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A Stokes-Parameter Technique for the Treatment of Polarization in Quantum Mechanics

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A technique is presented which adapts the Stokes method to quantum mechanics and serves to calculate polarization effects by means of Pauli matrices. Illustrative examples indicate that this technique may be not only convenient mathematically but also physically more transparent than previous methods.

1. INTRODUCTION

THE treatment of the polarization of electromagnetic radiation in terms of Stokes parameters[1] has been drawing increased attention, probably because it relies on operational concepts and therefore is particularly suited to quantum physics.[2] Recent successes in experimenting with the polarization of high-energy radiations[3] have stimulated theoretical studies of polarization effects and thereby increased the desirability of more powerful techniques for these studies.

No general technique for the quantum-mechanical application of the Stokes method seems to be available in the literature, even though the ideas of this method are being utilized by an increasing number of workers. Since such a technique is in fact amenable to a rather simple formulation, it may be worthwhile to present it in this paper. The method constitutes no more than a transcription of established quantum mechanical theory. A few very simple examples will illustrate the operation of the technique.

2. THE STOKES PARAMETER METHOD

(a) Stokes Parameters

The intensity and the kind and degree of polarization of a beam of light (or other electromagnetic radiation) can be represented by four parameters as follows. Take two orthogonal unit vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ perpendicular to the beam direction as a frame of reference. The parameters are:

(1) the total light intensity I_0;

(2) the difference I_1 between the light intensity transmitted by a filter (Nicol prism) which accepts the linear polarization $\mathbf{A}_1$ and the intensity transmitted by the "opposite" filter which accepts $\mathbf{A}_2$;

(3) the difference I_2 between the light intensities transmitted by a filter which accepts the linear polarization $(\mathbf{A}_1+\mathbf{A}_2)/\sqrt{2}$ and by the opposite filter;

(4) the difference I_3 between the intensities transmitted by a filter which accepts circular polarization rotating from $\mathbf{A}_1$ to $\mathbf{A}_2$ and by the opposite filter.

For convenience in the following applications we write

$$I_1=I_0P_\zeta,\quad I_2=I_0P_\xi,\quad I_3=I_0P_\eta \tag{1}$$

and regard the set of parameters as a four component vector $I_0(1, \mathbf{P})$, where $\mathbf{P}=(P_\zeta, P_\xi, P_\eta)$ is a vector of the three-dimensional Poincaré representation,[2] which describes the kind and degree of polarization ($P\leq 1$).[4]

(b) Density Matrix

Since two independent states of light polarization constitute a complete set, the quantum-mechanical treatment of polarization is mathematically equivalent (isomorphic) to the treatment of the orientation of a spin $\frac{1}{2}$ particle. Therefore the density matrix which represents the polarization state of a light beam according to quantum mechanics is a 2×2 matrix, which can be resolved into the sum of a unit matrix $\mathbb{1}$ and of Pauli matrices $(\omega_\zeta, \omega_\xi, \omega_\eta)=\boldsymbol{\omega}$. (These matrices are taken in the usual representation, with ω_ζ diagonal.) The coefficients of this sum are the Stokes parameters and we write the density matrix[5] in the form

$$\mathcal{I}=\tfrac{1}{2}I_0(\mathbb{1}+\mathbf{P}\cdot\boldsymbol{\omega}). \tag{2}$$

(c) Response of a Light Detector

A detector that serves as a polarization analyzer responds to light of different polarizations with different efficiency. Maximum and minimum efficiencies ϵ_M and ϵ_m correspond to completely polarized beams which have opposite polarizations with Poincaré vectors $\mathbf{P}$ equal, respectively, to $\mathbf{Q}$ and $-\mathbf{Q}$ ($Q=1$). Quantum-mechanically, the detector is represented by an operator with the eigenstates $\mathbf{Q}$ and $-\mathbf{Q}$ and with the eigen-

[1] G. G. Stokes, Trans. Cambridge Phil. Soc. **9**, 399 (1852).

[2] See, e.g., U. Fano, J. Opt. Soc. Am. **39**, 859 (1949).

