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Operators and Observables in Isotopic Spin Space*

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The usefulness of the conventional isotopic spin formalism is somewhat marred by the fact that it contains superfluous formal elements which have no physical counterpart. It is demonstrated that application of the "superselection principle" for total electric charge, recently suggested by Wick, Wightman, and Wigner, is sufficient to eliminate these superfluous elements and thus reduce the content of the theory to that of the more conventional formalism in which neutrons and protons are distinguished *ab initio*. This superselection principle applied to the conventional state-function representation of quantum mechanics requires that admissable state functions be eigenfunctions of the total charge and that operators representing observables commute with the total charge. The disjointness of the various subspaces in Hilbert space corresponding to different total charge eigenvalues, demanded by the superselection principle, leads to the result that many observables which are outwardly different in form are actually essentially equivalent. The construction of all inequivalent observables compounded from nucleonic isotopic spin operators only is carried out and it is shown that all such observables are simply functions of the square of the total isotopic spin and its z component. The essentially uniqueness of the charge parity operator introduced by Kroll and Foldy is established.

1. INTRODUCTION

THE isotopic spin formalism[1] has proved to be a very convenient means of dealing with the similarities in properties of the proton and neutron, and its employment has paid ample rewards as an aid in recognizing and formulating the consequences of these similarities for nuclear structure, nuclear interactions, and meson-nucleon interactions.[2-5] However, these advantages are partly cancelled by the fact that the isotopic spin formalism contains superfluous formal elements which have no known physical counterparts. These difficulties are best known in the following formulation: while the projection of a vector in isotopic spin space on the z axis has a definite physical interpretation in terms of the electric charge of the system being described, still the azimuth of the vector appears to have no such absolute significance and only relative azimuths of vectors having the same z component have at present a clearly identifiable physical meaning.

The program of the present paper consists in the separation of the physical from the nonphysical elements in the isotopic spin formalism by application of the recently suggested[6] "superselection principle" for electric charge. It is likely that many, if not all, of the specific results obtained here represent a transcription into isotopic spin notation of well-known results from the theory of symmetry groups. However, since the avowed purpose of the present paper is to learn how to work within the isotopic spin formalism itself, and since all of our results are obtained by the use of rather elementary operator algebra without direct resort to formal group-theoretical concepts, we feel that no apology is necessary in case of such duplication.

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2. THE SUPERSELECTION PRINCIPLE FOR ELECTRIC CHARGE

One of the fundamental principles of quantum mechanics is the principle of superposition of states according to which the states of a system form a linear vector space. Thus if ψ and ϕ represent two possible states of a system then a linear combination of them,

$$\Psi=a\psi+b\phi,$$

where a and b are complex numbers, is also a possible state of the system. It is usually understood that two distinct state vectors represent two different physical states of the system except when one is a complex numerical multiple of the other, in which case they represent exactly the same state. Thus the linear vector space is to be interpreted as a "ray space" in which different rays are associated with physically distinct states. Hence a second linear combination of the states ψ and ϕ,

$$\Psi'=a'\psi+b'\phi,$$

will represent a different physical state than that represented by Ψ if, and only if, $(a'/b')\neq(a/b)$. The existence of an experiment which allows one to measure the *relative* phase of any two states of a physical system is therefore usually regarded as a necessary implication of the conventional formulation of quantum mechanics.

Recently, Wick, Wightman, and Wigner[6] have pointed out that there exist situations in which the above conditions are not satisfied. They have shown, in particular, that in a situation in which the system described has accessible to it both integral and half (odd) integral eigenvalues of the total angular momentum, to grant the possibility of measuring the relative phase of two states, one belonging to an integral, the other to a half-integral, value for the total angular momentum, would be tantamount to a violation of covariance with respect to Lorentz transformations. They also suggest that a similar situation may arise with respect to systems which have accessible to them different eigenvalues of the total electric charge (or different numbers of "heavy particles"), in that it may be impossible to measure the relative phase of states belonging to different eigenvalues of these quantities. Thus, within the usual framework of quantum electrodynamics, this inability to measure the relative phase of states of different total charge follows directly from gauge invariance considerations. For by a gauge transformation of a particularly simple character (constant gauge function) which reflects no physical change in a system but only a change in its *description* the relative phase of two states of different total charge may be changed arbitrarily.

