Nuclear Matrix Elements in Beta Decay

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By using the *j-j* coupling model, all the β-decay nuclear matrix elements are calculated (in terms of radial integrals) for one- and two-nucleon configurations. The operators in terms of which one can describe the entire theory are of five types. Three of these, involving the nucleon momentum operator, replace the operators which, in the conventional representation of the theory, appeared as odd Dirac operators. The operators in the present representation, which is most naturally expressed in terms of spherical notation and angular momentum eigenfunctions are explicitly related to those which appeared in the older notation as cartesian tensor components. The results for both one- and two-nucleon configurations are expressed in terms of reduced matrix elements which, in turn, can be written in terms of Racah coefficients and other coefficients derived from them. All these coefficients, and thereby the reduced matrix elements, can be written in terms of comparatively simple algebraic formulas which cover all cases of interest. A brief discussion of the implications of these results for spectral shapes and comparative half-lives is given.

I. INTRODUCTION AND DEFINITION OF MATRIX ELEMENTS

IN the preceding paper[1] it has been shown that the entire theory of forbidden transitions can be formulated in terms of a representation in which only even Dirac operators occur in the nucleon space. This reformulation, which proceeds by means of a Foldy-Wouthuysen transformation,[2] implies that one needs only nonrelativistic nuclear wave functions for the description of all beta transitions. More specifically, if one adopts a particular coupling model, the opportunity for calculating nuclear matrix elements for forbidden as well as for allowed transitions is at hand. As has been emphasized by others, a particularly valuable application of the theory and experimental results of beta decay lies in the possibility of obtaining information of greater detail with regard to nuclear forces, by comparing ratios of matrix elements deduced from observed shapes of spectra with the results calculated on the basis of some model. The results presented below should facilitate such a comparison.

One should distinguish between a physical model in which an assertion is made concerning the coupling of the angular momentum vectors, and one in which additional assumptions concerning nuclear forces are made. We shall make a specific assumption concerning the vector coupling and shall consider the *j-j* coupling model in this paper. However, unless additional information descriptive of the nuclear force model is introduced, the matrix elements are reduced to radial integrals about which no very quantitative statements can be made. For specified angular momenta of the initial and final states, there are only a comparatively small number of such radial integrals (three for a given order of forbiddenness). In one well-known case the radial integral, in fact, reduces to the normalization integral.[3] Of greater interest is the fact that for a number of cases the radial integrals, associated with two interfering matrix elements, are identical, so that the shape of the spectrum can be specified with no arbitrariness. A comparison with experimental spectra in such cases provides a check on the validity of the coupling model. In all remaining cases the radial integrals play a nontrivial role and the shape of the beta spectrum is conditioned by the more detailed aspect of nuclear forces. The problem of evaluating the radial integrals by more detailed assumptions concerning nuclear forces is deferred for later consideration. In any case the results presented below exhibit explicitly the nuclear-force parameter (ratio of radial integrals) which may be ascertained from the analysis of observed shapes of beta spectra.

In Sec. II the nuclear matrix elements are expressed in terms of reduced matrix elements and the latter are evaluated for the single particle case (closed shells$\pm$one nucleon). In Sec. III it is shown that for even-mass nuclei, where two nucleons may participate in the beta transition, the nuclear matrix elements may be easily obtained from the single particle reduced matrix elements of Sec. II. In the remainder of this section we define the matrix elements, or combinations thereof, which are pertinent for our considerations. For this purpose we restrict our attention to forbiddenness order $n\leqslant 2$, although it is a trivial matter to extend the discussion to higher n.

It was shown in I, Eq. (21I), that in the new (even) formulation of the theory one makes the following replacements for the odd operators: $\boldsymbol{\alpha}\rightarrow-\mathbf{p}/M$, $i\beta\boldsymbol{\alpha}\rightarrow\boldsymbol{\sigma}\times\mathbf{p}/M$, $\gamma_5\rightarrow\boldsymbol{\sigma}\cdot\mathbf{p}/M$, where M is the nucleon mass and $\mathbf{p}=-i\nabla$. Then, all higher-rank tensors are

[1] M. E. Rose and R. K. Osborn [Phys. Rev. **93**, 1315 (1953)]. References to this paper are denoted by I.

[2] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

[3] Allowed transitions and light nuclei for which Coulomb effects are of minor importance and for which charge symmetry seems to be justified; see O. Kofoed-Hansen and A. Winther, Phys. Rev. **86**, 428 (1952). The allowed matrix elements, in the *j-j* coupling model, have also been thoroughly investigated by I. Talmi, Phys. Rev. **91**, 122 (1953).

formed exactly as usual.[4] Of course, some care must be exercised because of the noncommutability of $\mathbf{r}$ and the operators involving $\mathbf{p}$.

The exact correspondence between operators can easily be obtained. One may use the Cartesian notation [see (23I)]

$$\mathcal{T}_{ij}(\mathbf{a},\mathbf{b})=\int(a_ib_j+a_jb_i-\tfrac{2}{3}\mathbf{a}\cdot\mathbf{b}\delta_{ij}), \tag{1}$$

for second-rank tensors, for example. However, we find it far more convenient to represent the tensor in the spherical form:

$$T_{\lambda L}{}^{m}(\mathbf{r},\mathbf{B})=\sum_{m'}C(1L\lambda;-m',m+m')\mathcal{Y}_L{}^{m'+m}(\mathbf{r})\mathcal{Y}_1{}^{-m'}(\mathbf{B}), \tag{2}$$

where $\mathcal{Y}_L$ is a solid spherical harmonic of degree L and the C coefficient is a vector addition (Wigner or Clebsch-Gordan) coefficient. All the tensors with which we have to deal can be represented in the form (2) or simply as a solid harmonic $\mathcal{Y}_\lambda(\mathbf{r})$, or finally as $\mathcal{Y}_\lambda(\mathbf{r})\boldsymbol{\sigma}\cdot\mathbf{p}$. Throughout the notation is such that λ is the tensor rank and L gives the parity π'. For $\mathbf{B}=\mathbf{p}$ and $\boldsymbol{\sigma}\times\mathbf{p}$, $\pi=(-)^{L+1}$ while for $\mathbf{B}=\boldsymbol{\sigma}$, $\pi=(-)^L$.

The matrix element combinations are then conveniently given in terms of the following notation:

$$\sum_m\int T_{\lambda L}{}^{m}(\mathbf{r},\mathbf{A})\left[\int T_{\lambda L'}{}^{m}(\mathbf{r},\mathbf{B})\right]^*\equiv I_\lambda(LL';\mathbf{AB}), \tag{3a}$$

where, to avoid redundancy, we write $I_\lambda(LL,\mathbf{AA})\equiv I_\lambda(L;\mathbf{A})$; and

$$\sum_m\int\mathcal{Y}_\lambda{}^{m}(\mathbf{r})\left[\int T_{\lambda L}{}^{m}(\mathbf{r},\mathbf{A})\right]^*\equiv J_\lambda(L;\mathbf{A}); \tag{3b}$$

$$\sum_m\left|\int\mathcal{Y}_\lambda{}^{m}(\mathbf{r})\right|^2\equiv K_\lambda; \tag{3c}$$

$$\sum_m\left|\int\mathcal{Y}_\lambda{}^{m}(\mathbf{r})\boldsymbol{\sigma}\cdot\mathbf{p}\right|^2\equiv\mathcal{K}_\lambda. \tag{3d}$$

Of course, the parity of the interfering operators must be the same. In all cases $-\lambda\leqslant m\leqslant\lambda$. The quantity $\mathcal{K}_\lambda$ enters in only the A interaction and with the restriction $n\leqslant 2$ only $\mathcal{K}_0$ is needed. We also write $\mathcal{K}_0{}^{\frac{1}{2}}\equiv\int\boldsymbol{\sigma}\cdot\mathbf{p}/(4\pi)^{\frac{1}{2}}$. In Tables I and II we give the relation between these quantities and the corresponding invariant matrix element combinations in the customary notation.[4,5] In each case we have given the normalization constant $\mathfrak{N}$ so defined that $\mathfrak{N}$ is the ratio of the entry in the first column to that in the second. Thus, as an example,

$$I_2(11;\boldsymbol{\sigma\sigma})=I_2(1;\boldsymbol{\sigma})=\frac{9}{64\pi^2}\sum_{ij}|B_{ij}|^2.$$

The tables also indicate the order of forbiddenness and the interactions in which the particular combination occurs. For convenience, the tabulation is separated into two parts: Table I for the square terms, Table II for the interference terms. Throughout we omit consideration of tensors of rank 0 and 1 for second forbidden transitions. These should be small corrections to the

TABLE I. Square matrix element combinations for order of forbiddenness $n\leqslant 2$. The normalization constant $\mathfrak{N}$ in column 3 is the ratio of first to second column entries. Column 5 gives the pertinent interactions.

Spherical notation	Cartesian notation	$\mathfrak{N}$	n	Interactions
$I_1(0;\boldsymbol{\sigma})$	$\lvert\int\boldsymbol{\sigma}\rvert^2$	$3/16\pi^2$	0	T, A
K_1	$\lvert\int\mathbf{r}\rvert^2$	$3/4\pi$	1	S, V
$I_1(0,\mathbf{p})$	$\lvert\int\boldsymbol{\alpha}\rvert^2$	$3M^2/(4\pi)^2$	1	V
$I_1(0,\boldsymbol{\sigma}\times\mathbf{p})$	$\lvert\int\beta\boldsymbol{\alpha}\rvert^2$	$3M^2/(4\pi)^2$	1	T
$\mathcal{K}_0$	$\lvert\int\gamma_5\rvert^2$	$M^2/4\pi$	1	A
$I_0(1;\boldsymbol{\sigma})$	$\lvert\int\boldsymbol{\sigma}\cdot\mathbf{r}\rvert^2$	$3/16\pi^2$	1	T, A, P
$I_1(1;\boldsymbol{\sigma})$	$\lvert\int\boldsymbol{\sigma}\times\mathbf{r}\rvert^2$	$9/32\pi^2$	1	T, A
$I_2(1;\boldsymbol{\sigma})$	$\Sigma_{ij}\lvert B_{ij}\rvert^2$	$9/64\pi^2$	1	T, A
K_2	$\Sigma_{ij}\lvert R_{ij}\rvert^2$	$15/8\pi$	2	S, V
$I_2(1;\mathbf{p})$	$\Sigma_{ij}\lvert A_{ij}\rvert^2$	$9M^2/64\pi^2$	2	V
$I_2(1;\boldsymbol{\sigma}\times\mathbf{p})$	$\Sigma_{ij}\lvert A_{ij}{}^{\beta}\rvert^2$	$9M^2/64\pi^2$	2	T
$I_2(2;\boldsymbol{\sigma})$	$\Sigma_{ij}\lvert T_{ij}\rvert^2$	$15/64\pi^2$	2	T, A
$I_3(2;\boldsymbol{\sigma})$	$\Sigma_{ijk}\lvert S_{ijk}\rvert^2$	$15/96\pi^2$	2	T, A

TABLE II. Interfering matrix element combinations for order of forbiddenness $n\leqslant 2$. The normalization constant $\mathfrak{N}$ in column 3 is the ratio of first to second column entries. X-Y in column 5 indicates that the matrix element product in column 1 (or 2) occurs in the interference between X and Y interactions. X indicates the appearance of interference for the pure interaction.

