Energy Spectrum Resulting from Electron Slowing Down*

L. V. SPENCER AND U. FANO
National Bureau of Standards, Washington, D. C.
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The equation for the slowing down of electrons is cast in a form suitable for numerical calculations. Pilot calculations have been performed for source energies of $4mc^2$ and $80mc^2$ in Al and Pb. The spectrum of slowed-down electrons, which would be the reciprocal of the stopping power under the assumption of continuous slowing down, departs substantially from this elementary solution at the upper and lower energy ends of the spectrum. The departures are large throughout when bremsstrahlung is important. The accumulation of secondary knock-on electrons is also included in the calculations.

1. INTRODUCTION

HIGH-ENERGY particles (or photons) traversing a material generally dissipate their energy through a succession of collisions. Therefore, even though the radiation source be monochromatic with energy E_0, particles of all energies $E \leqslant E_0$ will be found in the material. The determination of the energy spectrum of all particles in the material, irrespective of their position and direction, constitutes a first task in a systematic study of the degradation, penetration, and diffusion of high-energy radiations.[1,2]

The spectrum resulting from the slowing down of neutrons has been studied by Placzek, in part by analysis and in part by numerical procedures.[3] The spectrum resulting from repeated Compton scattering of x-ray photons is derived by straightforward numerical procedures.[4] The problem of the slowing down of charged particles appears trivial only under the assumption of continuous slowing down; it has been discussed extensively in F (Secs. 1, 2, and 3), but accurate results were obtained in that paper only for heavy charged particles. The slowing down of electrons will be treated in the present paper. The solution of this problem may find direct application, for example, to the study of the total x-ray emission of electrons and to the analysis of the chemical and biological actions of x, γ, and β rays which depend on the spectrum of electrons traversing a material. However, the present work was undertaken primarily as a stepping stone to the theory of penetration and diffusion.

In the overwhelming majority of collisions a fast electron loses a minimal fraction of its energy. If larger energy losses were disregarded, the slowing down of electrons could be treated as a continuous process and the resulting spectrum would be simply the reciprocal of the stopping power (see, e.g., F, Sec. 3a). This solution of the problem is often accepted as self-evident. However, occasional large energy losses due to knock-on collisions against atomic electrons produce an error which has been discussed, but not calculated adequately, in F. This error is corrected in the present paper. It amounts to a few percent over most of the spectrum but becomes much larger in the portions of the spectrum near the source energy and at low energies. At high energies, bremsstrahlung emission makes the continuous slowing-down model unrealistic.

The probability of large energy losses by knock-on or by radiative collisions is given by rather complicated analytical formulas. Hence, the effects of these collisions should be treated, presumably, by numerical methods of direct integration similar to those employed in the x-ray problem.[4] On the other hand, the overwhelming number of small energy losses makes such a direct procedure unworkable (see also F, Sec. 3b). Therefore, our problem consists of developing a procedure suited to the contrasting requirements of large and of small energy losses.

This objective appears to have been achieved, and a few pilot calculations have been performed numerically. Source energies of $4mc^2$ and $80mc^2$ were considered, and aluminum and lead were chosen as representative materials. The calculations were performed by first disregarding and then taking into account the energy jumps due to bremsstrahlung. Knock-on collisions were first treated only as a source of large energy losses, but then also as additional sources of fast secondary electrons. For consistency, one should also include as sources the Compton and pair electrons generated by the bremsstrahlung and thereby solve the complete problem for a cascade shower; however, the application has not been carried that far. Some error has been

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[1] See, e.g., U. Fano, Phys. Rev. **92**, 328 (1953), which will be referred to as F.

[2] The following experimental arrangement may serve as a means of investigating the spectrum of slowed-down particles and of their secondaries, if any, which flow through a material loaded with a distributed source of particles. If a cavity is dug inside the material (the cavity being empty or filled with a gas of the same composition as the material), the flux in the cavity will be exactly the same as in the surrounding material, as shown in a general manner, by one of us [U. Fano, *Radiation Research* (to be published)]. A small aperture through the cavity walls will let through the same flux of radiation that would flow through an equal area within the material, except for edge effects which can be minimized. The emerging flux can be measured and analyzed.

[3] G. Placzek, Phys. Rev. **69**, 423 (1946); see also R. E. Marshak, Revs. Modern Phys. **19**, 185 (1947).

[4] P. R. Karr and J. E. Lamkin, Phys. Rev. **76**, 1843 (1949); the same procedure has been applied routinely in a large number of later applications.

incurred by disregarding the "Blunck-Leisegang correction."[5] This is discussed in the Appendix.

The numerical procedures were eventually so developed as to reduce the labor for each calculation to moderate proportions (see Sec. 8). Most of the remaining labor goes into the tabulation of collision cross sections; given these tabulations, some of the curves in Figs. 1–4 were calculated by one man in a day.

2. THE DEGRADATION EQUATION

Assuming that electrons are generated within a uniform infinite medium with initial energy T_0,[6] we inquire about the resulting spectrum $y(T)$ of electrons. Call $k(T,\tau)$ the probability per unit path length that an electron of energy T experiences an energy loss τ. The statistical balance of electrons of energy T is governed by the equation

$$y(T)\int_0^T k(T,\tau)d\tau=\int_0^\infty y(T+\tau)k(T+\tau,\tau)d\tau +S\delta(T_0-T), \quad (1)$$

with the boundary condition $y(\infty)=0$, where S represents the strength of the electron source and the Dirac δ represents the source spectrum.

The normalization of $y(T)$ depends on the physical situation and proves liable to misunderstandings. If the source is uniform in space and constant in time, S is expressed in particles cm^{-3} sec^{-1}; $y(T)$ is then expressed in particles cm^{-2} sec^{-1} $energy^{-1}$ and represents the flux of electrons of energy T traversing a small spherical probe per unit cross-sectional area of the probe, per unit time per unit energy range of the spectrum. The spectral density of electrons in space (particles cm^{-3} $energy^{-1}$) is given by $y(T)/v$, where v is the velocity corresponding to T. However, the time element is irrelevant and S may be expressed as particles cm^{-3} emitted at all times; $y(T)$ has then the dimensions particles cm^{-2} $energy^{-1}$ and represents the flux through the probe integrated over the time of traversal. The space element is irrelevant also, and S may represent the total number of particles emitted over the whole medium without regard to uniformity; $y(T)$ has then the dimensions cm $energy^{-1}$ and represents the track length covered by electrons of the stated energy, per unit energy range. This is actually the same quantity that is called differential track length in shower theory. This last normalization is adopted here, with $S=1$ (see also F, Sec. 2).