To describe the situation arising when for some reason it is impossible to measure the relative phase of certain states of a system these authors have coined the phrase: a "superselection principle" operates with respect to the quantity whose eigenvalues distinguish the states whose relative phase cannot be measured. To understand clearly the nature of a superselection principle it is imperative to distinguish between it and a universal conservation law. A conservation law, such as that for total momentum, total angular momentum, electric charge, space parity, energy, etc., is a condition which is satisfied by the Hamiltonian of the system being described. It does not of itself inhibit the measurement of relative phases; in fact, the measurement of such relative phases of states belonging to different eigenvalues of conserved quantities is a regular concomitant of many physical measurements. For example, the measurement of an angular distribution in a scattering experiment determines the relative phases of states of different angular momentum. But a superselection principle is more than a condition satisfied by the Hamiltonian describing a system; it denies the existence of apparatus which is capable of performing certain types of measurements on the system. Some understanding of how such limitations on the intrinsic capacity of available measuring apparatus may arise in a natural way has been obtained by Wigner in a recent investigation.[7]

The existence of a superselection principle for the total electric charge of a system (which is the only superselection principle with which we shall be concerned in this paper) has a profound influence on the quantum-mechanical interpretation of a system which is being described by means of the isotopic spin representation. The object of the isotopic spin representation is to encompass the dynamical behavior of systems of different particles in a single dynamical scheme by treating the different particles as different states of the same particle. Such a general scheme has decided advantages in at least two cases (which often occur together). The first occurs when the different particles, though distinguishable, possess certain similarities which it is of interest to exploit in the development of the theory, while the second occurs when actual transformations of a particle from one type to another is a natural part of the dynamical development of the system. Since in the principal applications of this formalism the different particles which are identified as different states of the same particle possess different electric charges, the abstract composite system with which one then deals has accessible to it states of different total electric charge; hence the superselection principle for electric charge immediately comes into play.

To prevent our considerations from becoming too abstract we shall limit ourselves now to the only case which we shall discuss in detail in the later parts of this paper, namely that in which we have a system of nucleons only, described in an isotopic spin formalism with neutron and proton regarded as two states of a nucleon. We consider first the case of a single nucleon, and in the familiar manner, represent the proton state of the nucleon by the function α and the neutron state by the function β. A state function representing a proton in a space-spin state $\phi(r,\sigma)$ would then be written $\phi\alpha$, while the state function representing a neutron in the same space-spin state would be written $\phi\beta$. The superposition principle would then suggest that $\phi(\alpha+\beta)$ is also a possible state function for the system. However, since the states $\phi\alpha$ and $\phi\beta$ are states of different total charge, their relative phase cannot be discerned by an experiment and hence the state $\phi(\alpha+\beta)$ would be indiscernable from the state $\phi(\alpha-\beta)$. In fact each of these would now be interpretable only as a statistical mixture (in contrast to a pure state) with equal probabilities of the system consisting of a proton or a neutron in the space-spin state ϕ. That one can hardly treat such state functions in the familiar manner in which one usually operates in quantum mechanics is made trivially, but strikingly, evident if one applies the usual rule for obtaining the probability than an observation will reveal that the system is in the state $\phi(\alpha-\beta)$ when it has been projected into the state $\phi(\alpha+\beta)$ by a previous preparation. In the customary procedure one forms the inner product of the two state functions (after they have been normalized) and identifies its absolute square with the desired probability. In this case such an inner product has the value zero; this in spite of the fact that the two states described by these functions are *observationally indistinguishable.*

In view of these difficulties it would seem appropriate to avoid the state function representation of the state of a physical system altogether and pass over to the statistical matrix representation which can cope with equanimity with both pure states and statistical

[7] E. P. Wigner, Z. Physik **133**, 101 (1952).

mixtures.[6] However, if one desires to remain on the more familiar quantum-mechanical ground of state function representations, it is obvious that one must modify the rules of the quantum-mechanical game. The simplest such change, and one which is tacitly observed when one works in situations where superselection principles operate, consists in limiting considerations to state functions which are eigenstates of the quantity with respect to which the superselection principle operates—in our particular case, the total electric charge. While state functions which have not this character might still be given interpretations as representing statistical mixtures, one nevertheless simply ignores them and omits them from the theory.