[3] Hoover, Faust, and Dohne, Phys. Rev. **85**, 58 (1952) (double Compton scattering); E. Bleuler and H. L. Bradt, Phys. Rev. **73**, 1938 (1948); R. C. Hanna, Nature **162**, 332 (1948); C. S. Wu and I. Shaknov, Phys. Rev. **77**, 136 (1950); F. L. Hereford, Phys. Rev. **81**, 482, 627 (1951) (all on polarization of annihilation quanta); F. Metzger and M. Deutsch, Phys. Rev. **78**, 551 (1950) (polarization-direction correlation of gamma-quanta); A. P. French and J. O. Newton, Phys. Rev. **85**, 1041 (1952); S. B. Gunst and L. A. Page, Phys. Rev. **92**, 970 (1953) (polarization by transmission through magnetized iron); F. L. Hereford and J. P. Keuper, Phys. Rev. **90**, 1043 (1953) (polarization effects in photoelectric effect); K. Phillips, Phil. Mag. **44**, 169 (1953) (polarization of bremsstrahlung); D. H. Wilkinson, Phil. Mag. **43**, 659 (1952); L. W. Fagg and S. S. Hanna, Phys. Rev. **88**, 1205 (1952) (polarization analysis by deuteron disintegration).

[4] Unitary transformations of the frame of reference $(\mathbf{A}_1, \mathbf{A}_2)$ are accompanied by rotations of the Poincaré axes (ζ, ξ, η). The formalism described in this paper is independent of the choice of the frame of reference.

[5] The equivalence between the Stokes parameters and the density matrix has been pointed out by D. L. Falkoff and J. E. Macdonald, J. Opt. Soc. Am. **41**, 862 (1951).

values ϵ_M and ϵ_m, i.e., by the matrix[6]

$$\mathcal{O}=\tfrac{1}{2}[(\epsilon_M+\epsilon_m)\mathbf{1}+(\epsilon_M-\epsilon_m)\mathbf{Q}\cdot\boldsymbol{\omega}]. \tag{3}$$

The probability of response of this detector to a light beam with the density matrix (2) is given by the trace of the product of the matrices (2) and (3), namely,

$$\mathrm{Tr}(\mathcal{O}\mathcal{J})=\tfrac{1}{2}I_0[(\epsilon_M+\epsilon_m)+(\epsilon_M-\epsilon_m)\mathbf{Q}\cdot\mathbf{P}]. \tag{4}$$

(d) Probability of Interactions with Matter

The calculation of photon emission, absorption, and scattering will be adapted to the Stokes formalism in close analogy to a well known technique for calculating the collisions of free Dirac electrons.[7] The quantum-mechanical expression of interaction probabilities is usually proportional to the square of a perturbation matrix element V_{fi} which relates to the transition from an initial state i to a final state f. If the initial and/or final states are not "pure states" (e.g., not completely polarized), the transition probability is proportional to a sum $\sum_f^*\sum_i^*|V_{fi}|^2$; the asterisks indicate that the sums must be carried out according to rules implied by the statement of the problem (e.g., over both photon polarizations if the polarization is irrelevant to the problem).

The perturbation matrix element V_{fi} for an interaction of photons with matter is a linear function of the polarization vector $\mathbf{A}_i$ for any incident radiation and of the polarization vector $\mathbf{B}_f^\dagger$ for any outgoing radiation (the dagger denotes Hermitian conjugation). In turn, $\mathbf{A}_i$ may be expressed in terms of unit polarization vectors $\mathbf{A}_\alpha$ ($\alpha=1, 2$) of the incident radiation and of a two-component wave function $a_{i\alpha}$,[8] $\mathbf{A}_i=\sum_\alpha a_{i\alpha}\mathbf{A}_\alpha$; similarly $\mathbf{B}_f^\dagger=\sum_\beta b_{f\beta}^\dagger\mathbf{B}_\beta$.

When the $\sum_f^*\sum_i^*|V_{fi}|^2$ is formed, the $\sum_i^*$ may be factored out as $\sum_i^* a_{i\alpha}a_{i\alpha'}^\dagger$ and constitutes in fact the density matrix (2) of the incident radiation, $\mathcal{J}_{\alpha\alpha'}=\sum_i^* a_{i\alpha}a_{i\alpha'}^\dagger$. Similarly the $\sum_f^* b_{f\beta'}b_{f\beta}^\dagger$ may be factored out and regarded as the matrix (3), $\mathcal{O}_{\beta'\beta}$, of an ideal detector which accepts outgoing radiation with the kind and degree of polarization specified in the statement of the problem.