Spherical notation	Cartesian notation	$\mathfrak{N}$	n	Interactions
$J_1(0;\mathbf{p})$	$i\int\mathbf{r}\cdot\int\boldsymbol{\alpha}^*$	$3iM/(4\pi)^{\frac{3}{2}}$	1	V, S-V
$J_1(0;\boldsymbol{\sigma}\times\mathbf{p})$	$i\int\mathbf{r}\cdot\int\beta\boldsymbol{\alpha}^*$	$-3M/(4\pi)^{\frac{3}{2}}$	1	S-T, V-T
$J_1(1;\boldsymbol{\sigma})$	$i\int\mathbf{r}\cdot\int\boldsymbol{\sigma}\times\mathbf{r}^*$	$-(27/2)^{\frac{1}{2}}/16\pi^2$	1	S-T, S-A V-T, V-A
$I_1(01;\mathbf{p},\boldsymbol{\sigma})$	$\int\boldsymbol{\alpha}\cdot\int\boldsymbol{\sigma}\times\mathbf{r}^*$	$iM(27/2)^{\frac{1}{2}}/16\pi^2$	1	V-T, V-A
$I_1(01;\boldsymbol{\sigma}\times\mathbf{p},\boldsymbol{\sigma})$	$\int\beta\boldsymbol{\alpha}\cdot\int\boldsymbol{\sigma}\times\mathbf{r}^*$	$M(27/2)^{\frac{1}{2}}/16\pi^2$	1	T, T-A
$\mathcal{K}_0{}^{\frac{1}{2}*}T_{01}{}^{0}(\mathbf{r},\boldsymbol{\sigma})$	$i\int\boldsymbol{\sigma}\cdot\mathbf{r}\int\gamma_5{}^*$	$iM3^{\frac{1}{2}}/(4\pi)^{\frac{3}{2}}$	1	A, T-A, P-A
$I_1(00;\mathbf{p},\boldsymbol{\sigma}\times\mathbf{p})$	$\int\boldsymbol{\alpha}\cdot\int\beta\boldsymbol{\alpha}^*$	$3iM^2/16\pi^2$	1	V-T
$J_2(2;\boldsymbol{\sigma})$	$i\Sigma_{ij}\,R_{ij}T_{ij}{}^*$	$15/(8\pi)^{\frac{3}{2}}$	2	S-T, S-A V-T, V-A
$J_2(1;\mathbf{p})$	$i\Sigma_{ij}\,R_{ij}A_{ij}{}^*$	$iM(135)^{\frac{1}{2}}/(8\pi)^{\frac{3}{2}}$	2	S-V, V
$J_2(1;\boldsymbol{\sigma}\times\mathbf{p})$	$i\Sigma_{ij}\,R_{ij}A_{ij}{}^{\beta*}$	$M(135)^{\frac{1}{2}}/(8\pi)^{\frac{3}{2}}$	2	S-T, V-T
$I_2(21;\boldsymbol{\sigma},\mathbf{p})$	$\Sigma_{ij}\,T_{ij}A_{ij}{}^*$	$-iM(135)^{\frac{1}{2}}/(8\pi)^2$	2	V-T, V-A
$I_2(21;\boldsymbol{\sigma},\boldsymbol{\sigma}\times\mathbf{p})$	$\Sigma_{ij}\,T_{ij}A_{ij}{}^{\beta*}$	$M(135)^{\frac{1}{2}}/(8\pi)^2$	2	T, T-A
$I_2(11;\boldsymbol{\sigma}\times\mathbf{p},\mathbf{p})$	$\Sigma_{ij}\,A_{ij}{}^{\beta}A_{ij}{}^*$	$-9iM^2/64\pi^2$	2	T-A

[4] E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. 60, 308 (1941). For interaction mixtures, see D. L. Pursey, Phil. Mag. 42, 1193 (1951), and A. M. Smith, Phys. Rev. 82, 955 (1951). While one can set $\beta=-1$ in our representation, this is not permissible in the conventional formulation, and appropriate changes are necessary in the results of the first and third of these references.

[5] In the customary (or cartesian) notation we have used such symbols as $\boldsymbol{\alpha}$ and A_{ij} to make comparison easier. Of course, the representations for the nuclear wave functions involved in the matrix elements of columns 1 and 2 are not the same.

contribution to the allowed matrix elements if both Fermi and Gamow-Teller interactions are present in appreciable strength. Otherwise, if these terms are considered it must be remembered that the conventional theory does not represent them properly.[1]

As an aid in the construction of Tables I and II we note that (see Appendix A),

$$T_{\lambda\lambda-1}{}^{m}(\mathbf{r},\mathbf{r}\times\mathbf{A})=i[6(2\lambda-1)]^{\frac{1}{2}}C(1\lambda-1\lambda;00)\times W(11\lambda\lambda-1;\lambda 1)T_{\lambda\lambda}{}^{m}(\mathbf{r},\mathbf{A}) \quad (4)$$
$$=i[(\lambda+1)/(2\lambda+1)]^{\frac{1}{2}}T_{\lambda\lambda}{}^{m}(\mathbf{r},\mathbf{A}).$$

Here W is a Racah coefficient,[6] and in this case a particularly simple one. We also observe that every entry in the second column in both Tables I and II is real and that therefore the corresponding statement applies to the first-column entry divided by $\mathfrak{N}$. In general, we recognize five different types of tensors:[7]

$$\begin{array}{ll} \mathcal{Y}_{\lambda}{}^{M}(\mathbf{r}) & \text{Type I,} \\ T_{\lambda L}{}^{m}(\mathbf{r},\boldsymbol{\sigma}) & \text{Type II,} \\ T_{\lambda L}{}^{m}(\mathbf{r},\mathbf{p}) & \text{Type III,} \\ \mathcal{Y}_{\lambda}{}^{m}(\mathbf{r})\boldsymbol{\sigma}\cdot\mathbf{p} & \text{Type IV,} \\ T_{\lambda L}{}^{m}(\mathbf{r},\boldsymbol{\sigma}\times\mathbf{p}) & \text{Type V.} \end{array} \quad (A)$$

If these are respectively designated as τ_{I}, τ_{II}, $\cdots\tau_{\mathrm{V}}$, then one gets real matrix element products for interference between any one of $i\tau_{\mathrm{I}}$, $i\tau_{\mathrm{II}}$, τ_{III}, τ_{IV}, and $i\tau_{\mathrm{V}}$, and the complex conjugate of any other.[8] Our results confirm this statement.

The order of forbiddenness for the five types of operators is $n=\lambda$, L, $L+1$, $\lambda+1$, and $L+1$, respectively.

II. SINGLE-PARTICLE MATRIX ELEMENTS IN *j-j* COUPLING

We now consider any one of the five operator types listed in Part A at the end of the foregoing section. Then, for $T_{\lambda L}{}^{-m}(\mathbf{r},\mathbf{B})$, say, the matrix element can always be expressed in the form

$$\int T_{\lambda L}{}^{-m}(\mathbf{r},\mathbf{B})\equiv(f|T_{\lambda L}{}^{-m}(\mathbf{r},\mathbf{B})|i)=C(j'\lambda j;\mu',-m)(f\|T_{\lambda L}\|i), \quad (5)$$

which is the Wigner-Eckart theorem. In (5) the entire magnetic quantum number dependence is contained in the vector addition coefficient and the remaining factor is a reduced matrix element. Also, j and $\mu=\mu'-m$ are the final state (f) quantum numbers; j', μ' refer to the initial state (i). Thus, m is the total z component of the angular momentum carried off by the electron and neutrino.[1] The parity quantum number, for which we have no immediate need, will be expressed explicitly in subsequent developments. A relation exactly like (5) holds for each of the other four types of operators.

What is pertinent for the transition probablity is the absolute square or cross product of two matrix elements summed over final magnetic substates (μ) and averaged over initial substates (μ'). Therefore we define

$$\langle I_{\lambda}(LL';\mathbf{AB})\rangle=\frac{1}{2j'+1}\sum_{m\mu\mu'}(j\mu|T_{\lambda L}{}^{-m}(\mathbf{r},\mathbf{A})|j'\mu')\times(j\mu|T_{\lambda L'}{}^{-m}(\mathbf{r},\mathbf{B})|j'\mu')^{*}. \quad (6)$$

Using (5), we can reduce Eq. (6) to

$$\langle I_{\lambda}(LL';\mathbf{AB})\rangle=\frac{2j+1}{2j'+1}(j\|T_{\lambda L}(\mathbf{r},\mathbf{A})\|j')\times(j\|T_{\lambda L}(\mathbf{r},\mathbf{B})\|j')^{*}. \quad (7)$$

Similarly,

$$\langle J_{\lambda}(L;A)\rangle=\frac{2j+1}{2j'+1}(j\|\mathcal{Y}_{\lambda}\|j')(j\|T_{\lambda L}(\mathbf{r},\mathbf{A})\|j')^{*} \quad (7a)$$

is the pertinent quantity for interference between type I and type II, III, or V operators. Note that type I and type IV operators cannot interfere. In the same way one finds

$$\langle K_{\lambda}\rangle=\frac{2j+1}{2j'+1}|(j\|\mathcal{Y}_{\lambda}\|j')|^{2}, \quad (7b)$$

etc. Thus, all that are required are the reduced matrix elements which we proceed to calculate by establishing the Wigner-Eckart theorem in each case. That is, we calculate $(f|T_{\lambda L}{}^{-m}|i)$ and compare the result with the right-hand side of Eq. (5). This is done in the following for each of the five types of operators.