If the electron source is not monochromatic but generates $S(T)dT$ electrons of energies between T and $T+dT$, the expected value $y(T)dT$ of the track length traveled by all electrons between T and $T+dT$ is the solution of

$$y(T)\int_0^T k(T,\tau)d\tau=\int_0^\infty y(T+\tau)k(T+\tau,\tau)d\tau+S(T). \quad (2)$$

[5] O. Blunck and S. Leisengang, Z. Physik **128**, 500 (1950).
[6] The letter T will serve to specify "kinetic energy."

The degradation equation (1) or (2) determines $y(T)$ directly once $y(T+\tau)$ is known for all $\tau>0$. This circumstance leads one naturally to stepwise methods of solution, beginning with the highest source energy T_0, such that $y(T_0+\tau)$ vanishes, and progressing from high to low energies along the course of energy degradation. Direct numerical evaluation of the integrals in (1) is, however, not workable for electrons owing to the very steep increases of $k(T,\tau)$ for small τ. Therefore, one must, in this range, manage to exploit analytical properties of the equation.

3. GENERAL FEATURES OF THE PROBABILITY OF SMALL AND MODERATE ENERGY LOSSES

We shall consider separately the contributions to $k(T,\tau)$ from bremsstrahlung $k_b(T,\tau)$ and from collisions against atomic electrons $k_c(T,\tau)$. For small τ, the bremsstrahlung contribution is always comparatively unimportant and can anyhow be represented in a simple analytical form, such as

$$k_b(T,\tau)=a/\tau+A, \text{ for } \tau\ll T, \quad (3)$$

where a and A depend on T only. The collisions against atomic electrons yield a simple Rutherford contribution for values of τ much larger than the binding energy I of the electrons and much smaller than T or mc^2, namely,

$$k_c(T,\tau)=(2\pi N_e e^4/mv^2)/\tau^2=\kappa/\tau^2, \quad \tau\gg I, \quad \tau\ll T, \quad \tau\ll mc^2. \quad (4)$$

Here e and m are the charge and rest mass of an electron, N_e is the number of atomic electrons per unit volume of the material, and v is the speed corresponding to T.

For $\tau\sim I$, $k_c(T,\tau)$ does not have a simple analytical form. However, small energy losses τ can be treated, in effect, according to a continuous slowing-down model so that the detailed dependence of k_c on τ is irrelevant but only the total rate of energy dissipation in low-τ collisions, $\int_0^\tau \tau' k_c(T,\tau')d\tau'$, matters. Analytical treatments which take advantage of this circumstance are presented in the following sections.

Owing to (4), we can write

$$\int_0^\tau \tau' k_c(T,\tau')d\tau'=\kappa\ln(\tau/\bar{Q}), \quad \tau\gg I, \quad \tau\ll T, \quad \tau\ll mc^2, \quad (5)$$

where $\bar{Q}$ is a constant determined by the theory of stopping power which, for $v\ll c$, equals Bethe's $Q_{\min}=(I_0Z)^2/2mv^2$, with $I_0\approx 10$ ev.[7] In so far as all relevant data on k_c for low τ are represented by (5), one may consider for analytical purposes that (4) holds down to $\tau=\bar{Q}$ and that k_c vanishes for $\tau<\bar{Q}$.

The preceding arguments do not hold as well for the

[7] H. A. Bethe, *Handbuch der Physik* (Springer, Berlin, 1933), Vol. 24, Part 1, 491 ff; Ann. Physik **5**, 325 (1930).

heaviest materials, where the binding energy of the K electrons is $\sim 0.2mc^2$ and there is hardly any range of τ for which the conditions of (3) are fully satisfied. (However, the K electrons in Pb are only 2 out of a total of 82.) This difficulty has been pointed out by Blunck and Leisegang[5] in connection with the Landau theory[8] of straggling of energy loss, which treats small energy losses in the manner indicated above. These authors gave a method for correcting the Landau theory. In the Appendix the size of the Blunck-Leisegang correction is estimated in connection with our results.

4. LANDAU-TYPE THEORY[9]

In order to learn how to solve the problem with regard to small energy losses without having yet to face the complications arising from the behavior of $k(T,\tau)$ for large τ, a solution of (1) was first sought in the range $T_0-T\ll T_0$, $T_0-T\ll mc^2$, $T_0-T\gg I$. Within this range one may perform the same simplifications as in the Landau theory,[8] i.e., assume $k(T,\tau)$ to be given by (4) from $\tau=\bar{Q}$ to $\tau=\infty$, and extend the integral on the left of (1) to $\tau=\infty$ since the resulting error is negligible. (See also F, Appendix V.) The equation becomes, then,

$$I(T_0-\Delta)\int_{\bar{Q}}^{\infty} d\tau/\tau^2=\int_{\bar{Q}}^{\Delta} I(T_0-\Delta+\tau)d\tau/\tau^2+\delta(\Delta), \quad (6)$$

where Δ represents T_0-T, and $I(T)$ represents $\kappa y(T)$.

The Laplace transform of $I(T)$ is then defined as follows:[10]

$$v(p)=\int_0^{\infty} d\Delta \exp(-p\Delta)I(T_0-\Delta). \quad (7)$$

Provided $\bar{Q}$ is effectively independent of T,[10] the transform of (6) is[11]

$$1=v(p)\int_{\bar{Q}}^{\infty} d\tau[1-\exp(-p\tau)]/\tau^2\approx v(p)p\ln(e/\gamma p\bar{Q}), \quad (8)$$

TABLE I. Comparison between approximate solutions to the Landau-type problem.

$(T_0-T)/\bar{Q}$	$I(T)$, expression (11)	$I(T)$, expression (24)
10	0.2069	0.2258
10^2	0.1435	0.1485
10^3	0.1086	0.1107
10^4	0.08718	0.08821
10^5	0.07272	0.07332
10^6	0.06235	0.06273

[8] L. Landau, J. Phys. (U.S.S.R.) **8**, 201 (1944).
[9] U. Fano and L. V. Spencer, Phys. Rev. **91**, 240 (1953).
[10] The possibility of incorporating the variable factor κ into the $I(T)$ extends the range of accuracy of the present calculation beyond that of the Landau theory. The limit of accuracy is now set by the assumed validity of (4) and the assumed constancy of $\bar{Q}$.
[11] The evaluation of the integral in this equation assumes that $p\bar{Q}\ll 1$, which is very well fulfilled for values of p of interest to this calculation.

with $e=2.718\cdots$ and $\gamma=1.781\cdots$. The solution of (6) is, therefore,[12]

$$I(T_0-\Delta)=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} dp \exp(p\Delta)/p\ln(e/\gamma p\bar{Q}). \quad (9)$$