In a scattering problem, which involves both incident and outgoing radiation, the perturbation matrix element V_{fi} is a linear function of both $\mathbf{B}_f^\dagger$ and $\mathbf{A}_i$. It can be expressed as

$$V_{fi}(\mathbf{B}_f^\dagger, \mathbf{A}_i)=\sum_\beta\sum_\alpha b_{f\beta}^\dagger V(\mathbf{B}_\beta, \mathbf{A}_\alpha)a_{i\alpha}, \tag{5}$$

where $V(\mathbf{B}_\beta, \mathbf{A}_\alpha)$ may be regarded as a matrix $V_{\beta\alpha}$ and resolved, if desired, into a sum of standard matrices:

$$\begin{aligned}V(\mathbf{B}_\beta, \mathbf{A}_\alpha)=\tfrac{1}{2}\{&[V(\mathbf{B}_1, \mathbf{A}_1)+V(\mathbf{B}_2, \mathbf{A}_2)]\,\mathbf{1}\\ &+[V(\mathbf{B}_1, \mathbf{A}_1)-V(\mathbf{B}_2, \mathbf{A}_2)]\omega_\zeta\\ &+[V(\mathbf{B}_1, \mathbf{A}_2)+V(\mathbf{B}_2, \mathbf{A}_1)]\omega_\xi\\ &+i[V(\mathbf{B}_1, \mathbf{A}_2)-V(\mathbf{B}_2, \mathbf{A}_1)]\omega_\eta\}. \end{aligned}\tag{5'}$$

Accordingly, the $\sum_f^*\sum_i^*|V_{fi}|^2$ takes the form of the trace of a product of matrices, quite analogous to (27) of reference 7:

$$\begin{aligned}\textstyle\sum_f^*\sum_i^*|V_{fi}|^2&=\textstyle\sum_{\alpha\alpha'\beta\beta'}\mathcal{O}_{\beta'\beta}V_{\beta\alpha}\mathcal{J}_{\alpha\alpha'}V_{\alpha'\beta'}^\dagger\\ &=\mathrm{Tr}(\mathcal{O}V\mathcal{J}V^\dagger)=\mathrm{Tr}(V^\dagger\mathcal{O}V\mathcal{J}).\end{aligned}\tag{6}$$

The core of the procedure suggested here is, then, to represent the perturbation matrix element as a polarization operator, according to (5) and (5′). In emission or absorption processes the procedure seems to fail because V_{fi} contains only one polarization vector, $B_f^\dagger$ or A_i, respectively, and cannot be reduced directly to a matrix $V_{\beta\alpha}$. However, in the event of emission, there is no density matrix $\mathcal{J}$ which depends on the polarization coordinates α, α'; the factors V and $V^\dagger$ in (6) constitute then a single polarization operator $V(\mathbf{B}_\beta)V^\dagger(\mathbf{B}_{\beta'})=(VV^\dagger)_{\beta\beta'}$ which can be resolved in the manner of (5′). In the event of absorption, the operator $\mathcal{O}$ disappears from (6) and one can construct the operator $(V^\dagger V)_{\alpha'\alpha}$.

A basic set of conventions for the treatment of polarization effects in terms of Stokes parameters and of matrices $\boldsymbol{\omega}$, according to (6), (2), (3), and (5′), has thus been completed. Products and traces of the matrices $\boldsymbol{\omega}$ are carried out according to the standard rules for Pauli matrices.

(e) Stokes Parameters of Emitted or Scattered Radiation

To display the intensity and the polarization of emitted or scattered radiation, it is not necessary to inquire about the probability of a specific event, such as the response of the detector represented by the matrix $\mathcal{O}$ in (6). Instead of constructing the full product $\mathcal{O}V\mathcal{J}V^\dagger$ in (6) and taking its trace, one may simply resolve the product $V\mathcal{J}V^\dagger$ into the sum of a unit matrix and of the polarization matrices $\boldsymbol{\omega}$. The coefficients of the matrices are proportional to the Stokes parameters of the emitted or scattered radiation. Indeed $V\mathcal{J}V^\dagger$ represents (here as well as in reference 7) the perturbed density matrix. Selection rules on the types of polarization resulting from particular processes become apparent upon inspection of the polarization operators which are contained in V and $V^\dagger$, as shown in the following examples.

3. EXAMPLES

(a) Dipole Emission of Light

In the theory of dipole emission, V is proportional to the displacement of electric charge, $\mathbf{r}$. In a transition

[6] If Q points along ζ, the matrix (3) is clearly diagonal with the stated eigenvalues; otherwise the diagonal form is achieved by a unitary transformation of ($\mathbf{A}_1, \mathbf{A}_2$), i.e., by a rotation of $\boldsymbol{\omega}$.