For a single particle the final-state wave function may be written in the form [see (I39)],

$$\psi_{f}=\mathcal{R}(r)\textstyle\sum_{\tau}C(l\tfrac{1}{2}j;\mu-\tau,\tau)\chi_{\frac{1}{2}}{}^{\tau}Y_{l}{}^{\mu-\tau}(\mathbf{r}). \quad (8)$$

Here $\mathcal{R}$ is a real radial wave function, $\chi_{\frac{1}{2}}{}^{\tau}$ (with $\tau=\pm\frac{1}{2}$) a spin eigenfunction, and $\mathbf{r}$ is the unit vector. For the initial state we replace $\mathcal{R}$, μ, l, j by $\mathcal{R}'$, μ', l', j', respectively. Since these single-particle wave functions are eigenfunctions of j, l, μ (for ψ_f), we write the matrix elements in the form $(jl\mu|T_{\lambda L}{}^{-m}|j'l'\mu')$ and the reduced matrix elements as $(jl\|T_{\lambda L}\|j'l')$, and similarly for the other tensor operators.

It should be noted that the reduced matrix elements are not Hermitian.[9] In fact, if we consider Eq. (5)

[6] G. Racah, Phys. Rev. **62**, 438 (1942); see also Biedenharn, Blatt, and Rose, Revs. Modern Phys. **24**, 249 (1952) (referred to as BBR), and L. C. Biedenharn, Oak Ridge National Laboratory Report ORNL-1098 (unpublished). The properties of the C coefficients needed here are also given by Racah. See also Rose, Biedenharn, and Arfken, Phys. Rev. **85**, 5 (1952).

[7] The tensors of types II and III are exactly those which occur in the emission of electromagnetic radiation by particles with spin.

[8] These results follow directly by using the properties of the time-reversal operator; see C. L. Longmire and A. M. L. Messiah, Phys. Rev. **83**, 464 (1951).

[9] Compare G. Racah, reference 6, Eqs. (31) and (31′). Our normalization of the reduced matrix elements differs from that

together with

$$(i|T_{\lambda L}{}^m|f)^* = (f|T_{\lambda L}{}^{m\dagger}|i) = (-)^{j'-j+m}\left(\frac{2j'+1}{2j+1}\right)^{\frac{1}{2}} \times C(j'\lambda j;\mu'-m)(i\|T_{\lambda L}\|f)^*, \quad (9)$$

and similar relations for the other tensor operators, we find the relation between a reduced matrix element and its Hermitian conjugate by considering $T_{\lambda L}{}^{m\dagger}$. In general, if Ω^m denotes any one of the five types of operators, we find

$$\Omega^{m\dagger} = (-)^m\{(-)^\eta\Omega^{-m} + \Gamma\Omega'^{-m}\}, \quad (10)$$

where Γ is a constant (including zero) and Ω' is another tensor operator listed in A. Thus,

$$\left(\frac{2j'+1}{2j+1}\right)^{\frac{1}{2}}(-)^{j'-j}(i\|\Omega\|f)^* = (-)^\eta(f\|\Omega\|i) + \Gamma(f\|\Omega'\|i). \quad (11)$$

For each of the five operators we give the results for $(-)^\eta$, Γ and Ω' in Table III. Of course, the parity and rank of Ω' is exactly the same as in Ω. In connection with the type III operator we shall not be interested in $T_{\lambda\lambda+1}(\mathbf{r}, \mathbf{p})$ since this occurs in terms which are small corrections to $T_{\lambda\lambda-1}(\mathbf{r}, \mathbf{p})$. The orders of forbiddenness corresponding to these two operators are $\lambda+2$ and λ, respectively. Also, the type IV operator, occurring in the A interaction, is of interest for the case $\lambda=0$ only. Thus, $\Gamma=0$ effectively for all but type V operators.[10] The appearance or absence of factors i is just what is required for the reality theorem quoted at the end of Sec. I. A derivation of a typical one of the results is given in Appendix A.

The results of Table III are useful as a check and also provide relationships between reduced matrix elements which simplifies the calculations and presentation of results. We now consider the explicit calculations of the matrix elements of the five operator types. We do not restrict the order of forbiddenness wherever it is just as convenient to give general results.

A. Type I Operator

We consider the $\mathcal{Y}_\lambda(\mathbf{r})$ operator which occurs in the S and V interactions. Then

$$(f|\mathcal{Y}_\lambda{}^{-m}|i) = \int r^{\lambda+2}\mathcal{R}\mathcal{R}'dr\int d\omega(\chi_\kappa{}^\mu|Y_\lambda{}^{-m}|\chi_{\kappa'}{}^{\mu'}),$$

where $\chi_\kappa{}^\mu$ is the 2-component spinor which represents the spin angular part of ψ_f [see Eqs. (2) and (39I)], κ represents both j and l, and $[j=|\kappa|-\frac{1}{2}=l-\frac{1}{2}\kappa/|\kappa|]$.

used by Racah; see his Eq. (29). Actually, all but one (type I) of the tensors listed above are more general than those discussed by Racah, and his rule for Hermitian conjugation does not apply in all cases. See Table III.

[10] Operators such as $\mathcal{Y}_1(\mathbf{r})\boldsymbol{\sigma}\cdot\mathbf{p}$ constitute second forbidden corrections to allowed transitions. However, such operators may play an important role in so-called l forbidden transitions.

TABLE III. Properties of the tensor operators under Hermitian conjugation; see Eq. (11).

Type	Ω^m	$(-)^\eta$	Γ	Ω'^{-m}
I	$\mathcal{Y}_\lambda{}^m$	1	0	
II	$T_{\lambda L}{}^m(\mathbf{r},\boldsymbol{\sigma})$	$(-)^{L+\lambda+1}$	0	
III	$T_{\lambda L}{}^m(\mathbf{r},\mathbf{p})$	$(-)^{L+\lambda+1}$	$i\left(\frac{3}{4\pi}\right)^{\frac{1}{2}}\left(\frac{L}{2L-1}\right)^{\frac{1}{2}} \times(2L+1)\delta_{\lambda,L-1}$	$\mathcal{Y}_\lambda{}^{-m}$
IV	$\mathcal{Y}_\lambda{}^m\boldsymbol{\sigma}\cdot\mathbf{p}$	1	$-i\left[\frac{4\pi}{3}\lambda(2\lambda+1)\right]^{\frac{1}{2}}$	$T_{\lambda\lambda-1}{}^{-m}(\mathbf{r},\boldsymbol{\sigma})$
V	$T_{\lambda L}{}^m(\mathbf{r},\boldsymbol{\sigma}\times\mathbf{p})$	$(-)^{L+\lambda+1}$	$(6L)^{\frac{1}{2}}(2L+1) \times W(11L-1L;1\lambda)$	$T_{\lambda L-1}{}^{-m}(r,\boldsymbol{\sigma})$

Using Eq. (45I), we find immediately that

$$(jl\|\mathcal{Y}_\lambda\|j'l') = (4\pi)^{-\frac{1}{2}}(-)^{l'+j-\frac{1}{2}}[(2\lambda+1)(2l+1)(2j'+1)]^{\frac{1}{2}} \times C(l\lambda l';00)W(ljl'j';\tfrac{1}{2}\lambda)\mathfrak{F}_\lambda, \quad (12)$$

where (5) has been used and $\mathfrak{F}_\lambda$ is a radial integral:[11]

$$\mathfrak{F}_\lambda = \int r^{\lambda+2}\mathcal{R}\mathcal{R}'dr. \quad (12')$$

Using the well-known symmetry properties of the Racah coefficients, we check the Hermitian conjugation property of Table III at once.

The special values in which we are interested correspond to $\lambda=0$, 1, and 2. For $\lambda=0$ we get $(4\pi)^{-\frac{1}{2}}\int 1 = (4\pi)^{-\frac{1}{2}}\mathfrak{F}_0$. For $\lambda=1$ one obtains $\int\mathbf{r}$ using Table I and the definitions given in Eq. (3); see also Eq. (7b). Similarly, for $\lambda=2$ one obtains R_{ij}. The specific results are easily obtained by using Table I of BBR[6] for the W coefficient with the following transcription of notation $(ljl'j'\lambda \leftrightarrow l_1J_1l_2J_2L)$ and

$$C(l1l';00) = \begin{cases} [(l+1)/(2l+1)]^{\frac{1}{2}}, & l'=l+1 \\ -[l/(2l+1)]^{\frac{1}{2}}, & l'=l-1, \end{cases} \quad (13a)$$

$$C(l2l';00) = \begin{cases} \left[\frac{3}{2}\frac{(l+1)(l+2)}{(2l+1)(2l+3)}\right]^{\frac{1}{2}}, & l'=l+2 \\ -\left[\frac{l(l+1)}{(2l-1)(2l+3)}\right]^{\frac{1}{2}}, & l'=l \\ \left[\frac{3}{2}\frac{l(l-1)}{(2l-1)(2l+1)}\right]^{\frac{1}{2}}, & l'=l-2. \end{cases} \quad (13b)$$

For higher than second forbidden transitions one needs the corresponding results for $L>2$. These can be easily obtained from BBR, Eq. (5). Of course, we have the

[11] These radial integrals have been evaluated by S. A. Moszkowski, Phys. Rev. 89, 474 (1952), using a number of rather simple models. See also H. Brysk, Phys. Rev. 86, 996 (1952). The use of relativistic nuclear wave functions as in this last reference is unnecessary here.

condition $\Delta(jj'\lambda)$ in all cases; i.e.,

$$|j-j'|\leqslant\lambda\leqslant j+j'. \quad (14)$$

As an example, we give $\langle\sum_{ij}|R_{ij}|^2\rangle$ (averaged and summed over initial and final states, respectively) for $l-\frac{1}{2}=j=j'+2=l'+\frac{3}{2}$. This is

$$\langle\sum_{ij}|R_{ij}|^2\rangle=\frac{l(l-1)}{(2l-1)(2l-3)}\mathfrak{F}_2^2. \quad (15)$$

As is to be expected, this vanishes for the specified quantum numbers unless $l\geqslant 2$.

B. Type II Operator

The operator $T_{\lambda L}{}^m(\mathbf{r},\boldsymbol{\sigma})$ occurs in T, A, and P interactions and, of course, in all interference terms of these among themselves and with S and V interactions. Here, and in the following, we give the results for the reduced matrix elements without exhibiting any of the details of the derivation. Such details for the type II operator matrix element are presented in Appendix B. The procedure for the remaining cases is very much the same.

