38}$$

Consequently sufficient conditions for a canonical transformation are that

$$\{p_{k\gamma}', p_{j\beta}'\}_\alpha{}^{q,p}=\{q_k', q_j'\}_\alpha{}^{q,p}=0, \tag{39}$$

$$\{q_k', p_{j\beta}'\}_\alpha{}^{q,p}=\delta_{jk}\delta_{\alpha\beta}. \tag{40}$$

Using Eqs. (21) and (22), one can easily verify that these conditions are also necessary, whether the transformation depends on the x_α explicitly or not. As a further justification for the above definition of Lagrange bracket, Eq. (36), it can be shown that these quantities are canonical invariants in the sense that they have the same value no matter what set of canonical variables are used to calculate them. One proves this easily by writing

$$\frac{\partial q_i}{\partial u_A}\frac{\partial p_{i\alpha}}{\partial u_B}=\left(\frac{\partial q_i}{\partial q_j'}\frac{\partial q_j'}{\partial u_A}+\frac{\partial q_i}{\partial p_{j\beta}'}\frac{\partial p_{j\beta}'}{\partial u_A}\right)\times\left(\frac{\partial p_{i\alpha}}{\partial q_k'}\frac{\partial q_k'}{\partial u_B}+\frac{\partial p_{i\alpha}}{\partial p_{k\gamma}'}\frac{\partial p_{k\gamma}'}{\partial u_B}\right) \tag{41}$$

so that

$$\begin{aligned}\{u_A, u_B\}_\alpha{}^{q,p}=&\{q_j', q_k'\}_\alpha{}^{q,p}\frac{\partial q_j'}{\partial u_A}\frac{\partial q_k'}{\partial u_B}\\&+\{q_j', p_{k\gamma}'\}_\alpha{}^{q,p}\frac{\partial q_j'}{\partial u_A}\frac{\partial p_{k\gamma}'}{\partial u_B}\\&+\{p_{j\gamma}', q_k'\}_\alpha{}^{q,p}\frac{\partial q_k'}{\partial u_B}\frac{\partial p_{j\gamma}'}{\partial u_A}\\&+\{p_{j\beta}', p_{k\gamma}'\}_\alpha{}^{q,p}\frac{\partial p_{k\gamma}'}{\partial u_B}\frac{\partial p_{j\beta}'}{\partial u_A}\\&=\{u_A, u_B\}_\alpha{}^{q',p'},\end{aligned} \tag{42}$$

where Eqs. (39) and (40) were used in the last step. In view of this fact the superscripts on the brackets will be omitted below.

The Poisson bracket between any two functions u_A and u_B, with respect to a certain set of coordinates and momenta, will be defined by

$$(u_A, u_B)_\alpha{}^{q,p}=\frac{\partial u_A}{\partial q_i}\frac{\partial u_B}{\partial p_{i\alpha}}-\frac{1}{4}\frac{\partial u_B}{\partial q_i}\frac{\partial u_A}{\partial p_{i\alpha}}. \tag{43}$$

With this definition the Poisson and Lagrange brackets are, in a sense, reciprocal:

$$\begin{aligned}\{u_A, u_B\}_\alpha(u_A, u_\Gamma)_\alpha{}^{q,p}&=\left(\frac{\partial q_i}{\partial u_A}\frac{\partial p_{i\alpha}}{\partial u_B}-\frac{\partial q_i}{\partial u_B}\frac{\partial p_{i\alpha}}{\partial u_A}\right)\left(\frac{\partial u_A}{\partial q_j}\frac{\partial u_\Gamma}{\partial p_{j\alpha}}-\frac{1}{4}\frac{\partial u_\Gamma}{\partial q_j}\frac{\partial u_A}{\partial p_{j\alpha}}\right)\\&=\frac{\partial p_{i\alpha}}{\partial u_B}\frac{\partial u_\Gamma}{\partial p_{i\alpha}}+\frac{\partial q_i}{\partial u_B}\frac{\partial u_\Gamma}{\partial q_i}=\delta_{B\Gamma}.\end{aligned} \tag{44}$$

Although in general these Poisson brackets are not canonical invariants, certain special cases of them do happen to be invariants. By thinking of the u_A as expressed in terms of the primed coordinates and momenta, one finds that

$$\begin{aligned}(u_A, u_B)_\alpha{}^{q,p}=&(q_j', q_k')_\alpha{}^{q,p}\frac{\partial u_A}{\partial q_j'}\frac{\partial u_B}{\partial q_k'}\\&+(q_j', p_{k\gamma}')_\alpha{}^{q,p}\frac{\partial u_A}{\partial q_j'}\frac{\partial u_B}{\partial p_{k\gamma}'}+(p_{j\beta}', q_k')_\alpha{}^{q,p}\frac{\partial u_B}{\partial q_k'}\frac{\partial u_A}{\partial p_{j\beta}'}\\&+(p_{j\beta}', p_{k\gamma}')_\alpha{}^{q,p}\frac{\partial u_B}{\partial p_{k\gamma}'}\frac{\partial u_A}{\partial p_{j\beta}'}.\end{aligned} \tag{45}$$

Also for a canonical transformation one finds, either directly from Eqs. (21) and (22) or by specializing Eq. (44) to the case when the u_A are the primed coordinates and momenta, that

$$(q_j', q_k')_\alpha{}^{q,p}=0, \tag{46}$$

$$(q_j', p_{k\gamma}')_\alpha{}^{q,p}=\delta_{jk}\delta_{\alpha\gamma}, \tag{47}$$

$$(p_{j\alpha}', q_k')_\alpha{}^{q,p}=-\delta_{jk}, \tag{48}$$

$$(p_{j\alpha}', p_{k\gamma}')_\alpha{}^{q,p}=0. \tag{49}$$

Therefore, in the special case when u_A is independent of the momenta, say $u_A=v(q, x)$,

$$(v, u_B)_\alpha{}^{q,p}=\frac{\partial v}{\partial q_j'}\frac{\partial u_B}{\partial p_{j\alpha}'}=(v, u_B)_\alpha{}^{q',p'}. \tag{50}$$