[7] See, e.g., W. Heitler, *Quantum Theory of Radiation* (Oxford University Press, London, 1944), second edition, p. 151 ff. The operator $(H_0+E_0)/4E_0$ of Eq. (27) of this reference represents, for unpolarized electrons of positive energy, the density matrix $\mathcal{J}$ which appears in (6); the operator $(H+E)/2E$ represents a detector $\mathcal{O}$ which accepts only positive energy electrons with any spin orientation.

[8] In the second quantization formalism, $a_{i\alpha}$ represents a destruction operator.

between fully specified states, $\mathbf{r}$ represents a specific matrix element; in general one may treat $\mathbf{r}\mathbf{r}^\dagger$ as an operator whose expectation value can be determined later by a procedure such as (6). Proceeding as suggested in 2e, we resolve $VV^\dagger$, or $\mathbf{r}\cdot\mathbf{B}\,\mathbf{r}^\dagger\cdot\mathbf{B}^\dagger$, which amounts to the same, into a sum of Pauli matrices. Equation (5′) shows that the Stokes parameters of the emitted light are proportional, respectively, to

$$\mathbf{r}\cdot\mathbf{B}_1\,\mathbf{r}^\dagger\cdot\mathbf{B}_1+\mathbf{r}\cdot\mathbf{B}_2\,\mathbf{r}^\dagger\cdot\mathbf{B}_2=\mathbf{r}\cdot\mathbf{r}^\dagger-\mathbf{r}\cdot\mathbf{n}\,\mathbf{r}^\dagger\cdot\mathbf{n}, \qquad (7)$$

$$\mathbf{r}\cdot\mathbf{B}_1\,\mathbf{r}^\dagger\cdot\mathbf{B}_1-\mathbf{r}\cdot\mathbf{B}_2\,\mathbf{r}^\dagger\cdot\mathbf{B}_2, \qquad (7')$$

$$\mathbf{r}\cdot\mathbf{B}_1\,\mathbf{r}^\dagger\cdot\mathbf{B}_2+\mathbf{r}\cdot\mathbf{B}_2\,\mathbf{r}^\dagger\cdot\mathbf{B}_1, \qquad (7'')$$

$$i(\mathbf{r}\cdot\mathbf{B}_1\,\mathbf{r}^\dagger\cdot\mathbf{B}_2-\mathbf{r}\cdot\mathbf{B}_2\,\mathbf{r}^\dagger\cdot\mathbf{B}_1)=i\mathbf{r}\times\mathbf{r}^\dagger\cdot\mathbf{n}, \qquad (7''')$$

where $\mathbf{n}=\mathbf{B}_1\times\mathbf{B}_2$ indicates the direction of emission. The expression (7) for the total emission along $\mathbf{n}$ is trivial and familiar; the expression (7‴) for the degree of circular polarization may be somewhat more transparent than usual.

(b) Rayleigh Scattering

In the nonrelativistic approximation, the essential factor of V is the scalar product of the polarization vectors of the incident and scattered waves. If the unit vectors $\mathbf{A}_1$, $\mathbf{B}_1$ are laid in the plane of scattering, Eq. (5′) yields that V is proportional to

$$\tfrac{1}{2}[(1+\cos\vartheta)\mathcal{I}-(1-\cos\vartheta)\omega_\zeta], \qquad (8)$$

where ϑ is the angle of scattering. The product $V\mathcal{J}V^\dagger$ is then proportional to

$$\begin{aligned}&\tfrac{1}{8}[(1+\cos\vartheta)\mathcal{I}-(1-\cos\vartheta)\omega_\zeta]I_0(\mathcal{I}+\mathbf{P}\cdot\boldsymbol{\omega})\\&\quad\times[(1+\cos\vartheta)\mathcal{I}-(1-\cos\vartheta)\omega_\zeta]\\&=\tfrac{1}{4}[1+\cos^2\vartheta-\sin^2\vartheta P_\zeta]\mathcal{I}+\tfrac{1}{4}[(1+\cos^2\vartheta)P_\zeta-\sin^2\vartheta]\omega_\zeta\\&\qquad+\tfrac{1}{2}\cos\vartheta P_\xi\omega_\xi+\tfrac{1}{2}\cos\vartheta P_\eta\omega_\eta. \qquad (9)\end{aligned}$$

The results contained in this formula are rather well known. Notice how the anticommutation property of the ω's operates as a selection rule: the coefficient of each ω vanishes for the scattered wave whenever it vanishes for both the incident wave and the operator V; one might say loosely that the output contains any one kind of polarization only if that kind is present in the input or generated by the interaction.