As shown in Appendix B we have

$$(jl\|T_{\lambda L}(\mathbf{r}\boldsymbol{\sigma})\|j'l')=\frac{3\sqrt{2}}{4\pi}(-)^{j'-j}[(2L+1)(2\lambda+1)(2l+1)(2j'+1)]^{\frac{1}{2}}\times C(lLl';00)X(\tfrac{1}{2}1\tfrac{1}{2};j\lambda j';lLl')\mathfrak{F}_L. \quad (16)$$

In (16) the X coefficient is a combination of Racah coefficients originally defined by Fano.[12] The precise relation is exhibited in Appendix B. All the X coefficients which may ever arise in β transitions can be expressed in comparatively simple algebraic form as follows. We consider four cases:

$$\begin{aligned}&\text{Case 1.}\quad j=l+\tfrac{1}{2},\quad j'=l'+\tfrac{1}{2};\\&\text{Case 2.}\quad j=l+\tfrac{1}{2},\quad j'=l'-\tfrac{1}{2};\\&\text{Case 3.}\quad j=l-\tfrac{1}{2},\quad j'=l'+\tfrac{1}{2};\\&\text{Case 4.}\quad j=l-\tfrac{1}{2},\quad j'=l'-\tfrac{1}{2}.\end{aligned} \quad (17)$$

Since we are interested in T_{10}, T_{11}, T_{22}, T_{21}, T_{32} whose matrix elements are related to $\int\boldsymbol{\sigma}$, $\int\boldsymbol{\sigma}\times\mathbf{r}$, T_{ij}, B_{ij}, S_{ijk}, respectively, in the usual notation (see Tables I and II) we consider $\lambda=L$ and $\lambda=L+1$. There is one further case of interest which corresponds to $\lambda=L-1$; $T_{01}(r,\boldsymbol{\sigma})\sim\boldsymbol{\sigma}\cdot\mathbf{r}$. This case is quite trivial. In fact,

$$(jl\|T_{01}(\mathbf{r},\boldsymbol{\sigma})\|j'l')=\frac{\sqrt{3}}{4\pi}\delta_{\kappa,-\kappa'}\mathfrak{F}_1, \quad (18)$$

where $\delta_{\kappa,-\kappa'}$ is the same as $\delta_{jj'}\delta_{|l-l'|,\pm1}$. Thus, all remaining cases can be classified as above.

For the four cases listed in (17) and for $\lambda=L$, $L+1$ (with $\lambda\geqslant1$ throughout), we have:

Case 1, $\lambda=L+1$,

$$X=\frac{1}{4}\left[\frac{2}{3}\frac{(l+l'+\lambda+2)(l+l'+\lambda+1)(l-l'+\lambda)(l'-l+\lambda)}{(l+1)(l'+1)(2l+1)(2l'+1)\lambda(2\lambda-1)(2\lambda+1)}\right]^{\frac{1}{2}}; \quad (18a)$$

Case 1, $\lambda=L$,

$$X=-\frac{1}{4}\left[\frac{2}{3}\frac{(l+l'+\lambda+2)(l+l'-\lambda+1)}{(l+1)(l'+1)(2l+1)(2l'+1)\lambda(\lambda+1)(2\lambda+1)}\right]^{\frac{1}{2}}(l-l'); \quad (18b)$$

Case 2, $\lambda=L+1$,

$$X=\frac{1}{4}\left[\frac{2}{3}\frac{(l+l'+\lambda+1)(l-l'+\lambda+1)(l-l'+\lambda)(l'+l-\lambda+1)}{l'(l+1)(2l+1)(2l'+1)\lambda(2\lambda-1)(2\lambda+1)}\right]^{\frac{1}{2}}; \quad (18c)$$

Case 2, $\lambda=L$,

$$X=\frac{1}{4}\left[\frac{2}{3}\frac{(\lambda-l+l')(\lambda-l'+l+1)}{l'(l+1)(2l+1)(2l'+1)\lambda(\lambda+1)(2\lambda+1)}\right]^{\frac{1}{2}}(l+l'+1); \quad (18d)$$

Case 3, $\lambda=L+1$,

$$X=-\frac{1}{4}\left[\frac{2}{3}\frac{(l'+l-\lambda+1)(l'+l+\lambda+1)(l'-l+\lambda+1)(l'-l+\lambda)}{l(l'+1)(2l+1)(2l'+1)\lambda(2\lambda-1)(2\lambda+1)}\right]^{\frac{1}{2}}; \quad (18e)$$

Case 3, $\lambda=L$,

$$X=\frac{1}{4}\left[\frac{2}{3}\frac{(l+\lambda-l')(l'+\lambda-l+1)}{l(l'+1)(2l+1)(2l'+1)\lambda(\lambda+1)(2\lambda+1)}\right]^{\frac{1}{2}}(l+l'+1); \quad (18f)$$

Case 4, $\lambda=L+1$,

$$X=-\frac{1}{4}\left[\frac{2}{3}\frac{(l'+l-\lambda+1)(l'+l-\lambda)(l'-l+\lambda)(l-l'+\lambda)}{ll'(2l+1)(2l'+1)\lambda(2\lambda-1)(2\lambda+1)}\right]^{\frac{1}{2}}; \quad (18g)$$

Case 4, $\lambda=L$,

$$X=-\frac{1}{4}\left[\frac{2}{3}\frac{(l'+l+\lambda+1)(l'+l-\lambda)}{ll'(2l+1)(2l'+1)\lambda(\lambda+1)(2\lambda+1)}\right]^{\frac{1}{2}}(l'-l). \quad (18h)$$

[12] U. Fano, National Bureau of Standards Report NBS-1214 (unpublished), p. 48; also A. Simon, Phys. Rev. **90**, 1036 (1953). The pertinent properties of this coefficient are as follows: Write the coefficient as $X(a_{11}a_{12}a_{13};\ a_{21}a_{22}a_{23};\ a_{31}a_{32}a_{33})$ and consider the

Returning to (16), one observes that again $\Delta(lLl')$ exists and that, moreover, $l+l'+L=$even integer. However, for even L one of the possibilities ($l=l'$) is ruled out for $\lambda=L$, cases 1 and 4. This implies the exclusion of $j=j'$ for T_{ij}. The other triangular rule expressing conservation of angular momentum $\Delta(jj'\lambda)$ is fulfilled automatically by virtue of the fact that the X coefficients either vanish or become imaginary when this conservation rule is violated. The X coefficients, like the Racah coefficients, are defined to vanish under such circumstances so that they are always real.

The results given in (18) and (13) define the type II reduced matrix element completely in terms of the radial integral $\mathfrak{F}_L$. The special case $\lambda=1$, $L=0$, giving the matrix element $\int\boldsymbol{\sigma}$, yields results which agree completely with those previously published,[3] except that we have not assumed complete overlap of the radial wave functions ($\mathfrak{F}_0\neq 1$ necessarily). To what extent the distinction between favored allowed and ordinary allowed transitions can be attributed to the radial matrix element is not yet clear.

C. Type III Operator

The operator $T_{\lambda L}{}^m(\mathbf{r},\mathbf{p})$, which occurs in the V interaction only, gives rise to the reduced matrix element

$$(jl\|T_{\lambda L}(\mathbf{r},\mathbf{p})\|j'l')$$
$$=\frac{\sqrt{3}i}{4\pi}(-)^{l'+j'-\frac{1}{2}}[(2L+1)(2\lambda+1)(2l+1)(2j'+1)]^{\frac{1}{2}}$$
$$\times W(ljl'j';\tfrac{1}{2}\lambda)\{(l'+1)^{\frac{1}{2}}C(lLl'+1;00)$$
$$\times W(L\lambda l'+1l';1l)\mathcal{G}_{Ll'}{}^-$$
$$-l'^{\frac{1}{2}}C(lLl'-1;00)W(L\lambda l'-1l';1l)\mathcal{G}_{Ll'}{}^+\},\quad(19)$$

where two new radial matrix elements have been introduced

$$\mathcal{G}_{Ll'}{}^-=\int r^{L+2}\mathcal{R}\left(\frac{d}{dr}-\frac{l'}{r}\right)\mathcal{R}'dr,\quad(20a)$$

$$\mathcal{G}_{Ll'}{}^+=\int r^{L+2}\mathcal{R}\left(\frac{d}{dr}+\frac{l'+1}{r}\right)\mathcal{R}'dr.\quad(20b)$$

Since these are real, interchange of initial and final states is equivalent to Hermitian conjugation and

$$(\mathcal{G}_{Ll'}{}^-)^\dagger=-\mathcal{G}_{L,l'+L+1}{}^+,\quad(20c)$$

$$(\mathcal{G}_{Ll'}{}^+)^\dagger=-\mathcal{G}_{L,l'-L-1}{}^-.\quad(20d)$$

In (19) the C coefficients can be obtained from (13) (note the parity rule: $l+L+l'=$odd integer), while the Racah coefficients can be obtained from Tables I and II of BBR with the notation transcription $(L\lambda l'\pm 1l'l)\leftrightarrow(l_1J_1l_2J_2L)$ for the second and third W's. We again note that $\Delta(\lambda L1)$ exists and that for our purposes ($n\leqslant 2$) only T_{10} and T_{21} (giving $-\int\mathbf{p}/M$, corresponding to $\boldsymbol{\alpha}$, and $\mathcal{T}_{ij}(\mathbf{r},\mathbf{p})$, corresponding to A_{ij}) are needed. For the first case ($L=0$) the results become much simpler:

$$(jl\|T_{10}(\mathbf{r},\mathbf{p})\|j'l')$$
$$=\frac{i(-)^{l+l'+j'-\frac{1}{2}}}{4\pi}[3(2j'+1)]^{\frac{1}{2}}W(ljl'j';\tfrac{1}{2}1)$$
$$\times\{l^{\frac{1}{2}}\mathcal{G}_{0l'}{}^-\delta_{l'+1,l}-l'^{\frac{1}{2}}\mathcal{G}_{0l'}{}^+\delta_{l'-1,l}\}.\quad(21)$$

The results of Table III are checked from (19) by using (20c) as well as Table II and Eq. (5) of BBR.