Deformation of the path of integration to loop along the negative real axis yields

$$I(T_0-\Delta)=\int_0^{\infty}(du/u)\exp(-u)/[\pi^2+\ln^2(e\Delta/\gamma u\bar{Q})]$$
$$=\pi^{-1}\int_0^{\infty} du \exp(-u) \times\arctan[\pi/\ln(e\Delta/\gamma u\bar{Q})]. \quad (10)$$

Since $\ln u$ is small throughout the important range of the last integration, the arctangent can be expanded into powers of $\ln u$. Recalling that

$$\int_0^{\infty} du \exp(-u)\ln^n u=\Gamma^{(n)}(1),$$

the nth derivative of the gamma function at $x=1$, and setting $\bar{B}=\ln(e\Delta/\gamma\bar{Q})$, we find

$$I(T_0-\Delta)=\{\pi^{-1}\arctan(\pi/\bar{B})+\Gamma^{(1)}(1)/(\pi^2+\bar{B}^2) +\bar{B}\Gamma^{(2)}(1)/(\pi^2+\bar{B}^2)^2+\cdots\}. \quad (11)$$

This expansion converges very rapidly. A short tabulation of the bracketed quantity in (11) is given in the second column of Table I. Notice that, for $\bar{B}\gg\pi$, $y(T_0-\Delta)\approx 1/\kappa\bar{B}$, which is the reciprocal of the stopping power calculated by excluding all energy losses $\tau>e\Delta/\gamma$. This consideration, as well as an inspection of (9), illustrates the relationship of the result of this calculation to the prediction of the model of continuous slowing down. Notice also the rise of y for very small Δ indicating that the "effective stopping power" becomes quite small. This rise constitutes the "transient" discussed in F, Sec. 3c.

5. GENERAL ANALYTICAL TREATMENT

We now want to return to the general problem of formulating an equation which is amenable to numerical treatment. To do this we utilize an integral form which is equivalent to (1), namely,

$$\int_T^{T_0} dT' y(T')K(T',T)=1, \quad (12)$$

where

$$K(T',T)=\int_{T'-T}^{\infty} k(T',\tau)d\tau \quad (13)$$

expresses the probability per unit path that an electron's

[12] This equation results from Landau's Eqs. (5) and (10) by integrating $f(x,\Delta)$ over all values of the path length x.

kinetic energy drops from T' to a value *below* T. [It is implied, of course, that $k(T,\tau)$ vanishes for $\tau>T$.] Equation (12) states that the energy of every electron must drop below T somewhere along its path. The form (12) minimizes an undesireable feature of Eq. (1): the integral on the left side of (1) represents the total probability of collision, which is excessively large owing to the contribution of small energy losses but whose value is irrelevant since only the rate of energy dissipation through small losses matters. The integral cross section $K(T',T)$ in (12) is large only for a very small range of values of T' (very near to T) and thus does not contribute excessively to the integral.

The integral form of Eq. (2) is

$$\int_T^\infty dT' y(T') K(T',T) = \int_T^\infty S(T') dT'. \quad (14)$$

In order to simplify the integration over T' in the critical range $T'\approx T$, we strive to smooth out and reduce in magnitude the integrand in this range by adding and subtracting from (12) some suitable expression. We indicate with $\bar{K}(T',T)$ a function[13] to be chosen according to criteria indicated below and write

$$y(T)\int_T^{T_0} dT' \bar{K}(T',T) + \int_T^{T_0} dT' [y(T')K(T',T) - y(T)\bar{K}(T',T)] = 1. \quad (15)$$

Consider now the expression in brackets. As T' approaches T, $y(T')$ approaches $y(T)$ because it is a continuous function. The bracket will then tend to be small and finite, as T' approaches T, provided $\bar{K}(T',T)$ approximates $K(T',T)$ in this range. More specifically, we are interested in minimizing the integral over the difference between these expressions, in the critical range. Accordingly we set as a main consideration for the choice of $\bar{K}(T',T)$ that[14]

$$\int_T^{T+\delta} dT' [K(T',T) - \bar{K}(T',T)] = 0, \quad \text{for } \delta \gg I,\ \delta \ll T,\ \delta \ll mc^2. \quad (16)$$

In terms of the differential cross section $k(T,\tau)$, this condition reads

$$\int_0^\delta \tau k(T+\tau,\tau) d\tau - \int_0^\delta \bar{K}(T+\delta, T) d\delta = -\delta \int_\delta^\infty k(T+\delta,\tau) d\tau + \int_T^{T+\delta} dT' (T'-T) \times \int_{T'-T}^{T'} [\partial k(T',\tau)/\partial T'] d\tau. \quad (17)$$

[13] One may express $\bar{K}(T',T)$ in the form $\int_{T'-T}^\infty \bar{k}(T,\tau) d\tau$ and regard $\bar{k}(T,\tau)$ as a "mock cross section."

[14] The variation of κ between T and $T+\delta$, which is small owing to $\delta \ll T$, will often be disregarded in the following formulas.

The terms on the right side of (15) are small or of incidental significance. The first of them, which yields κ in the limit of small δ, materially improves the convergence of the numerical procedure; otherwise it could have been removed by modifying details of the condition (16). The essential point of the condition is that the integral over $\bar{K}(T',T)$ must approximate the other term on the left side of (17), namely the *stopping power, limited to small or moderate energy losses*. In addition, it will clearly be convenient that $\bar{K}(T',T)$ remain rather close to $K(T',T)$ even for $T'\gg T$ and that it be integrable analytically.

Setting

$$F(T_0,T) = \int_T^{T_0} dT' \bar{K}(T',T), \quad (18)$$

we recast (15) in the form

$$y(T) = [1/F(T_0,T)] \times \left\{1 - \int_T^{T_0} dT' [y(T')K(T',T) - y(T)\bar{K}(T',T)]\right\}, \quad (19)$$

which is suited to numerical integration. In fact, one may regard $1/F(T_0,T)$ as a first approximation estimate of $y(T)$. The first term on the right of (19), which is the largest one, represents indeed a reciprocal stopping power. The important point is then that the whole integral in the braces of (19) be much smaller than 1. Thus the structure of (19) lends itself also to an iteration procedure. In fact (19) can be integrated directly by numerical methods proceeding stepwise from high to low energies, as in the x-ray problem,[4] iteration being used to a minor extent in performing each integration step. The important conditions on the choice of $\bar{K}$, as stated before, are (1) that the numerical integration over T' be easy, and (2) that the whole integral over T' remain much smaller than 1.