Now the simple change suggested above has some consequences for the theory which are not trivially evident. These arise with respect to the role played by Hermitian operators as representatives of observables in the theory. It is immediately evident that if a superselection principle operates with respect to a quantity Q, then no operator which has matrix elements connecting states belonging to different eigenvalues of Q can represent an observable; otherwise the measurement of such an observable would be tantamount to an observation of the relative phase of the two states, in contradiction to the superselection principle. Hence, in general, only Hermitian operators which commute with Q, the total charge in our case, represent observables, and, in the absence of any other superselection principle, we assume that every Hermitian operator which does commute with Q does indeed represent an observable. In actuality, however, in working in an isotopic spin representation another principle closely akin to a superselection principle does intervene. For if one is working in such a representation it is essential that the antisymmetrization (Pauli) principle be observed if one is working with a system of particles satisfying Fermi-Dirac statistics or that the symmmetrization principle be observed in the case of Bose-Einstein statistics if one is to obtain only the manifold of state functions which one finds by observation. This then means that, provided no further superselection principle intervenes, only *symmetric* Hermitian operators which commute with Q will represent observables and conversely.

Now the observation which is not so trivially evident is that within these restrictions many operators which satisfy the above criteria and which are outwardly quite different in form actually represent essentially the same observable. To economize in our language concerning this point let us regard the function space for the system under consideration as decomposed into the various subspaces each associated with a given eigenvalue of Q, or in view of our later considerations, a given eigenvalue of the z component of isotopic spin T_z which is linearly related to the total charge Q. Then the admissible state functions lie entirely in one or another of these subspaces T_z. Let $\mathcal{O}$ be a symmetric Hermitian operator which commutes with T_z and consequently represents an observable. Then the operator $\mathcal{O}f(T_z)=f(T_z)\mathcal{O}$ where $f(T_z)$ is any real function of its argument, is also an observable which is essentially equivalent to $\mathcal{O}$ in the following sense. In each subspace T_z, $\mathcal{O}f(T_z)$ is simply a numerical multiple of $\mathcal{O}$ itself, though it will in general be a different numerical multiple in each different subspace. Since one admits state functions lying entirely in one of these subspaces only, nothing is essentially gained by discriminating between these operators, and it becomes of some importance to recognize whether two outwardly different operators represent equivalent observables.

Actually our restriction that in order for an operator to represent an observable it must commute with T_z and hence leave all the subspaces T_z invariant is more stringent than is necessary. It is certainly necessary if the operator is to represent an observable in all subspaces T_z. Such observables we shall call complete observables. On the other hand there may exist operators which leave one or more, but not all, subspaces T_z invariant and these will represent perfectly good observables in these particular subspaces. Such observables we shall call incomplete observables; an example is provided by the charge-parity operator.[5] Actually the distinction between complete and incomplete observables is largely a matter of convenience, since to every incomplete observable we can always find an equivalent complete observable which vanishes in all subspaces which the original does not leave invariant. One has simply to postmultiply such an incomplete observable by a polynomial in T_z which vanishes in every such noninvariant subspace.

Before proceeding further we make some remarks concerning our notation. We employ the conventional Pauli matrices to represent the three components τ_x^n, τ_y^n, τ_z^n of the isotopic spin vector of the nth nucleon, and in addition we shall sometimes use the notation τ_0^n for the unit isotopic spin matrix. The total isotopic spin vector $\mathbf{T}$ is defined by

$$\mathbf{T}=\tfrac{1}{2}\sum_n \boldsymbol{\tau}^n,$$

and its components are designated by T_x, T_y, and T_z. In addition we shall write

$$\tau_\pm^n=\tfrac{1}{2}(\tau_x^n\pm i\tau_y^n),\quad T_\pm=T_x\pm iT_y.$$

While our earlier remarks are applicable to systems composed of both nucleons and mesons, or perhaps even more general systems, our further considerations will be limited to nuclear systems composed of a number A of nucleons.

3. INEQUIVALENT OBSERVABLES

We shall first determine all of the *linearly independent* observables formed from the isotopic spin operators only. To do this we note that the most general operator formed from isotopic spin operators will be a linear combination of terms, each of which consists of A (the total number of nucleons in the system) factors chosen from the τ_0^n, τ_+^n, τ_-^n, τ_z^n with one and only one τ for each nucleon. All such terms are clearly linearly independent and their total number is 4^A. The linearly independent observables must then be *symmetric* functions formed from these by linear combination. Each such linearly independent symmetric function can clearly be designated by giving only the total number of τ_0's, τ_+'s, τ_-'s, and τ_z's occurring in each term of the sum; that is, by a *partition* $n_0+n_++n_-+n_z=A$. We denote each such symmetrical operator (with unit coefficient for each term) by the symbol (n_0,n_+,n_-,n_z).