D. Type IV Operator

The operator $\mathcal{Y}_\lambda(\mathbf{r})\boldsymbol{\sigma}\cdot\mathbf{p}$ occurs in the A interaction only. The reduced matrix element is

$$(jl\|\mathcal{Y}_\lambda(\mathbf{r})\boldsymbol{\sigma}\cdot\mathbf{p}\|j'l')$$
$$=i(-)^{l-l'}\left[\frac{3}{2\pi}(2l+1)(2\lambda+1)(2j'+1)\right]^{\frac{1}{2}}$$
$$\times\{(l'+1)^{\frac{1}{2}}C(l\lambda l'+1;00)W(1\tfrac{1}{2}l'j';\tfrac{1}{2}l'+1)$$
$$\times W(l'+1j'lj;\tfrac{1}{2}\lambda)\mathcal{G}_{\lambda,l'}{}^--l'^{\frac{1}{2}}C(l\lambda l'-1;00)$$
$$\times W(1\tfrac{1}{2}l'j';\tfrac{1}{2}l'-1)W(l'-1j'lj;\tfrac{1}{2}\lambda)\mathcal{G}_{\lambda l'}{}^+\}.\quad(22)$$

Again the C coefficient may be obtained from (13) and the parity rule is $l+\lambda+l'=$odd integer, as expected. The Racah coefficients can be obtained directly from Table I of BBR.

For $\lambda=0$ the result (22) simplifies to

$$(jl\|\mathcal{Y}_0(\mathbf{r})\boldsymbol{\sigma}\cdot\mathbf{p}\|j'l')$$
$$=-i\left(\frac{3}{2\pi}\right)^{\frac{1}{2}}(-)^{l+j'-\frac{1}{2}}\delta_{jj'}W(1\tfrac{1}{2}l'j';\tfrac{1}{2}l)$$
$$\times[l^{\frac{1}{2}}\delta_{l,l'+1}\mathcal{G}_{0l'}{}^--l'^{\frac{1}{2}}\delta_{l,l'-1}\mathcal{G}_{0l'}{}^+].\quad(22a)$$

E. Type V Operator

The operator $T_{\lambda L}{}^m(\mathbf{r},\boldsymbol{\sigma}\times\mathbf{p})$ occurs in only the T interaction. In the calculation of the reduced matrix element the operator $\mathcal{Y}_1(\boldsymbol{\sigma}\times\mathbf{p})$ is replaced by $T_{11}(\boldsymbol{\sigma},\mathbf{p})$ as in Appendix A, and finally by a product of two solid harmonics $\mathcal{Y}_1(\boldsymbol{\sigma})\mathcal{Y}_1(\mathbf{p})$. The essential complication introduced by the form of this operator is reflected in the fact that the magnetic quantum number sums involve seven C coefficients. [These may be indicated schematically by the couplings $\mathbf{l}+\frac{1}{2}\to\mathbf{j}$, $\mathbf{l}'+\frac{1}{2}\to\mathbf{j}'$, $\mathbf{1}+\mathbf{L}\to\boldsymbol{\lambda}$, $\mathbf{1}+\mathbf{1}\to\mathbf{1}$, $\mathbf{l}'+\mathbf{1}\to(\mathbf{l}'+\mathbf{b})$, $\mathbf{l}+(\mathbf{l}'+\mathbf{b})\to\mathbf{L}$, and $\frac{1}{2}+\frac{1}{2}\to\mathbf{1}$. All of these are intended as vector additions.

nine arguments arranged in a square array like the matrix a_{ij}. Then, the interchange of any two rows or columns multiplies X by $(-)^E$ where $E=\sum_{ij}a_{ij}$. Interchange of rows and columns ($a\to$transpose of a) leaves X unchanged. With the aid of this rule one verifies the result of Table III. A triangular condition exists between the elements of any row or any column of the matrix a. If one element of a is zero, one has

$$X=X(a_{11}a_{12}a_{13};a_{21}a_{22}a_{13};a_{31}a_{31}0)$$
$$=(-)^{a_{13}+a_{31}-a_{11}-a_{22}}[(2a_{13}+1)(2a_{31}+1)]^{-\frac{1}{2}}W(a_{11}a_{12}a_{21}a_{22};a_{13}a_{31}).$$

The coupling $\mathbf{l}'+1\to(\mathbf{l}'+\mathbf{b})$ with $b=\pm1$ is the result of the $\mathbf{p}$ operator acting on the orbital angular momentum of the initial state and $\mathbf{l}+(\mathbf{l}'+\mathbf{b})\to\mathbf{L}$ implies that $Y_{l'+b}$ is coupled to Y_l of the final state to give a resultant Y_L, so that orbital angular momentum and parity are conserved. The coupling $\frac{1}{2}+\frac{1}{2}\to1$ represents the fact that the initial and final spin functions $\chi_{\frac{1}{2}}{}^{\tau}$ and $\chi_{\frac{1}{2}}{}^{\tau'}$ must couple together to make a resultant with intrinsic spin of 1.] As a consequence it is not surprising that the reduced matrix element in this case is somewhat more complex than any previously encountered. A new combination of Racah coefficients enters and this new coefficient (M coefficient) will be defined below.

The reduced matrix element is

$$
\begin{aligned}
(jl\|T_{\lambda L}(\mathbf{r},\boldsymbol{\sigma}\times\mathbf{p})\|j'l')=&(-)^{l'+1+\lambda}(3/2\pi)\\
&\times[3(2\lambda+1)(2L+1)(2l+1)(2j'+1)]^{\frac{1}{2}}\\
&\{(l'+1)^{\frac{1}{2}}C(lLl'+1;00)M(l\lambda l'+1,1L\tfrac{1}{2},j'jl')\mathcal{G}_{Ll'}{}^{-}\\
&-l'^{\frac{1}{2}}C(lLl'-1;00)M(l\lambda l'-1,1L\tfrac{1}{2},j'jl')\mathcal{G}_{Ll'}{}^{+}.
\end{aligned}\tag{23}
$$

Here the only new quantity is the M coefficient which is defined as[13]

$$
\begin{aligned}
M(l\lambda l'+b,&\,1L\tfrac{1}{2},\,j'jl')\\
&=\textstyle\sum_s(-)^s(2s+1)W(l\lambda l'+b1;sL)W(l\lambda\tfrac{1}{2}j';sj)\\
&\qquad\times W(1l'\tfrac{1}{2}j';s\tfrac{1}{2})W(1l'1l'+b;s1).
\end{aligned}\tag{24}
$$

If we do not consider any transitions beyond second forbidden, we are interested in the reduced matrix elements for $\lambda=1$, $L=0$ (giving $\int\boldsymbol{\sigma}\times\mathbf{p}/M$ corresponding to $i\int\beta\boldsymbol{\alpha}$) and for $\lambda=2$, $L=1$ (corresponding to $A_{ij}{}^{\beta}$). For $L=0$, $\lambda=1$ one has

$$
\begin{aligned}
M(l1l'+b,&\,10\tfrac{1}{2},\,j'jl')\\
&=(-)^{l'}\delta_{l,l'+b}[3(2l+1)]^{-\frac{1}{2}}X(\tfrac{1}{2}1\tfrac{1}{2};j1j';l1l'),
\end{aligned}\tag{25}
$$

and numerical values can thus be obtained from (18b, d, f, h) with $\lambda=L=1$. For any other case it is possible to find a fairly simple algebraic form for the M coefficient. This is obtained by noting that in (24) one has $s=j'\pm\frac{1}{2}$. The two-term sum is then simplified by the use of Tables I and II of BBR. One finds, with the designations (17):

Case 1, $b=1$,

$$M=\frac{(-)^{l'}}{24}\frac{3l'-l+\lambda+1}{(l'+1)(2l'+1)}\left[\frac{(l+l'+\lambda+2)(l-l'+\lambda-1)(l+l'-\lambda+2)(l-l'+\lambda)}{\lambda(2\lambda-1)(2\lambda+1)(l+1)(2l+1)(2l'+3)}\right]^{\frac{1}{2}};\tag{26a}$$

Case 1, $b=-1$,

$$M=\frac{(-)^{l'+1}}{24(2l'+1)}\left[\frac{(l+l'+\lambda+2)(l+l'+\lambda+1)(l+l'+\lambda)(l+l'-\lambda+1)(l'-l+\lambda)(l'-l+\lambda-1)}{\lambda(2\lambda-1)(2\lambda+1)l'(l'+1)(l+1)(2l'-1)(2l+1)}\right]^{\frac{1}{2}};\tag{26b}$$

Case 2, $b=1$,

$$M=\frac{(-)^{l'+1}}{24(2l'+1)}\left[\frac{(l+l'-\lambda+2)(l+l'-\lambda+1)(l-l'+\lambda+1)(l-l'+\lambda)(l-l'+\lambda-1)(l'-l+\lambda)}{\lambda(2\lambda-1)(2\lambda+1)l'(l'+1)(l+1)(2l+1)(2l'+3)}\right]^{\frac{1}{2}};\tag{26c}$$

Case 2, $b=-1$,

$$M=\frac{(-)^{l'}(3l'+l-\lambda+2)}{24l'(2l'+1)}\left[\frac{(l+l'+\lambda+1)(l+l'+\lambda)(l'-l+\lambda-1)(l-l'+\lambda+1)}{\lambda(2\lambda-1)(2\lambda+1)(l+1)(2l+1)(2l'-1)}\right]^{\frac{1}{2}};\tag{26d}$$

Case 3, $b=1$,

$$M=\frac{(-)^{l'}(3l'+l+\lambda+2)}{24(l'+1)(2l'+1)}\left[\frac{(l'+l-\lambda+2)(l'-l+\lambda+1)(l-l'+\lambda-1)(l+l'-\lambda+1)}{\lambda(2\lambda-1)(2\lambda+1)l(2l+1)(2l'+3)}\right]^{\frac{1}{2}};\tag{26e}$$

Case 3, $b=-1$,

$$M=\frac{(-)^{l'}}{24(2l'+1)}\left[\frac{(l+l'+\lambda+1)(l+l'+\lambda)(l'-l+\lambda)(l'-l+\lambda-1)(l'-l+\lambda+1)(l-l'+\lambda)}{\lambda(2\lambda-1)(2\lambda+1)ll'(l'+1)(2l+1)(2l'-1)}\right]^{\frac{1}{2}};\tag{26f}$$

Case 4, $b=1$,

$$M=\frac{(-)^{l'-1}}{24(2l'+1)}\left[\frac{(l+l'-\lambda+2)(l+l'-\lambda+1)(l-l'+\lambda)(l-l'+\lambda-1)(l+l'+\lambda+1)(l+l'-\lambda)}{\lambda(2\lambda-1)(2\lambda+1)ll'(l'+1)(2l+1)(2l'+3)}\right]^{\frac{1}{2}};\tag{26g}$$

Case 4, $b=-1$,

$$M=\frac{(-)^{l'}}{24l'(2l'+1)}(3l'-l-\lambda+1)\left[\frac{(l'+l+\lambda)(l'-l+\lambda)(l'-l+\lambda-1)(l'+l-\lambda)}{\lambda(2\lambda-1)(2\lambda+1)l(2l+1)(2l'-1)}\right]^{\frac{1}{2}}.\tag{26h}$$

[13] There is no special significance about the ordering of the nine arguments on the left-hand side of Eq. (24) nor is there any reason, other than convenience of reading, for separating the arguments into triads by commas. The M coefficient is a special

Here we have used the parity rule: $l+l'+\lambda=$ even integer. These results, which apply for $\lambda=L+1$, together with Eq. (5) of BBR, are all that is needed to obtain the reduced matrix elements in terms of the radial integrals for any order of forbiddenness.