(c) Bremsstrahlung

The bremsstrahlung cross section has been calculated relativistically in the Born approximation, taking into account polarization effects, by May[9] and by Gluckstern, Hull, and Breit.[10,11] Equation (15) of reference 11 shows this cross section as consisting of four terms. The last term, independent of polarization, arises from spin effects (these effects should also yield a polarization if the spin orientations were not averaged out in both the initial and final electron states). The first three terms depend on the components, in the direction of the polarization $\mathbf{B}$, of the electron momentum before and after the collision, p_{0l} and p_l; they are proportional, respectively, to p_{0l}^2, $2p_{0l}p_l$, and p_l^2, which are written in our notations as $\mathbf{p}_0\cdot\mathbf{B}\,\mathbf{p}_0\cdot\mathbf{B}^\dagger$, $\mathbf{p}_0\cdot\mathbf{B}\,\mathbf{p}\cdot\mathbf{B}^\dagger+\mathbf{p}\cdot\mathbf{B}\,\mathbf{p}_0\cdot\mathbf{B}^\dagger$, and $\mathbf{p}\cdot\mathbf{B}\,\mathbf{p}\cdot\mathbf{B}^\dagger$. These quantities can be expressed as a sum of Pauli matrices by means of (5′). Calling $\mathbf{n}$ the direction of the bremsstrahlung, ϑ_0 and ϑ the angles between $\mathbf{p}_0$ and $\mathbf{n}$ and between $\mathbf{p}$ and $\mathbf{n}$, and φ the angle between the planes $(\mathbf{p}_0\mathbf{n})$ and $(\mathbf{p}\mathbf{n})$ and laying $\mathbf{B}_1$ in the plane $(\mathbf{p}_0\mathbf{n})$, with $\mathbf{p}_0\cdot\mathbf{B}_1>0$, we find

$$\begin{aligned}&\mathbf{p}_0\cdot\mathbf{B}\,\mathbf{p}_0\cdot\mathbf{B}^\dagger=\tfrac{1}{2}[\mathcal{I}+\omega_\zeta]\sin^2\vartheta_0,\\&\mathbf{p}_0\cdot\mathbf{B}\,\mathbf{p}\cdot\mathbf{B}^\dagger+\mathbf{p}\cdot\mathbf{B}\,\mathbf{p}_0\cdot\mathbf{B}^\dagger\\&\qquad=\tfrac{1}{2}\sin\vartheta_0\sin\vartheta[(\mathcal{I}+\omega_\zeta)\cos2\varphi+\omega_\xi\sin2\varphi], \qquad (10)\\&\mathbf{p}\cdot\mathbf{B}\,\mathbf{p}\cdot\mathbf{B}^\dagger=\tfrac{1}{2}\sin^2\vartheta[\mathcal{I}+\omega_\zeta\cos2\varphi+\omega_\xi\sin2\varphi].\end{aligned}$$

The Stokes parameters of the bremsstrahlung are obtained by replacing p_{0l}^2, $2p_{0l}p_l$, and p_l^2 in (15) of reference 11 with the expressions (10) and separating out the coefficients of the unit matrix and of the various Pauli matrices. The coefficient of the unit matrix is the Bethe-Heitler formula for the total intensity. The coefficients of ω_ζ and ω_ξ characterize the partial linear polarization (the coefficient of ω_ξ is bound to vanish when the direction of $\mathbf{p}$ is averaged out). The absence of ω_η in (10) indicates the total lack of circular polarization which is stressed in reference 11.

(d) Compton Effect

The relativistic treatment of Compton scattering yields a polarization which differs from that of Rayleigh scattering owing to the effect of electron spin. If the spin orientation before and after scattering is averaged out, the spin effects modify (9) only by the insertion of a new term in the coefficient of the unit matrix and in the coefficient of $P_\eta\omega_\eta$. The results for a nonrandom initial spin orientation are more complicated and are given in reference 2. The calculation for a nonrandom initial orientation combined with an analysis of the final orientation is still more complicated; it is being completed now by Tolhoek and Lipps[12] by means of a technique of the type presented in this paper.

[9] M. May, Phys. Rev. **84**, 265 (1951).

[10] Gluckstern, Hull, and Breit, Science **114**, 480 (1951).

[11] Gluckstern, Hull, and Breit, Phys. Rev. **90**, 1026 (1953).

[12] I wish to thank Dr. Tolhoek and Mr. Lipps for information on their work and for participating, together with Dr. S. Berko and Dr. F. L. Hereford, in stimulating discussions.