However, not all of these symmetrical operators will be observables since they do not all commute with T_z. In view of the discussion in the last section we may limit our attention to complete observables. A simple calculation gives us for the commutator of any of these operators with T_z:

$$[(n_0,n_+,n_-,n_z),\ T_z]=(n_--n_+)(n_0,n_+,n_-,n_z),$$

whence it immediately follows that to obtain complete observables which do not vanish in *every* subspace T_z we must include in our basic linearly independent set only those (n_0,n_+,n_-,n_z) for which $n_+=n_-=N\leqslant A/2$. Hence we may now designate our basic set by (n_0,N,n_z), each associated with a partition $n_0+2N+n_z=A$.

Now, while each of the operators of our basic set are observable, many of these will be equivalent in the sense of the discussion in the preceding section. To select a basic set of observables which are inequivalent we note that

$$(n_0,N,n_z)T_z=\tfrac{1}{2}n_0(n_0-1,N,n_z+1)+\tfrac{1}{2}n_z(n_0+1,N,n_z-1).$$

Therefore $(n_0-1,N,1)$ is equivalent to $(n_0,N,0)$ except possibly in the subspace $T_z=0$ where the first vanishes while the second may not. Repeated application of the above recursion relation allows us to express (n_0,N,n_z) as $(n_0+n_z,N,0)$ multiplied by a function of T_z. Examination of these results shows that we will lose no nonvanishing observables in any subspace if we choose as a basic set of inequivalent observables the operators $(n_0,N,0)$ with n_0 running through all integral values from 0 to A. In other words, any observable in any particular subspace T_z can be written as a linear combination of the $(n_0,N,0)$ with fixed coefficients, or any complete observable can be written as a linear combination of the $(n_0,N,0)$ in all subspaces with coefficients which are functions (polynomials) in T_z. We may now designate our basic set of linearly independent inequivalent observables simply by $[N]$, where N designates the number of τ_+'s and τ_-'s occurring as factors in each term. The total number of observables $[N]$ will be $(A+1)/2$ or $(A+2)/2$ according as A is odd or even ($[0]$ is included in this count).

While we have determined the linearly independent, inequivalent observables, it does not follow that they are completely independent. Actually they are not, for we shall now show that all of them may be written as functions of T^2 and T_z from which it follows that any complete observable composed of isotopic spin operators only can be expressed as functions of T^2 and T_z.

For the proof we shall require the following lemma: any polynomial in the operators T_+, T_-, and T_z in which each term is of equal degree in T_+ and T_- is a function of T^2 and T_z only. The proof is very elementary. We begin by noting that $T^2=\frac{1}{2}(T_+T_-+T_-T_+)+T_z^2$, whence, in view of the commutation relations satisfied by these operators, $T_+T_-=T^2-T_z^2+T_z$. Now in the given polynomial consider first the terms of highest degree, say s, in T_+ and T_-. By the use of the commutation relations rearrange each such term so that all of the T_z's stand to the right of all T_+'s and T_-'s. All terms resulting from this process will still be of degree s in T_+ and T_-. Now by further use of the commutation relations rearrange each such term so that it is of the form $T_+T_-T_+\cdots T_+T_-$ times a power of T_z standing to the right. In this process new terms will be generated which are of degree s–1, s–2,$\cdots$ in T_+ and T_-. The terms of degree s will now be functions of (T_+T_-) and T_z only, and consequently by our earlier result can be written as functions of T^2 and T_z only. One now carries out the same process on terms of degree s–1, s–2, etc., whereupon the lemma is established.

Finally, to prove our main theorem we note that $[N]$ can be written as the coefficient of p^Nq^N in the expansion of

$$S=\prod_{n=1}^{A}(1+p\tau_+^n+q\tau_-^n).$$

Letting

$$p=(x/y)^{\frac{1}{2}}\tanh(xy)^{\frac{1}{2}},\quad q=(y/x)^{\frac{1}{2}}\tanh(xy)^{\frac{1}{2}},$$

we may rewrite S as follows:

$$\begin{aligned}S&=[\cosh(xy)^{\frac{1}{2}}]^{-A}\prod_{n=1}^{A}\left[\cosh(xy)^{\frac{1}{2}}+(x\tau_+^n+y\tau_-^n)\frac{\sinh(xy)^{\frac{1}{2}}}{(xy)^{\frac{1}{2}}}\right]\\&=[\cosh(xy)^{\frac{1}{2}}]^{-A}\prod_{n=1}^{A}\exp(x\tau_+^n+y\tau_-^n)\\&=[\cosh(xy)^{\frac{1}{2}}]^{-A}\exp[2(xT_+^n+yT_-^n)].\end{aligned}$$

Note that xy is a function of pq only, and that x and y are equal to p and q, respectively, divided by a function of pq. Then it follows that the expansion of S in powers of p and q will have for the coefficient of p^Nq^N a polynomial formed from the sum of terms in each of which the degrees of T_+ and T_- are equal. The theorem then follows from our lemma.