It is of interest to note that on the basis of the present model we can confirm certain results which have been quoted in the literature. We designate the terms in the β interaction which were originally even by e and those which were originally odd by o. Then the contributions of e terms to the transition probability will be denoted by ee, those of the o terms alone by oo, and the cross terms by eo. For first-forbidden transitions with $|\Delta j| =0$ or 1, it is well known that the correction factors are of order $(\alpha Z/\rho)^2$, where ρ is the nuclear radius. For most spectra of interest $\alpha Z/\rho W_0 \gg 1$, and this is of decisive importance for explaining the allowed shape of first forbidden spectra[14] (except RaE[1]). For eo terms and oo terms the correction factors are of order $\alpha Z/\rho$ and 1, respectively. This is valid for pure as well as mixed interactions. Nevertheless, all three types of terms are of the same order and, leaving aside the P interaction, are essentially energy independent. This comes about because the o-matrix elements are larger than the e matrix elements by a factor of order $\alpha Z/\rho$. This has already been pointed out in the literature.[15] In the present model this result may be seen as follows. The e and o matrix elements involve $\mathfrak{F}_1$ and $\mathcal{G}_{0l'}^{\pm}$, respectively. A relation between these is derived by forming the radial wave equations from Eq. (11 I). With

$$U_f=(\chi_\kappa{}^\mu,\ V_f\chi_\kappa{}^\mu),$$
$$U_i=(\chi_{\kappa'}{}^{\mu'},\ V_i\chi_{\kappa'}{}^{\mu'}), \tag{27}$$

where angular integration is included in the scalar product, one finds

$$\frac{d^2\mathfrak{R}}{dr^2}+\frac{2}{r}\frac{d\mathfrak{R}}{dr}+\left[2M(W_f+\Delta-U_f)-\frac{l(l+1)}{r^2}\right]\mathfrak{R}=0, \tag{28a}$$

and

$$\frac{d^2\mathfrak{R}^1}{dr^2}+\frac{2}{r}\frac{d\mathfrak{R}'}{dr}+\left[2M(W_i-U_i)-\frac{l'(l'+1)}{r^2}\right]\mathfrak{R}=0, \tag{28b}$$

with $\Delta=M_n-M_p$, the neutron-proton mass difference. Using the fact that U_i and U_f are Hermitian radial operators, elementary operations yield

$$(L+1)\mathcal{G}_{Ll'}{}^-=\tfrac{1}{2}(l-l'-L-1)(l+l'+L+2)\mathfrak{F}_{L-1}+M(W_i-W_f-\Delta)\mathfrak{F}_{L+1}-M\mathfrak{D}_L,$$
$$(L+1)\mathcal{G}_{Ll'}{}^+=\tfrac{1}{2}(l-l'+L+1)(l+l'-L)\mathfrak{F}_{L-1}+M(W_i-W_f-\Delta)\mathfrak{F}_{L+1}-M\mathfrak{D}_L, \tag{29}$$

where

$$\mathfrak{D}_L=\int\mathfrak{R}(r^{L+3}U_i-U_fr^{L+3})\mathfrak{R}'dr. \tag{29a}$$

If one defines[15]

$$\frac{1}{2}\Lambda\frac{\alpha Z}{\rho}=\frac{-i\int\boldsymbol{\alpha}}{\int\mathbf{r}}=\frac{-i\int\gamma_5}{\int\boldsymbol{\sigma}\cdot\mathbf{r}}=\frac{-\int\beta\boldsymbol{\alpha}}{\int\boldsymbol{\sigma}\times\mathbf{r}}, \tag{30}$$

then from the results of this section one has for these first forbidden matrix elements, $l=l'\mp1$,

$$\Lambda=\frac{2\rho}{\alpha ZM}\frac{\mathcal{G}_{0l'}{}^{\pm}}{\mathfrak{F}_1}. \tag{31}$$

In this model one can envisage three possible contributions to $\mathfrak{D}_0$. The first is the Coulomb interaction between odd nucleon and core which contributes zero to V_i and $\alpha Z/r$ to V_f. The second is the spin-orbit coupling which we write as $V_0\boldsymbol{\sigma}\cdot\mathbf{r}\times\mathbf{p}$. The third, which is rather unlikely and will be ignored, is the possible appearance of the radial component of the momentum operator in V. From the first, one obtains a contribution $-\alpha Z\mathfrak{F}_0$ to $\mathfrak{D}_0$ and from the second $-(\kappa-\kappa')\Delta E/(2l+1)$, where $\Delta E=E_{j=l-\frac{1}{2}}-E_{j=l+\frac{1}{2}}$ is the spin-orbit splitting. Consequently, from (29) and (31) one finds in all cases:

$$\Lambda=2\left\{\rho\frac{\mathfrak{F}_0}{\mathfrak{F}_1}+\frac{\rho}{\alpha Z}\left[\frac{\kappa-\kappa'}{2l+1}\Delta E+W_i-W_f-\Delta\right]\right\}. \tag{32}$$

Roughly $\rho\mathfrak{F}_0/\mathfrak{F}_1\approx1$ and $\Delta E\approx2$ Mev (i.e., ≈4 in our units); the second and third terms are negligible and $\Lambda\sim1$ to 3. Since $\mathfrak{F}_1$ is the average value of r (with $\mathfrak{R}\mathfrak{R}'$ as a weight function), one would expect $\rho\mathfrak{F}_0/\mathfrak{F}_1>1$, if anything ($\rho\mathfrak{F}_0/\mathfrak{F}_1=4/3$ for $\mathfrak{R}\mathfrak{R}'=$const), and the upper value appears somewhat more plausible. This slightly weakens the argument against rejection of the V-T mixture in the β interaction.[14]

A similar situation holds for second forbidden transitions so far as the relative order of magnitude of the ee, eo, and oo contributions is concerned. Thus, for S-T interference, which is the interesting case, we can obtain the ratio of squares of o-matrix elements to e-matrix elements. These are matrix elements of $T_{21}(\mathbf{r},\boldsymbol{\sigma}\times\mathbf{p})$ and $\mathcal{Y}_2$, respectively. For simplicity, we consider the two cases $l-\frac{1}{2}=j=l'+\frac{3}{2}=j'+2$ and $l-\frac{1}{2}=j=l'-5/2=j'-2$, in which cases only one term of (29) enters. The terms in $\mathfrak{F}_{L-1}$ in (29) also vanish. Then, for both

case of a more general coefficient of the form given by the right-hand side of Eq. (24) but with fewer relations between the arguments. For instance, in the coupling of particles with intrinsic spin other than $\frac{1}{2}$ (arbitrary channel spin, say) a more general coefficient would appear. Note that the application of the results shown in Table III would provide a relation between M coefficients and the X coefficient.

[14] H. M. Mahmoud and E. J. Konopinski, Phys. Rev. 88, 1266 (1952).

[15] D. L. Pursey, Phil. Mag. 42, 1193 (1951); T. Ahrens and E. Feenberg, Phys. Rev. 86, 64 (1952).

cases, we obtain

$$\sum_{ij}|A_{ij}{}^{\beta}|^2/\sum_{ij}|R_{ij}|^2 \cong \left(\frac{\alpha Z}{\rho}\right)^2\left(\frac{\rho\mathfrak{F}_1}{\mathfrak{F}_2}\right)^2, \tag{33}$$

where only the dominant Coulomb contribution has been considered in $\mathfrak{D}_1$, and the relatively small contribution of $\mathfrak{F}_2$ has also been neglected. The ratio $\rho\mathfrak{F}_1/\mathfrak{F}_2$ should be about unity or somewhat larger. The correction factors contain $\alpha Z/\rho$ in such a way as to cancel the dominance of the *o-o* matrix element over the *e-e* matrix element. Similar results apply to the cross term and to the comparison of other pairs of second-rank tensors. For example, for the cases cited above,[16]

$$k^2 \equiv \sum_{ij}|A_{ij}{}^{\beta}|^2/\sum_{ij}|T_{ij}|^2 = \left(\frac{\alpha Z}{2\rho}\right)^2\left(\frac{\rho\mathfrak{F}_1}{\mathfrak{F}_2}\right)^2. \tag{33a}$$

If one considers different orbitals, the first term of (29) need not vanish and an extra term $\mathfrak{F}_{L-1}/M\mathfrak{F}_{L+1}$ is added to the Coulomb contribution. This added term is of order $1/M\rho^2$ (i.e., essentially the kinetic energy of a nucleon at the nuclear surface), which is $\sim 40/A^{\frac{2}{3}}$, and this is almost always small compared to $\alpha Z/\rho \sim 2Z/A^{\frac{1}{3}}$ for all but the very light nuclei.

For the ratio of two *e*-matrix elements one notes that the radial integrals cancel out for the following cases: (a) the ratio of $(f\|\mathbf{r}\|i)$ to $(f\|\boldsymbol{\sigma}\times\mathbf{r}\|i)$ in first forbidden, and (b) the ratio of $(f\|\mathcal{Y}_2\|i)$ to $(f\|T_{22}(\mathbf{r},\boldsymbol{\sigma})\|i)$ in second forbidden. The ratio (*a*) is obtained from Eq. (12) for $\lambda=1$, and Eq. (16) for $\lambda=L=1$ [see Eq. (55 I)], and results for special cases previously given are confirmed.[15] The ratio (*b*), which is obtained from Eq. (12) with $\lambda=2$ and Eq. (16) with $\lambda=L=2$, gives $\sum_{ij}R_{ij}T^*_{ij}/\sum_{ij}|T_{ij}|^2$ or $\sum_{ij}|R_{ij}|^2/\sum_{ij}|T_{ij}|^2$, which is involved in *S* and/or *V* interactions mixed with *T* and/or *A*.