6. A SIMPLE APPLICATION TO THE LANDAU-TYPE PROBLEM

As a first test of (19), this equation was solved under the assumption that the probability of energy losses is the same as in the Landau-type theory of Sec. 4, i.e., $k(T,\tau)=\kappa(T)/\tau^2$ for $\tau \gtrsim \bar{Q}$. Equation (13) yields, then,

$$\begin{aligned} K(T',T) &= \kappa(T')/(T'-T), \quad \text{for } T'-T \gtrsim \bar{Q}, \\ K(T',T) &= K(T+\bar{Q}, T), \quad \text{for } T'-T \lesssim \bar{Q}. \end{aligned} \quad (20)$$

We choose $\bar{K}(T',T)$, for this problem, as identical with $K(T',T)$ except for the replacement of $\kappa(T')$ with $\kappa(T)$, namely,

$$\begin{aligned} \bar{K}(T',T) &= \kappa(T)/(T'-T), \quad \text{for } T'-T \gtrsim \bar{Q}, \\ \bar{K}(T',T) &= \bar{K}(T+\bar{Q}, T), \quad \text{for } T'-T \lesssim \bar{Q}. \end{aligned} \quad (21)$$

Equation (18) yields, then,

$$F(T_0,T) = \kappa(T)\{1+\ln[(T_0-T)/\bar{Q}]\}. \quad (22)$$

If we enter (20), (21), and (22) into (19) and define, as in Sec. 4, $I(T)=\kappa(T)y(T)$, (19) reduces to

$$I(T)=\{1+\ln[(T_0-T)/\bar{Q}]\}^{-1}\times\left\{1-\int_0^{T_0-T}(d\tau/\tau)[I(T+\tau)-I(T)]\right\}. \quad (23)$$

A first approximation to $I(T)$ is obtained by neglecting the integral in comparison with unity

$$I(T)\approx\{1+\ln[(T_0-T)/\bar{Q}]\}^{-1}. \quad (24)$$

Table I compares $I(T)$ obtained from this expression with values for $I(T)$ obtained from (11). The agreement is excellent.[15] Further, expressions (24) and (11) can be shown to be asymptotically the same as (T_0-T) increases.

7. DISCUSSION OF CROSS SECTIONS

For the purpose of a realistic calculation, we have utilized cross sections with the following improvements with respect to the Landau-type theory. (1) The nonrelativistic Rutherford cross section for knock-on collisions has been replaced by the relativistic Møller formula. The energy loss is limited to $\tau\leqslant T/2$ for electrons of energy T since an electron emerging from a knock-on collision with energy lower than $T/2$ is classified as a secondary. (2) Energy losses by bremsstrahlung have been added. (3) The production of secondary electrons has been entered as an additional source on the right side of (14). Notice that the production of secondaries of energy T depends upon the presence of primaries of energy greater than $2T$ and therefore on previous determination of $y(T')$ for $T'>2T$.

For very small energy losses the Møller formula has been arbitrarily assumed to remain valid down to $\tau=\bar{Q}$ for the reasons discussed in Sec. 3, thus causing the small error considered in the Appendix.

A. Probability of Knock-on Collisions

The probability of collisions with energy loss τ, according to the Møller relativistic calculation,[16] may be written

$$k_M(T,\tau)=(2\pi N_e r_0^2/\beta^2)\{\tau^{-2}+(T-\tau)^{-2}-[(2+T^{-1})/(T+1)^2]\times[\tau^{-1}+(T-\tau)^{-1}]+(T+1)^{-2}\}, \quad (25)$$

for $\tau\leqslant T/2$, where T and τ are in mc^2 units, $r_0=e^2/mc^2$ and $\beta=v/c=[T(T+2)]^{\frac{1}{2}}/(T+1)$. As in Secs. 3 and 6 we assume this law to hold for small τ down to $\tau=\bar{Q}$, where $\bar{Q}$ is chosen so as to give the correct stopping power, limited to moderate energy losses. The value of $\bar{Q}$ was chosen, on the basis of stopping power theory,[7] supplemented by experimental data, as

$$\bar{Q}=\tfrac{1}{2}(ZI_0/mc^2)^2[T(T+2)]^{-1}\exp(\beta^2), \quad (26)$$

in units of mc^2, with I_0 equal to 11.5 ev for Al and to 10.2 ev for Pb. This value of $\bar{Q}$ fails to take into account the density effect, a simplification which seemed warranted in a pilot calculation.[17]

According to (13), the contribution of collisions to $K(T',T)$ is, then,

$$K_c(T',T)=0, \quad \text{for } T'\geqslant 2T,$$

$$K_c(T',T)=\int_{T'-T}^{T'/2}k_M(T',\tau)d\tau$$

$$=(2\pi N_e r_0^2/\beta'^2)\{(T'-T)^{-1}-T^{-1}-[(2+T'^{-1})/(T'+1)^2]\ln[T/(T'-T)]+(T'+1)^{-2}(T-T'/2)\}, \quad (27)$$

$$\text{for } T+\bar{Q}\leqslant T'\leqslant 2T,$$

$$K_c(T',T)=K_c(T+\bar{Q},T), \quad \text{for } T'\leqslant T+\bar{Q}.$$

For the corresponding function $\bar{K}_c(T',T)$ we took

$$\bar{K}_c(T',T)=0, \quad \text{for } T'\geqslant 2T,$$

$$\bar{K}_c(T',T)=(2\pi N_e r_0^2/\beta^2)\{(T'-T)^{-1}-T^{-1}-[(2+T^{-1})/(T+1)^2]\ln[T/(T'-T)]+(T+1)^{-2}(T-T'/2)\}, \quad \text{for } T+\bar{Q}\leqslant T'\leqslant 2T, \quad (28)$$

$$\bar{K}_c(T',T)=\bar{K}(T+\bar{Q},T), \quad \text{for } T'\leqslant T+\bar{Q}.$$

The contribution to $F(T_0,T)$ arising from $\bar{K}_c(T',T)$ is

$$F(T_0,T)=\int_T^{T_0}dT'\bar{K}_c(T',T)$$

$$=(2\pi N_e r_0^2/\beta^2)\{1+\ln(\Delta/\bar{Q})-\Delta/T-[(2+T^{-1})/(T+1)^2]\Delta[1+\ln(T/\Delta)]+(T+1)^{-2}(\Delta/2)(T-\Delta/2)\}, \quad (29)$$

where Δ equals T_0-T or T, whichever is smaller, and terms of the order of $\bar{Q}$ have been disregarded in the braces.

B. Production of Secondary Electrons

The Møller expression (25), $k_M(T,\tau)$, also represents the probability of production of secondary electrons of energy $T-\tau$ when evaluated for $\tau>T/2$. Therefore, the additional "source" of secondary electrons, to be entered as $S(T')$ in (14) is given by $\int_{2T'}^{\infty}dT''y(T'')$

[15] A second approximation can easily be calculated by inserting the approximate solution (24) into the integral in (23). This second approximation, which agrees to at least four significant figures with column 1 of Table I, in the following:

$$I(T)\approx\{1+\ln(\Delta/\bar{Q})\}^{-1}\{1-(\pi^2/6)[1+\ln(\Delta/\bar{Q})]^{-2}\},$$

where $\Delta=T_0-T$.