Another special case is when four of the u_A, say w_α, depend on no other momenta than the $p_{i\alpha}$ and are linear in those, so that

$$\partial w_\alpha/\partial p_{i\beta}=\tfrac{1}{4}\delta_{\alpha\beta}\partial w_\gamma/\partial p_{i\gamma}. \tag{51}$$

Then one sees from Eqs. (45) to (49) that

$$(w_\alpha, u_B)_\alpha{}^{q,p} = (q_j', q_k')_\alpha{}^{q,p} \frac{\partial w_\alpha}{\partial q_j'} \frac{\partial u_B}{\partial q_k'} + (q_j', p_{k\gamma}')_\alpha{}^{q,p} \frac{\partial w_\alpha}{\partial q_j'} \frac{\partial u_B}{\partial p_{k\gamma}'} + \tfrac{1}{4}(p_{j\alpha}', q_k')_\alpha{}^{q,p} \frac{\partial u_B}{\partial q_k'} \frac{\partial w_\beta}{\partial p_{j\beta}'} + \tfrac{1}{4}(p_{j\alpha}', p_{k\gamma}')_\alpha{}^{q,p} \frac{\partial u_B}{\partial p_{k\gamma}'} \frac{\partial w_\beta}{\partial p_{j\beta}'} = (w_\alpha, u_B)_\alpha{}^{q',p'}. \quad (52)$$

In these two special cases, then, the Poisson brackets are invariants, and the superscripts may be omitted. These two special types of Poisson brackets correspond to partial differentiation with respect to momenta and coordinates, for it is easily seen that

$$\partial K/\partial p_{i\alpha} = (q_i, K)_\alpha, \quad (53)$$

$$\partial K/\partial q_i = -(p_{i\alpha}, K)_\alpha, \quad (54)$$

where $K(q, p, x)$ is any function. One can then write the Hamiltonian equations, Eqs. (16) and (17), in the form

$$dq_i/dx_\alpha = (q_i, H)_\alpha, \quad (55)$$

$$dp_{i\alpha}/dx_\alpha = (p_{i\alpha}, H)_\alpha. \quad (56)$$

In fact, for the two special types of function introduced above,

$$dv/dx_\alpha = (v, H)_\alpha + \partial v/\partial x_\alpha, \quad (57)$$

$$dw_\alpha/dx_\alpha = (w_\alpha, H)_\alpha + \partial w_\alpha/\partial x_\alpha. \quad (58)$$

V. INTEGRALS OF MOTION

In the mechanics of point particles and rigid bodies, one is concerned with functions of the time $q_i(t)$ and one calls a function $G(q, p, t)$ an integral of the motion if

$$(D/Dt)G(q(t), p(t), t) = 0, \quad (59)$$

where the symbol D is used to indicate total differentiation with respect to the only argument. In the mechanics of fields one is concerned with functions of the space-time $q_i(x_\alpha)$ and, as a generalization of the point-mechanical idea, a function $G_{\alpha\beta\ldots}(q, p, x)$ will be called an integral of the motion if

$$(d/dx_\alpha)G_{\alpha\beta\ldots}(q(x), p(x), x) = 0. \quad (60)$$

This generalization is appropriate in view of the well-known theorem that if $G_{\alpha\beta\ldots}$ is a covariant quantity which satisfies Eq. (60) and which is zero except in a region of space near the origin, then

$$\bar{G}_{\beta\ldots}(t) = \int_{\substack{\text{all}\\ \text{space}}} G_{4\beta\ldots} d^3x \quad (61)$$

is a covariant quantity of one lower rank, and furthermore

$$(D/Dt)\bar{G}_{\beta\ldots}(t) = 0. \quad (62)$$

In this field mechanics, in parallel with point mechanics, if the functional dependence of the Hamiltonian on its arguments is unchanged during an infinitesimal contact transformation, then the generating vector of the transformation is an integral of the motion. One sees this easily by noting that, when the Hamiltonian is in this sense invariant, then to first order

$$\delta H = H(q', p', x) - H(q, p, x) = \frac{\partial H}{\partial p_{i\alpha}} \delta p_{i\alpha} + \frac{\partial H}{\partial q_i} \delta q_i. \quad (63)$$

Next, using Eqs. (33), (34), and (35) to rewrite this in terms of the generating vector, one finds that

$$\epsilon \frac{\partial G_\alpha}{\partial x_\alpha} = -\epsilon \frac{\partial H}{\partial p_{i\alpha}} \frac{\partial G_\alpha}{\partial q_i} + \frac{\epsilon}{4} \frac{\partial H}{\partial q_i} \frac{\partial G_\alpha}{\partial p_{i\alpha}} = -\epsilon (G_\alpha, H)_\alpha. \quad (64)$$

Finally, since $G_\alpha(q, p, x) = s_i(q, x)p_{i\alpha} + g_\alpha(q, x)$, one can apply Eq. (58) with the result that

$$\frac{dG_\alpha}{dx_\alpha} = (G_\alpha, H)_\alpha + \frac{\partial G_\alpha}{\partial x_\alpha} = 0. \quad (65)$$

The converse theorem of point mechanics, that each integral of the motion is the generator of a canonical transformation which leaves the functional form of the Hamiltonian unchanged, does not apply here since only generating vectors which give point transformations are allowed.