A second question is the matter of fluctuation of *ft* values for a given order of forbiddenness. One may ask whether this fluctuation is or is not almost entirely due to the radial integrals. To take a simple case, it is clear that for unique spectra (G-T transitions with $|\Delta j| = n+1$), one obtains a measure of the radial integrals by comparison of the foregoing results with measured values of $\log_{10}ft$ or, somewhat better, $\log_{10}[(W_0{}^2-1)ft]$. Thus, if the transition probability is represented in terms of

$$\frac{2j+1}{2j'+1}(jl\|T_{\lambda L}(\mathbf{r},\boldsymbol{\sigma})\|j'l')^2 = A(jl,j'l')\mathfrak{F}_L{}^2,$$

one could compute the values of $\log_{10}[(W_0{}^2-1)ftA]$. As is to be expected, the scatter is not much improved as compared to the values with $A=1$.[17] For example, with $\lambda=2$, $L=1$, $A=15/22\pi^2$ for a $g_{7/2}\to h_{11/2}$ transition, and $A=9/10\pi^2$ for a $p_{\frac{1}{2}}\to d_{5/2}$ transition. Thus, on the basis of the *j-j* coupling model, and for the cases wherein a one- or two-particle configuration (see Sec. III) is applicable, one can obtain, from a comparison of $\log[(W_0{}^2-1)ft]$ values, an indication of differences in nuclear structure as reflected in the radial integrals.

III. MATRIX ELEMENTS FOR TWO-NUCLEON CONFIGURATIONS

We employ the isotopic spin notation and designate the charge wave function for a single nucleon by $\varphi_i{}^{\nu}$ with $i=\frac{1}{2}$. The values $\nu=\frac{1}{2}$, $-\frac{1}{2}$ correspond to neutron, proton, respectively. For the two-nucleon configuration the isotopic spin wave function is $\Phi_I{}^N$, with $I=0$ or 1 and $N=\frac{1}{2}(N_n-N_p)$, where N_n and N_p are, respectively, the number of neutrons and protons in the configuration. As before, primes are used for the initial state. Then, for the final state,

$$\Phi_I{}^N = \sum_{\nu} C(\tfrac{1}{2}\tfrac{1}{2}I;\nu N-\nu)\varphi_i{}^{\nu}(1)\varphi_i{}^{N-\nu}(2). \tag{34}$$

On interchange of the arguments 1 and 2, which designate the two particles, $\Phi_I{}^N$ goes to $(-)^{I+1}\Phi_I{}^N$ as expected.

The total wave function for the final state with total angular momentum *J* and projection quantum number *M* is now

$$\Psi_f = \Phi_I{}^N\Psi_J{}^M = \frac{1}{\sqrt{2}}\Phi_I{}^N\sum_m C(j_1j_2J;mM-m) \times[\psi_{\kappa_1}{}^m(1)\psi_{\kappa_2}{}^{M-m}(2)+(-)^I\psi_{\kappa_1}{}^{M-m}(2)\psi_{\kappa_2}{}^m(1)], \tag{35}$$

which is antisymmetric with respect to interchange of all coordinates. In Eq. (35) κ_i stands for j_i, l_i. The wave function Ψ_f is normalized to unity if $\kappa_1\neq\kappa_2$ (i.e., $j_1\neq j_2$ and/or $l_1\neq l_2$). If $\kappa_1=\kappa_2$, one has $(\Psi_f,\Psi_f)=2$ and the normalization factor is to be changed, see Eq. (36) below.

If, in the initial state, $\kappa_1'\neq\kappa_2'$, the wave function Ψ_i is obtained from (35) by priming all the quantum numbers. On the other hand, let us assume that $\kappa_1'=\kappa_2'$; that is, the two nucleons in the initial state are in the same orbital. Then,

$$\Psi_i = \Phi_{I'}{}^{N'}\Psi_{J'}{}^{M'} = \Phi_{I'}{}^{N'}\sum_{m'} C(j'j'J';m'M'-m') \times\psi_{\kappa'}{}^{m'}(1)\psi_{\kappa'}{}^{M'-m'}(2), \tag{36}$$

and over all antisymmetry is assured if

$$(-)^{J'+I'} = -1.$$

We consider three cases: (a) In the initial state the two orbitals are identical and in the final state they are different. (b) The orbitals are the same for the final state and different in the initial state. (c) Both initial and final states are characterized by different orbitals but at least one orbital in initial and final states must

[16] While we recognize that the single-particle model cannot be directly applied to Cl^{36} or Tc^{99}, it is interesting to note that (33a) gives $k^2\cong 30$ and $k^2\cong 90$ $(\rho\mathfrak{F}_1/\mathfrak{F}_2=1)$, respectively, while the empirical values are $k^2=18$ and 45, respectively; see L. Feldman and C. S. Wu, Phys. Rev. **87**, 1091 (1952). One does not expect the order of magnitude to change radically when more complicated configurations are considered. In fact, it is fairly easy to see that the results given above [Eq. (31)–(33a)] also hold for two-nucleon configurations; see Sec. III.

[17] See, e.g., Mayer, Moszkowski, and Nordheim, Revs. Modern Phys. **23**, 315 (1951).

be the same. Where $\kappa_1=\kappa_2$ and $\kappa_1'=\kappa_2'$, the matrix elements are easily obtained as special cases of the result of (a) or (b).

Considering any operator of the five types listed in A and designating the operator by $\Omega_\lambda^{-M''}$, one has

$$(f|\sum_k \Omega_\lambda^{-M''}(k)Q_k|i) \\ =C(J'\lambda J; M', -M'')\delta_{M'-M'',M}(J\|\Omega_\lambda\|J'), \quad (37)$$

where the sum over k is over particles 1 and 2. For the transition probability, one needs

$$\langle|(f|\sum\Omega_\lambda^{-M''}(k)Q_k|i)|^2\rangle=\frac{2J+1}{2J'+1}|(J\|\Omega_\lambda\|J')|^2. \quad (38)$$

This is what one designates by $|\int\Omega_\lambda|^2$ in the usual notation. For interference terms between operators Ω and Ω' the absolute square of the reduced matrix element is replaced by $(J\|\Omega_\lambda\|J')(J\|\Omega_\lambda'\|J')^*$.

Considering case (a) first, the wave functions (35) and (36) are substituted in (37) and the result is reduced to the form of the right-hand side, whereby identification of the reduced matrix element is made. Using

$$Q_k\varphi_{\frac{1}{2}}^{\nu}(k)=(\nu+\tfrac{1}{2})\varphi_{\frac{1}{2}}^{-\nu}(k), \quad (39)$$

and with the notation

$$\Theta=\frac{1}{\sqrt{2}}[(\Phi_I^N, Q_1\Phi_{I'}^{N'})+(-)^{I+I'}(\Phi_I^N, Q_2\Phi_{I'}^{N'})],$$

one finds that $\Theta=1$ for the transitions $I'=1$, $N'=1\rightarrow I=1$, $N=0$; $(I'N')=(10)\rightarrow(IN)=(1-1)$; $(I'N')=(00)\rightarrow(IN)=(1-1)$. For the only other transition possible, $(I'N')=(11)\rightarrow(IN)=(00)$, one has $\Theta=-1$. It can be seen that Θ occurs as a factor of the reduced matrix element in every case, and we can take $\Theta=1$ since a phase which is the same for all interfering matrix elements is irrelevant.

Using Eq. (5), together with the orthonormality of the single nucleon ψ_κ^m, a simple Racah recoupling gives

$$\begin{aligned}(f\|\Omega_\lambda\|i)=(-)^{\lambda-J}(2J'+1)^{\frac{1}{2}}\{&(2j_1+1)^{\frac{1}{2}}\delta_{\kappa_2\kappa'}\\ &\times W(j'j_1J'J;\lambda j')(\kappa_1\|\Omega_\lambda\|\kappa')\\ &+(2j_2+1)\delta_{\kappa_1\kappa'}(-)^{j_2+j'+J+I}\\ &\times W(j'j_2J'J;\lambda j')(\kappa_2\|\Omega_\lambda\|\kappa')\}. \quad (40)\end{aligned}$$

In (40) the single-particle reduced matrix elements are $(\kappa_1\|\Omega_\lambda\|\kappa')\equiv(j_1l_1\|\Omega_\lambda\|j'l')$ and similarly for $(\kappa_2\|\Omega_\lambda\|\kappa')$. These are given in Sec. II. Under the assumption that $\kappa_1\neq\kappa_2$, only one term of (40) will be nonvanishing in any particular case. If, however, $\kappa_1=\kappa_2=\kappa$ (i.e., $j_1=j_2=j$, $l_1=l_2=l$), one obtains

$$(f\|\Omega_\lambda\|i)=(-)^{\lambda-J}[(2J'+1)(2j+1)]^{\frac{1}{2}} \\ \times W(j'jJ'J;\lambda j')(\kappa\|\Omega_\lambda\|\kappa'). \quad (40a)$$

For tensors of rank $\lambda\leqslant 2$ the Racah coefficient can be obtained from BBR, Tables II and IV. For higher-rank tensors the numerical tables may be used.[6] If one or more of j_1, j_2, $j'=\frac{3}{2}$, Table III of BBR may be used.

For case (b), where the orbitals are identical in the final state, one finds

$$\begin{aligned}(f\|\Omega_\lambda\|i)=(-)^\lambda[(2J'+1)(2j+1)]^{\frac{1}{2}}\\ \times\{(-)^{j-j_1'-J}\delta_{\kappa\kappa_2'}W(j_1'jJ'J;\lambda j)(\kappa\|\Omega_\lambda\|\kappa_1')\\ +(-)^{J'+I'+I}\delta_{\kappa\kappa_1'}W(j_2'jJ'J;\lambda j)(\kappa\|\Omega_\lambda\|\kappa_2')\}. \quad (41)\end{aligned}$$

Here $\kappa_i'=j_i'$, l_i' which designate the orbital of particle i in the initial state. Again, with $\kappa_1'\neq\kappa_2'$ only one term enters and the Racah coefficients can be obtained as described above. For $\kappa_1'=\kappa_2'$, the result (41) reduces to (40a) when proper account is taken of the necessary renormalization of Ψ_i.