[16] C. Møller, Z. Physik **70**, 686 (1931).

[17] Owing to the density effect, $\bar{Q}$ should not decrease indefinitely as T increases but should approach the minimum value $2\pi e(137)^2N_e r_0^2$, with $e=2.71828\cdots$.

$\times k_M(T'', T''-T')$. When this expression is entered in (14) together with the "true" source $S(T')=\delta(T'-T_0)$ we find, by changing the order of integrations,

$$\int_T^\infty S(T')dT' = 1+\int_{2T}^{T_0} dT' y(T') K_s(T',T), \quad \text{for } T<T_0/2, \quad (30)$$

where

$$K_s(T',T) = \int_{T'/2}^{T'-T} k_M(T',\tau)d\tau$$
$$= (2\pi N_e r_0^2/\beta'^2)\{T^{-1}-(T'-T)^{-1} - [(2+T'^{-1})/(T'+1)^2]\ln[(T'-T)/T] + (T'+1)^{-2}(T'/2-T)\}. \quad (31)$$

Notice that $k_M(T,\tau)=k_M(T,T-\tau)$, and $K_s(T',T) = \int_T^{T'/2} k_M(T',\tau)d\tau$.

C. Bremsstrahlung Energy Losses

The cross section for energy losses by bremsstrahlung has been calculated by Bethe and Heitler.[18] This treatment relies upon the Born approximation and yields a cross section of the form

$$k_b(T,\tau) = (ZN_e r_0^2/137)[28.1-(16/9)\ln Z] \times [T/(T+1)][\tau^{-1}-f(T,\tau)], \quad (32)$$

where T,τ are in mc^2 units and $f(T,\tau)$ is a positive function, containing no singularity, which is known only numerically.[19] Since $k_b(T,\tau)$ vanishes for $\tau>T$ the integral (13) over this cross section yields

$$K_b(T',T) = (ZN_e r_0^2/137)[28.1-(16/9)\ln Z] \times [T'/(T'+1)]\left\{\ln[T'/(T'-T)] - \int_{T'-T}^{T'} d\tau f(T',\tau)\right\}. \quad (33)$$

Since $K_c(T',T)$ vanishes for $T'>2T$, according to (27), and since $K_b(T',T)$ is small in the same range of $T'>2T$, it is possible and, in fact, convenient, to modify again the integral of Eq. (14). We limit the integral on the left side to $T\leqslant T'\leqslant 2T$ and transfer the remaining portion, $\int_{2T}^\infty y(T')K_b(T',T)dT'$ to the right side, to be treated in effect as a "negative source."

Following this idea, we take

$$\bar{K}_b(T',T)=0, \quad \text{for } T'>2T,$$
$$\bar{K}_b(T',T) = (ZN_e r_0^2/137)[28.1-(16/9)\ln Z] \times [T/(T+1)]\left\{\ln[T'/(T'-T)] - \int_{T'-T}^{T'} d\tau f(T',\tau)\right\}, \quad \text{for } T'<2T, \quad (34)$$

and, therefore,

$$F_b(T_0,T) = \int_T^{T_0} dT' \bar{K}_b(T',T)$$
$$= (ZN_e r_0^2/137)[28.1-(16/9)\ln Z] \times [T/(T+1)]\left\{\Delta \ln[(\Delta+T)/\Delta] + T\ln[(\Delta+T)/T] - \int_T^{T+\Delta} dT' \int_{T'-T}^{T'} d\tau f(T',\tau)\right\}, \quad (35)$$

where Δ equals T_0-T or T, whichever is smaller, as in (29). In the integrations over $f(T',\tau)$, the approximate analytical expressions by Schiff[20] were entered instead of the numerical tables of Bethe and Heitler.

D. Final Form of the Equation

We combine the preceding formulas by setting

$$K(T',T) = K_c(T',T)+K_b(T',T), \quad \text{for } T'\leqslant 2T, \quad (36)$$
$$\bar{K}(T',T) = \bar{K}_c(T',T)+\bar{K}_b(T',T), \quad \text{for } T'\leqslant 2T, \quad (37)$$
$$K^*(T',T) = K_s(T',T)-K_b(T',T), \quad \text{for } 2T\leqslant T'\leqslant T_0, \quad (38)$$
$$F(T_0,T) = F_c(T_0,T)+F_b(T_0,T), \quad (39)$$

where K_c, $\bar{K}_c$, K_s, K_b, $\bar{K}_b$, F_c, and F_b are given, respectively, by (27), (28), (31), (33), (34), (29), and (35). We take (30) as the source term and treat the large bremsstrahlung losses as a "negative source," as explained above. Equation (19) takes now the form

$$y(T) = [F(T_0,T)]^{-1}\left\{1+\int_{2T}^{T_0} dT' K^*(T',T)y(T') - \int_T^{T+\Delta} dT'[y(T')K(T',T)-y(T)\bar{K}(T',T)]\right\}, \quad (40)$$

where Δ equals T_0-T or T, as before.

8. SOME DETAILS OF THE CALCULATION

The usual method for solving numerically a Volterra integral equation like that of expression (40) involves writing finite sums for the integrals. This reduces the integral equation to a triangular system of linear equations which can be easily solved in succession. A set of points T_i must be chosen, one per interval of integration, at which the integrand is evaluated. A suitable integration formula must likewise be selected (e.g., trapezoidal, Simpson's Rule). The first step in the integration (i.e., the first linear equation) often requires special attention.

[18] H. Bethe and W. Heitler, Proc. Roy. Soc. (London) **A146**, 83 (1934).

[19] Notice that bremsstrahlung in the field of an electron has not been included in (32).

[20] L. I. Schiff, Phys. Rev. **83**, 252 (1951).

In our calculations the T_i were distributed evenly on a scale of $\log T$, that is, according to the formula T_0, ζT_0, $\zeta^2 T_0$, etc. There were three reasons for this: (1) the population of secondary electrons might increase so rapidly at low energies that the solution would diverge, perhaps as some inverse power of T, as $T\rightarrow 0$. If this happened, the integration could nevertheless proceed owing to the smaller and smaller intervals of T. (2) The choice of ζ according to $\zeta^n=\frac{1}{2}$, where n is an integer, would enable us to take into account very simply the special behavior at $T'=2T$. (3) Owing to the logarithmic distribution, the same set of points is equally suited to different primary energies.

As a first trial, ζ was chosen so that $\zeta^3=\frac{1}{2}$. (Too small a value for ζ means coarse intervals which tend to yield inaccurate results. On the other hand, a ζ near unity means many steps and a long calculation. We decided that for the first calculation it would be better to err in the direction of small ζ.) Fortunately, later experience indicated that this was a fairly good choice, and most of the calculations were accomplished using this interval.