4. UNIQUENESS OF THE CHARGE PARITY OPERATOR

As an example of the preceding ideas, we shall consider the question[8] of the uniqueness of the charge parity operator[5] introduced by Kroll and Foldy as a formal means of expressing the consequences of charge symmetry for nuclear structure and nuclear reactions. Formally this operator is defined to be one which for a particular nuclear system changes all neutrons to protons and all protons to neutrons. The above authors elected to write this operator as

$$P_x=\prod_{n=1}^{A}(i\tau_x{}^n)=\exp(i\pi T_x),$$

in which case it may be interpreted as the operator which performs a rotation of 180° about the x axis in isotopic spin space. It is clearly an observable only in the subspace $T_z=0$, and is therefore an incomplete observable. Applied to a state function of a nucleus in a state of nonvanishing T_z it generates a state function representing the corresponding state of the mirror nucleus.

Now it is readily observed that there exist many operators which perform the same function of changing all protons to neutrons and all neutrons to protons. In fact any product of A factors chosen from the $\tau_x{}^n$ and $\tau_y{}^n$ with one and only one τ referring to each nucleon will accomplish this result, as will any linear combination of such products. Since there are 2^A such operators which are linearly independent it is not at all clear that the charge parity operator is a unique quantity, and questions may be raised as to whether a unique charge parity assignment can be made for states of a self-conjugate nucleus when charge symmetry obtains.

In the present section we shall show that no real ambiguity exists and that the charge parity operator is essentially unique. We note first that all of the 2^A operators formed as above are not observables since they are not symmetric. However, by forming linear combinations one can obtain $A+1$ linearly independent symmetric operators each of which may be specified by the number n_x of τ_x's and the number $A-n_x$ of τ_y's contained in each term of the symmetric sum. We shall now show that all of these symmetric operators are equivalent in the sense defined earlier. Thus, take any of these symmetric operators and replace in each term each of the τ_y's contained in it by its equivalent $-i\tau_x\tau_z$. This symmetric operator then takes the form of the operator P_x above multiplied by a symmetric function of the τ_z's. To complete our proof we show that every symmetric function of the τ_z's is a function simply of T_z. To accomplish this we note that the linearly independent symmetric functions of the τ_z's can be obtained as the coefficients of x^s in the expansion of

$$Q=\prod_{n=1}^{A}(1+ix\tau_z{}^n).$$

However, the following manipulations indicate that this generating function is a function of T_z only:

$$\begin{aligned}Q&=(1+x^2)^{A/2}\prod_{n=1}^{A}(\cos\theta+i\tau_z{}^n\sin\theta)\\&=(1+x^2)^{A/2}\prod_{n=1}^{A}\exp(i\theta\tau_z{}^n)\\&=(1+x^2)^{A/2}\exp(i\theta T_z),\end{aligned}$$

where $\sin\theta=x/(1+x^2)^{\frac{1}{2}}$. Our result then follows.

Thus all of the $A+1$ symmetric operators are simply multiples of the Kroll-Foldy parity operator[9] in the subspace $T_z=0$. The argument further demonstrates the essential uniqueness of the operator which transforms any nucleus into its mirror nucleus, apart from an irrelevant (because of the superselection principle for total charge) phase factor. This last fact is useful in establishing relationships between matrix elements referring to pairs of mirror nuclei.

We may note also that the charge parity operator may be converted into a complete observable by multiplying it by the factor:

$$(T_z{}^2-1)(T_z{}^2-4)(T_z{}^2-9)\cdots(T_z{}^2-A^2).$$

Our results obtained in Sec. 5 also tell us that this complete observable (which now vanishes in every subspace except $T_z=0$) should be a function of T^2 only. That this is the case was noted in reference 5 where it was shown that in the subspace $T_z=0$, P_x has the eigenvalue $+1$ in a state in which $T^2=t(t+1)$ with t an even integer and the eigenvalue -1 in a state in which t is an odd integer.