Finally, the case (c) we use a wave function like (35) for both initial and final states. The reduced matrix element is

$$\begin{aligned}(f\|\Omega_\lambda\|i)=(-)^{\lambda+1}(2J'+1)^{\frac{1}{2}}\{&(-)^{J+j_2'+j_1'}(2j_1+1)^{\frac{1}{2}}\\ &\times\delta_{\kappa_2\kappa_2'}W(j_1j_1'JJ';\lambda j_2)(\kappa_1\|\Omega_\lambda\|\kappa_1')\\ &+(-)^{I+I'+j_1+j_2+J'}(2j_2+1)^{\frac{1}{2}}\delta_{\kappa_1\kappa_1'}\\ &\times W(j_2j_2'JJ';\lambda j_1)(\kappa_2\|\Omega_\lambda\|\kappa_2')\\ &+(-)^{I'+J+J'}(2j_1+1)^{\frac{1}{2}}\delta_{\kappa_2\kappa_1'}\\ &\times W(j_1j_2'JJ';\lambda j_2)(\kappa_1\|\Omega_\lambda\|\kappa_2')\\ &+(-)^{I-j_2+j_1'}(2j_2+1)^{\frac{1}{2}}\delta_{\kappa_1\kappa_2'}\\ &\times W(j_2j_1'JJ';\lambda j_1)(\kappa_2\|\Omega_\lambda\|\kappa_1')\}. \quad (42)\end{aligned}$$

The reduced matrix element (42) vanishes if all four κ's are different, as is obvious from the fact that the beta interaction is a sum of single-particle operators. A nonvanishing result with $\kappa_1\neq\kappa_2$ and $\kappa_1'\neq\kappa_2'$ is obtained if: $\kappa_1=\kappa_1'$ and $\kappa_2=\kappa_2'$ in which the first two terms coexist, $\kappa_1'=\kappa_2$, $\kappa_2'=\kappa_1$, for which case the last two terms contribute; or only one pair of κ's can be equal, in which case any one of the four terms may contribute. Thus, at most, two terms contribute.

For configurations of more than two nucleons the procedure is somewhat more complicated but the reduced matrix elements can still be expressed as linear combinations of single-nucleon reduced matrix elements. For this purpose one may use the coefficients of fractional parentage[18] as explained by Talmi.[3] Further applications to experimental results must await a continuation of the calculations along these lines.

APPENDIX A. HERMITIAN CONJUGATION PROPERTY OF THE TENSOR OPERATORS

As an illustration of the derivation of the results given in Table III, we consider the case of the type V operator: $T_{\lambda L}^m(\mathbf{r}, \boldsymbol{\sigma}\times\mathbf{p})$. From Eq. (2),

$$T_{\lambda L}^{m\dagger}(\mathbf{r}, \boldsymbol{\sigma}\times\mathbf{p})=(-)^m\sum_{m'} C(1L\lambda; -m', m+m') \\ \times\mathcal{Y}_1^{m'}(\boldsymbol{\sigma}\times\mathbf{p})\mathcal{Y}_L^{-m'-m}(\mathbf{r}). \quad (A.1)$$

[18] See, for example, A. R. Edmonds and B. H. Flowers, Proc. Roy. Soc. (London) 214, 515 (1952).

From Eq. (I 55), we obtain

$$\mathcal{Y}_1^{m'}(\boldsymbol{\sigma}\times\mathbf{p})=(3/4\pi)^{\frac{1}{2}}(\boldsymbol{\sigma}\times\mathbf{p})_{m'}$$
$$=\sqrt{2}i(4\pi/3)^{\frac{1}{2}}T_{11}^{m'}(\boldsymbol{\sigma},\mathbf{p}),\quad (A.2)$$

and if Eq. (2) is used to express $T_{11}(\boldsymbol{\sigma},\mathbf{p})$ in terms of $\mathcal{Y}_1(\boldsymbol{\sigma})$, $\mathcal{Y}_1(\mathbf{p})$, we find

$$T_{\lambda L}^{m\dagger}(\mathbf{r},\boldsymbol{\sigma}\times\mathbf{p})=\sqrt{2}i(4\pi/3)^{\frac{1}{2}}(-)^m$$
$$\times\sum_{m'M}C(1L\lambda,-m',m+m')C(111;-m,m'+M)$$
$$\times\mathcal{Y}_1^{m'+M}(\boldsymbol{\sigma})\mathcal{Y}_1^{-M}(\mathbf{p})\mathcal{Y}_L^{-m'-m}(\mathbf{r}).\quad (A.3)$$

We note that $\mathcal{Y}_1(\boldsymbol{\sigma})$ commutes with the other operators in (A.3) and from Eq. (I 47) we have

$$(\mathcal{Y}_1^{\mu}(\mathbf{p}),\mathcal{Y}_L^{\mu'}(\mathbf{r}))=i\left(\frac{3}{4\pi}\right)^{\frac{1}{2}}(2L+1)\left(\frac{L}{2L-1}\right)^{\frac{1}{2}}$$
$$\times C(1LL-1;\mu\mu')\mathcal{Y}_{L-1}^{\mu+\mu'}(\mathbf{r}),\quad (A.4)$$

where (A,B) is the commutator of A and B. Using (A.4) in (A.3) and the symmetry relations of the C coefficients,[6] one finds that

$$T_{\lambda L}^{m\dagger}(\mathbf{r},\boldsymbol{\sigma}\times\mathbf{p})=(-)^{m+L+\lambda+1}T_{\lambda L}^{-m}(\mathbf{r},\boldsymbol{\sigma}\times\mathbf{p})$$
$$+(-)^m\Gamma\Omega'^{-m},\quad (A.5)$$

where, with $M+m'=M'$,

$$\Gamma\Omega'^{-m}=-\left(\frac{2L}{2L-1}\right)^{\frac{1}{2}}(2L+1)$$
$$\times\sum_{m'M'}C(1L\lambda;-m',m+m')C(111;-M'+m',M')$$
$$\times C(1LL-1;-M'+m',-m'-m)$$
$$\times\mathcal{Y}_{L-1}^{-m-M'}(\mathbf{r})\mathcal{Y}_1^{M'}(\boldsymbol{\sigma}).\quad (A.6)$$

We now use Eq. (I 44) applied to the second and third C coefficients so as to produce one C coefficient free of m'. The sum over m' is then easily carried out by the orthogonality property of the unitary C coefficients, and the sum over M' is used with Eq. (2) to introduce the tensor operator $\Omega'=T_{\lambda L-1}(\mathbf{r},\boldsymbol{\sigma})$. One then finds

$$\Gamma=(6L)^{\frac{1}{2}}(2L+1)W(11L-11;1\lambda).\quad (A.7)$$

Thus, $\Gamma\neq0$ only if $\lambda=L$ or $L-1$, $(L>0)$. For type V operators the practical case is $\lambda=L+1$.

APPENDIX B. THE MATRIX ELEMENT OF THE TYPE II OPERATOR

Employing Eqs. (2) and (8), one may immediately write

$$(jl\mu|T_{\lambda L}^{-m}(\mathbf{r},\boldsymbol{\sigma})|j'l'\mu')$$
$$=\sum_{\tau\tau'm'}C(l\tfrac{1}{2}j;\mu-\tau,\tau)C(l'\tfrac{1}{2}j';\mu'-\tau',\tau')$$
$$\times C(1L\lambda;-m',m'-m)(\chi_{\frac{1}{2}}^{\tau},\mathcal{Y}_1^{-m'}(\boldsymbol{\sigma})\chi_{\frac{1}{2}}^{\tau'})$$
$$\times\mathfrak{F}_L(Y_l^{\mu-\tau},Y_L^{m'-m}Y_{l'}^{\mu'-\tau'}),\quad (B.1)$$

where the notation of Eq. (12′) has been used. The scalar product in spin-space is readily evaluated, yielding

$$(\chi_{\frac{1}{2}}^{\tau},\mathcal{Y}_1^{-m'}(\boldsymbol{\sigma})\chi_{\frac{1}{2}}^{\tau'})$$
$$=\frac{3}{(4\pi)^{\frac{1}{2}}}(-1)^{-m'}C(\tfrac{1}{2}1\tfrac{1}{2};\tau'-m',m')\delta_{\tau,\tau'-m'}.\quad (B.2)$$

Using this result, along with Eq. (I 43) for the coupling between a product of two orbital angular momentum wave functions in the same space, one obtains, after performing the trivial sum over τ,

$$(jl\mu|T_{\lambda L}^{-m}(\mathbf{r}\boldsymbol{\sigma})|j'l'\mu')$$
$$=\frac{3}{4\pi}(2L+1)^{\frac{1}{2}}(-)^{l+l'}C(lLl';0,0)$$
$$\times\delta_{m,\mu'-\mu}\mathfrak{F}_L\sum_{m'}(-)^{m'}C(1L\lambda;-m',m'-m)$$
$$\times\sum_{\tau'}C(l\tfrac{1}{2}j;\mu-\tau'+m',\tau'-m')$$
$$\times C(l'\tfrac{1}{2}j';\mu'-\tau',\tau')C(\tfrac{1}{2}1\tfrac{1}{2};\tau'-m',m')$$
$$\times C(Ll'l;m'-m,\mu'-\tau').\quad (B.3)$$

Employing Eq. (I 44) twice, the four Clebsch-Gordan coefficients in the τ' sum may be transformed to a new set of four C coefficients and two Racah coefficients. Only two of the new Clebsch-Gordan coefficients depend upon τ', and these yield an orthogonality relation enabling the τ' sum and one of the recoupling sums to be done immediately. The remaining two C coefficients depend upon m' and after using (I 44) once more one is able, in an entirely similar fashion, to carry out the m' sum and another recoupling sum. One then finds

$$(jl\mu|T_{\lambda L}^{-m}(\mathbf{r}\boldsymbol{\sigma})|j'l'\mu')$$
$$=(3\sqrt{2}/4\pi)[(2L+1)(2j'+1)(2\lambda+1)(2l+1)]^{\frac{1}{2}}$$
$$\times C(lLl';0,0)\delta_{m,\mu'-\mu}(-)^{\lambda+1}$$
$$\times C(j'\lambda j;\mu',-m)\mathfrak{F}_L\sum_s(2s+1)W(1jLj';s\lambda)$$
$$\times W(j1l\tfrac{1}{2};s\tfrac{1}{2})W(j'L\tfrac{1}{2}l;sl').\quad (B.4)$$

One now notes that the sum over s yields, by definition,[12]

$$(-1)^{\lambda+L+j'+j+l'+l}X(\tfrac{1}{2}1\tfrac{1}{2};j\lambda j';lLl'),$$

so that finally the reduced matrix element becomes

$$(jl\|T_{\lambda L}(\mathbf{r},\boldsymbol{\sigma})\|j'l')$$
$$=(3\sqrt{2}/4\pi)[(2\lambda+1)(2L+1)(2l+1)(2j'+1)]^{\frac{1}{2}}$$
$$\times(-)^{j'-j}C(lLl';0,0)X(\tfrac{1}{2}1\tfrac{1}{2};j\lambda j';lLl')\mathfrak{F}_L,\quad (B.5)$$

in accordance with Eq. (16).