For an integration formula we took Simpson's Rule which means approximation by parabolas. Simpson's Rule is especially suited to an even number of integration intervals as well as to *equal-sized* intervals. We therefore modified Simpson's Rule to apply to equal-sized intervals in a logarithmic sense. Where the number of integration intervals was odd, we fitted the last two intervals with a parabola and integrated this parabola over the last interval only.

For the first calculations we tabulated $K(T',T)$, $\bar{K}(T',T)$, and $K^*(T',T)$, neglecting the bremsstrahlung part, from $T_0=80$ (mc^2 units) to $T=1.25$. (Later this was extended to $T=0.0124$.) In these tabulations the points $T'=T$ were left blank, since for $T'\rightarrow T$,

$$[y(T')K(T',T)-y(T)\bar{K}(T',T)]\rightarrow (\partial/\partial T)\{2\pi N_e r_0^2 y(T)/\beta^2\}. \quad (41)$$

(See the Appendix.)

While accomplishing the first solutions, we tabulated the quantity $[y(T')K(T',T)-y(T)\bar{K}(T',T)]$ for each step $T=T_n$ in order to make sure that it is sufficiently well behaved to allow an accurate evaluation of the integral over T' by our numerical integration scheme. We also decided to use a step-by-step iteration scheme as follows. Having determined $y(T)$ at T_0, T_1, $\cdots T_{n-1}$ we estimated what $y(T_n)$ will be. Making use of this, we then tabulate the quantity $[y(T')K(T',T)-y(T)\times\bar{K}(T',T)]$ for $T=T_n$, and for $T'=T_n$, T_{n-1}, T_{n-2}, T_{n-3}.[21] We then evaluate the integral and thereby obtain from (40) a new estimate of $y(T_n)$. These estimates converge extremely rapidly to a unique value.

The remaining questions concern the start of the integration from $T\sim T_0$. The approximate solutions (24) and (11) are accurate near T_0 and can be used to give nearly correct values for the $y(T_i)$ for T_i near T_0. Unfortunately, $y(T)$ has a peak of logarithmic type at T near T_0. However, this peak contributes a negligible amount to the integral. In most calculations it was found adequate to clip off the peak at the level of $y(0.95T_0)$, the more so since the integral of $[y(T')\times K(T',T)-y(T)\bar{K}(T',T)]$ is very small compared with unity near T_0. For greater accuracy at the beginning of the calculation an extra point was inserted between T_0 and ζT_0.

9. FURTHER DETAILS—BREMSSTRAHLUNG

Calculations involving bremsstrahlung have two unpleasant features: (1) the quantity $\int_{T'-T}^{T'} f(T',\tau)d\tau$ must be tabulated, and (2) the approximate solution, $1/F(T_0,T)$, is less accurate than in calculations not involving bremsstrahlung.[22]

To tabulate the quantity $\int_{T'-T}^{T'} f(T',\tau)d\tau$, we established a fairly coarse grid of values of $f(T',\tau)$. We then carried out the integration numerically by fitting parabolas. Finally, we did a double interpolation, first in the variable $(1-T/T')$ and then in the variable T', in order to establish the table needed for the solution of the integral equation.[23] The tabulations were not carried below $T=1.25mc^2$. At this energy the sudden omission of bremsstrahlung introduces a discontinuity in $y(T)$ which amounts to about 5 percent in Pb and to less than 1 percent in Al. We made a rough extrapolation of the bremsstrahlung correction for energies somewhat lower than $1.25mc^2$ in order to reduce the discontinuity in Pb to a smaller value.

In problems involving bremsstrahlung, the integration was started at energies so near T_0 that bremsstrahlung modified the $y(T)$ given by expressions (24) and (11) by no more than a percent or so. In the $80mc^2$ Pb problem, this meant setting $T_1/T_0\approx 0.998$. Thereafter, to expand the interval of integration, the T_i's were distributed uniformly in a scale of $\log(T_0-T)$. In order to join this system of exponentially increasing intervals with the set of *decreasing* intervals previously established for lower energies, the last of the increasing intervals was arranged to coincide with the interval $\zeta T_0 \geqslant T \geqslant \zeta^2 T_0$, i.e., with the second interval of the set discussed in the preceding section. This procedure proved workable because $1/F(T_0,T)$ is always a fairly good estimate of $y(T)$ for $T>T_0/2$.

An appropriate integration formula was established for this rather unusual sequence of intervals, a weight being assigned to each pair of values (T',T).

[21] To calculate the derivative in (41) we merely fitted a parabola to the points T_n, T_{n-1}, and T_{n-2} and differentiated. This proved adequate because the solution is not very sensitive to the value of the derivative.

[22] For example, in the $80mc^2$ calculation with bremsstrahlung in Pb, $y(T)$ and $1/F(T_0,T)$ differ by as much as 12 percent, whereas neglecting bremsstrahlung they differ by no more than a percent or so.

[23] The tabulation of $f(T',\tau)$ for Pb utilized the spectrum of Schiff with $\theta_0=0$. The $f(T',\tau)$ so obtained was normalized to agree with that of Bethe and Heitler at $\tau=0$. This procedure appeared justified by expedience in a first pilot calculation. The tabulation for Al used the Schiff spectrum integrated over all angles.

As mentioned previously, much of the numerical work involves tabulating and integrating numerically the function $[y(T')K(T',T)-y(T)\bar{K}(T',T)]$. Having determined that this is a reasonably well-behaved function, the integrals $\int_T^{T_0,2T} dT' y(T')K(T',T)$ and $\int_T^{T_0,2T} dT' \bar{K}(T',T)$ were calculated separately, using the same integration weights for both integrals and assuming that the first integrand is

$$(\partial/\partial T)[2\pi N_e r_0^2 y(T)/\beta^2]$$

at $T'=T$, whereas the second integrand is zero at that point. This separation makes it possible to perform first the integrations and then independently the determination of $y(T_n)$.

TABLE II. Comparison between $y(T)$ and the reciprocal stopping power neglecting both bremsstrahlung and secondary electrons.