5. CONCLUSION

We may summarize our results by the statement that the overgenerality of the usual isotopic spin formalism represents a purely formal difficulty which can easily be overcome without destroying the conveniences of the formalism. The apparently superfluous elements contained in the usual isotopic spin formalism may be

[8] The question of the uniqueness of the charge parity operator was raised with the author by Professor E. L. Hill.

[9] Of course the uniqueness of the charge parity operator is established here only to within a constant multiplying factor (of, say, P_x). If one further requires the operator to be unitary, that is that it preserve the normalization of the wave function, then the constant multiplying factor is restricted to modulus unity. If the operator is further restricted to be hermitian (in the subspace $T_z=0$) then the factor must be ±1. The choice between these two is a matter of convention. The important point is that the operator discriminate through its eigenvalues states of different parity. However, note that if the charge parity operator is defined as a rotation of 180° about an axis lying in the x-y plane in isotopic spin space, then there exists no ambiguity, even with respect to sign, to be resolved by convention.

eliminated by proper application of the superselection principle for electric charge and the recognition of the consequent existence of equivalent observables. Under these restrictions the isotopic spin formalism becomes equivalent to a conventional representation in which neutrons and protons are discriminated *ab initio*. It is conceivable that someday new phenomena may appear which will require the full potential content of the isotopic spin formalism for its description, but at present this does not appear likely.

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Multipole Singularities of Classical Scalar and Pseudoscalar Meson Fields*

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The general form of the equations of motion of a particle possessing multipole singularities of a neutral scalar or pseudoscalar meson field has been found by Harish-Chandra on the basis of Dirac's method. In this paper the general form of the multipole moment compatible with these equations is established under the assumption that the spin and the 2^n-pole moments of the particle are of constant magnitude and have only spatial components in the system in which the particle is at rest. Then the general form of the equations of motion and of the multipole moment compatible with them is established for point particles interacting with a charge-symmetric scalar or pseudoscalar meson field. It is found that 2^n-pole moments of different types are possible for arbitrary n, and that a particle can carry an arbitrary combination of such moments.

I. INTRODUCTION

SEVERAL methods have been suggested recently to avoid the infinities associated with point singularities in classical field theory. The method first used by Dirac[1] for the case of point charges interacting with an electromagnetic field was extended by Harish-Chandra[2] to fields of any integral spin. He succeeded in obtaining the general form of the equations of motion of point multipoles of such (neutral) fields and the explicit form of the equations for point charges and dipoles in fields of spin zero and one.

Although the general form of the equations of motion is thus known, this in itself does not mean that a self-consistent theory of arbitrary point singularities is possible. Except in the case of spin zero, the field equations impose certain restrictions on the charge density; furthermore the equations of motion interrelate the momentum, angular momentum, and charge density of the particle. In addition it appears to be desirable for the physical interpretation of the theory to impose further restrictions on the spin and the 2^n-pole moments of the particle. It is not inherent in the method that it should be possible to satisfy all these relations for any type of charge singularity.

Although many different types of "elementary" particles have been found in nature, it appears that present theories (quantum as well as classical) are far too wide in allowing an infinite variety of such particles. The present study was undertaken in the hope that it might be possible to exclude at least some types of particles by proving that they could not satisfy all the conditions required. In the case of the scalar and pseudoscalar meson fields it was found, on the contrary, that all the conditions can be fulfilled for multipole singularities of arbitrary order and also for arbitrary combinations of such poles by explicit construction of the most general form of such poles. The results obtained in the case of neutral fields are summarized in Theorems I–IV of Secs. III–VI. Then the theory is extended to charge-symmetric fields; the results are summarized in Theorem V of Sec. VII.

II. THE EQUATIONS OF MOTION IN NEUTRAL FIELDS

We shall first outline the field-theoretical derivation of the general form of the equations of motion of a point particle interacting with a neutral scalar or pseudoscalar meson field as given by Harish-Chandra.[2] Except for minor changes we shall use his notation.

We consider a four-space with coordinates x_μ, Greek letters taking the values 0, 1, 2, 3, where x_0 is the time coordinate. Repetition of an index implies summation

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[1] P. A. M. Dirac, Proc. Roy. Soc. (London) A167, 148 (1938).
[2] Harish-Chandra, Proc. Roy. Soc. (London) A185, 269 (1946).