T, mc^2 units	Reciprocal stopping power, Pb	$y(T)$, Pb	Reciprocal stopping power, Al	$y(T)$, Al
79.86	0.0293	0.0366	0.0892	0.1065
63.50	0.0302	0.0304	0.0914	0.0921
50.40	0.0310	0.0307	0.0936	0.0927
40.00	0.0319	0.0314	0.0960	0.0947
20.00	0.0350	0.0344	0.1039	0.1024
10.00	0.0385	0.0378	0.1126	0.1109
5.00	0.0421	0.0414	0.1211	0.1192
2.50	0.0448	0.0440	0.1263	0.1242
1.25	0.0443	0.0433	0.1218	0.1195
0.625	0.0387	0.0379	0.1037	0.1020
0.3125	0.0295	0.0290	0.0769	0.0757
0.1563	0.0203	0.0199	0.0510	0.0502
0.07813			0.0318	0.0310
0.03906			0.0193	0.0186
0.01953			0.0116	0.0110

TABLE III. Comparison between $y(T)$ and the reciprocal stopping power when bremsstrahlung is included but secondary electrons are not.

T, mc^2 units	Reciprocal stopping power, Pb	$y(T)$, Pb	Reciprocal stopping power, Al	$y(T)$, Al
79.86	0.00584	0.0336	0.0555	0.1065
63.50	0.00703	0.00704	0.0617	0.0658
50.40	0.00853	0.00668	0.0679	0.0633
40.00	0.0104	0.00779	0.0740	0.0664

10. DISCUSSION OF RESULTS

Tables II and III and Figs. 1 through 4 summarize the results of our calculations. In all tables and figures the differential track length in units of cm/mc^2 is presented as a function of the electron kinetic energy in mc^2 units.

Each of the Figs. 1 through 4 presents the electron slowing down spectra calculated with different refinements, namely, neglecting both secondary electrons and bremsstrahlung (curve I), neglecting only the bremsstrahlung (curve II), and including both bremsstrahlung and secondary electrons (curve III). In Fig. 4 the bremsstrahlung influence is so small that curves II and III coincide. In all figures curves I and II coincide

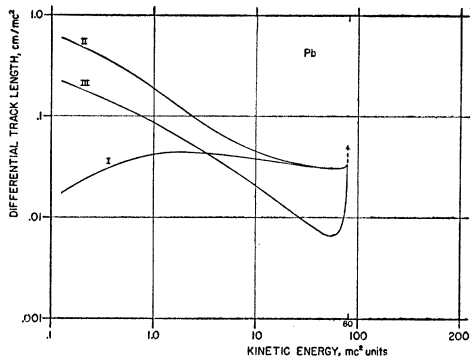

FIG. 1. Electron slowing-down spectra for an $80mc^2$ source in Pb.

at energies higher than one-half the initial energy since all secondaries have energies less than one half the initial energy.

A comparison of curves I and II shows the buildup of secondary electrons. All four figures indicate that neglecting bremsstrahlung the secondary electrons become as numerous as the primary electrons at about 4 percent of the initial energy. There are 10 percent as many secondaries as primaries at about 16 percent of the initial energy. Notice also that at the lowest energies there are up to 300 times as many secondary electrons as there are primary electrons.

A comparison of curves II and III demonstrates the influence of bremsstrahlung. This is overwhelming at high energies in Pb and negligible at low energies in Al. Bremsstrahlung has been treated only as a mechanism for energy loss and electrons resulting from cascade processes have *not* been included.

Notice that curves II and III tend to separate at energies at which the secondaries begin to dominate ($\sim 1mc^2$ for the $80mc^2$ source problems). This is because the generation of secondaries of energy T depends upon the spectrum at energies greater than $2T$. In the bremsstrahlung calculations there are always fewer electrons

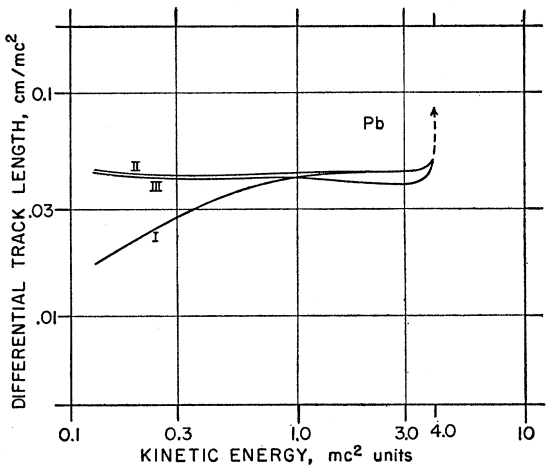

FIG. 2. Electron slowing-down spectra for a $4mc^2$ source in Pb.

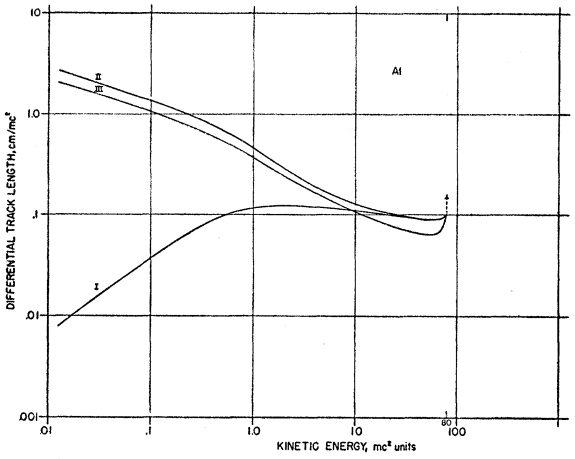

FIG. 3. Electron slowing-down spectra for an $80mc^2$ source in Al.

at high energies than in the bremsstrahlung-neglected calculations, hence fewer secondaries are generated and the spectrum at low energies rises somewhat more slowly. On the other hand, at *very* low energies curves II and III tend to become parallel, indicating an approach to an equilibrium spectrum.

Table II compares spectra (which do not involve secondary electrons or bremsstrahlung) with the reciprocal stopping power, which is the spectrum obtained on the assumption that the electrons lose their energy continuously. The model of continuous slowing down disregards the possibility that an electron loses much energy in a single process and thereby overlooks the interlinkage between the values of $y(T)$ for different values of T. It stands to reason that this oversimplification will not have drastic effects in regions of the spectrum where $y(T)$ remains fairly constant. The data in Table II verify this surmise.

The model of continuous slowing down was improved in F, Sec. 5, by a procedure of successive approximations. This procedure takes partial account of the interlinkage between different portions of the spectrum by considering successive derivatives of the stopping power and of higher moments of the energy loss distribution, at each energy. The spectrum $y(T)$ is thereby given by an expansion, Eq. (30), of F whose first term is the reciprocal stopping power. For electron slowing down, the convergence of this expansion cannot be depended upon because extreme fluctuations of energy loss in individual collisions are too important. Nevertheless, it is interesting to compare the first corrective term of the expansion with the discrepancy between the reciprocal stopping power and the "exact" value of $y(T)$ calculated in this paper. The first correction reduces the reciprocal stopping power by a factor of approximately $[1-1/4B]$, where B is the stopping number $[B\sim 2\ln(T/I_0Z)$ is of the order of 10 or 20$]$. Comparison with Table II shows that this correction has the proper sign and order of magnitude for most values of T and displays the increasing trend of the discrepancy at lower values of T. The large discrepancy between the reciprocal stopping power and $y(T)$ in the upper portion of the spectrum derives from the "transient" effect for which no allowance was made in Eq. (30) of F.

Table III gives a comparison similar to that of Table II between the reciprocal stopping power and $y(T)$ when bremsstrahlung is included in the calculation. As in Table II, the continuous slowing down model gives a low value in the transient range, near T_0, and a high value in the broad range of lower energies. However, in this range, the discrepancy is quite large, when bremsstrahlung is important, as expected in view of the extreme type of energy fluctuations that is characteristic of the bremsstrahlung process.

We wish to thank Miss Ida E. Hornstein and Mr. John Hubbell for much help in the numerical work.

APPENDIX

The Blunck-Leisegang correction concerns the evaluation of the integral of $[y(T')K(T',T)-y(T)K(T',T)]$ for $T'\sim T$ in (19). We rewrite this expression in the form

$$[y(T')-y(T)]K(T',T)+y(T)[K(T',T)-\bar{K}(T',T)]. \tag{42}$$

The second bracket in this expression can be taken as zero at $T'=T$ owing to the condition (16).[14] Since $K(T',T)$ peaks near $T'=T$, the first bracket in (42) must be replaced with $[dy(T)/dT](T'-T)$ for $T'\sim T$. (Higher-order terms in the expansion of $y(T')-y(T)$ are not significant in this approximation.) Since the important quantity is not (42) itself but its integral over a range of T' near T we calculate, as in (17),

$$\left(\frac{dy}{dT}\right)\int_T^{T+\delta} dT'(T-T')K(T',T)$$
$$=\frac{1}{2}\left(\frac{dy}{dT}\right)\left\{\int_0^{\delta}\tau^2 k(T+\tau,\tau)d\tau + \delta^2\int_{\delta}^{T+\delta} k(T+\delta,\tau)d\tau - \int_T^{T+\delta} dT'(T'-T)^2 \times\int_{T'-T}^{T'}[\partial k(T',\tau)/\partial T']d\tau\right\}. \tag{43}$$

For the purpose of evaluating the integral in (19) we are interested only in those terms of (43) that are small of order δ as δ grows smaller. The last term in the braces of (43) is, accordingly, negligible. The second term in the braces yields $\kappa(T)\delta$, if one assumes k to be given by (4) for the relevant values of the variables.[14] With regard to the first term, we have been assuming since Sec. 3 that $k(T,\tau)$ is represented adequately by (4) also

for low values of τ when it is multiplied by τ^n, with $n>1$, and integrated. Under this assumption, the first term in the braces yields $\kappa(T)\delta$, like the second term, and the total value of (43), to within δ^2 corrections, is $(dy/dT)\kappa(T)\delta$, which justifies (41).[14]

The Blunck-Leisegang correction consists in evaluating $\int_0^\delta \tau^2 k(T+\tau, \tau)d\tau$ to a better accuracy than is done by taking k according to (4). The following evaluation is believed to represent an improvement with respect to the rough estimation given by the original authors.

The collision probability is given, for low or moderate τ, by the Bethe formula[7]

$$k(T,\tau)=\kappa\int_{Q_{\min}}^{Q_{\max}}|F_\tau(Q)|^2 dQ/Q^2. \tag{44}$$

Here, $|F_\tau(Q)|^2$ represents the generalized form factor, i.e., the probability that an atom absorbs the energy τ when one of its electrons has absorbed the momentum corresponding to a free recoil energy Q. Since, for large Q, $|F_\tau(Q)|^2\sim\delta(Q-\tau)$, and since $Q_{\min}$, which depends on τ, may be replaced with an effective value $\bar{Q}$, we have with adequate accuracy

$$\int_0^\delta \tau^2 k(T+\tau, \tau)d\tau\sim\kappa\int_{\bar{Q}}^\delta dQQ^{-2}\int_0^\infty |F_\tau(Q)|^2\tau^2 d\tau. \tag{45}$$

The last integral over τ can be evaluated by a closure theorem (sum rule) and yields $Q^2+(4/3)Q\langle T\rangle_{\text{at}}$, where $\langle T\rangle_{\text{at}}$ is the mean kinetic energy of the atomic electrons. When this expression is entered in (45), the first term Q^2 yields the same result that was obtained by taking $k(T,\tau)$ according to (4) instead of (44). The second term, $(4/3)Q\langle T\rangle_{\text{at}}$, yields the desired correction. We have then

$$\int_0^\delta \tau^2 k(T+\tau, \tau)d\tau\sim\kappa[\delta+(4/3)\langle T\rangle_{\text{at}}\ln(\delta/\bar{Q})], \tag{46}$$

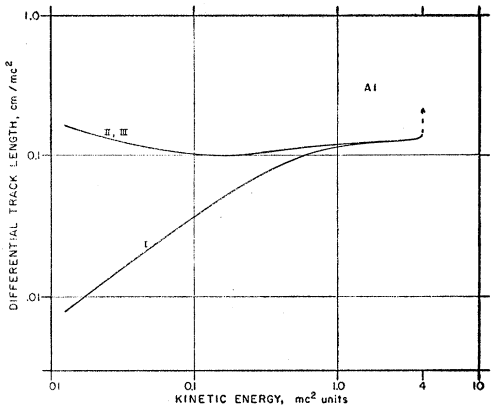

FIG. 4. Electron slowing-down spectra for a $4mc^2$ source in Al.

where $\bar{Q}$ has been disregarded as compared to δ. Thus, recalling (42) and (43), we find that the integral in (19) has been underestimated, in the range $T'\sim T$, by an amount approximately equal to

$$\tfrac{1}{2}(dy/dT)\kappa(4/3)\langle T\rangle_{\text{at}}\ln(\delta/\bar{Q}). \tag{47}$$

This quantity may be expressed, according to (5), in terms of the stopping power of nonradiative collisions

$$(2/3)(dy/dT)\langle T\rangle_{\text{at}}\int_0^\delta k_c(T,\tau)\tau d\tau. \tag{48}$$

The residual dependence of (47) and (48) on δ indicates that the departures from the free-electron collision formula (4) due to the bound motion of atomic electrons cause our earlier evaluation of (19) to be somewhat in error even for T' substantially larger than T. The total error is given approximately by (48) with the integral replaced with the full stopping power of nonradiative collisions. Therefore, it is clearly of the order of magnitude of the relative variation of $y(T)$ over a spectral interval $\langle T\rangle_{\text{at}}$. According to the Thomas-Fermi model $\langle T\rangle_{\text{at}}$ is of the order of $10Z^{4/3}$ ev.