the Brownian particles are in kinetic equilibrium with the surrounding medium, their initial velocities are symetrically distributed and they are all initially located at the origin. Then for Eq. (2) the initial value is $\rho(x,0)=\delta(x)$, and for Eq. (5) the values are $\rho(x,0)=\delta(x)$, $\rho_t(x,0)=0$. Equation (2) is now the heat equation, and its solution for above initial values is given by

$$\rho_1(x,t)=\left(4\pi\frac{kT}{m\beta}t\right)^{-\frac{1}{2}}\exp\left[-x^2\bigg/\left(2\frac{kT}{m\beta}t\right)\right]. \quad (7)$$

Equation (5) is now the telegraph equation, and its solution is given by

$$\rho_2(x,t)=\begin{cases}\frac{1}{4}e^{-\frac{1}{2}\beta t}\beta[I_0(X)+\frac{1}{2}\beta t\, I_1(X)/X]; & x<(kT/m)^{\frac{1}{2}}t\\ 0; & x>(kT/m)^{\frac{1}{2}}t\\ \frac{1}{2}e^{-\frac{1}{2}\beta t}\delta; & x=\pm(kT/m)^{\frac{1}{2}}t\end{cases} \quad (8)$$

and

$$X=\tfrac{1}{2}\beta(t^2-mx^2/kT)^{\frac{1}{2}},$$

where I_0 and I_1 represents the Bessel functions of imaginery argument. For large time ρ_2 is asymptotic to ρ_1, which is to be expected since both Eqs. (2) and (5) have the same steady-state solution.

As stated in the first paragraph, Eq. (1) represents a simple Markoff process-type diffusion in phase space. However when the problem is reduced to configuration space in the representational form of Eq. (5), the diffusion is a Markoff process of order two. In fact it is possible to derive the Telegraph equation from a random-walk problem in configuration space which is a multiple Markoff process. This was first done by Goldstein.[3]

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PHYSICAL REVIEW VOLUME 93, NUMBER 6 MARCH 15, 1954

Quantum Statistics of Closed and Open Systems

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(Received September 21, 1953)

In this note an ideal Bose-Einstein and an ideal Fermi-Dirac gas are considered in the "limit L," i.e., in the limit in which the number of particles N and the volume V of the system are allowed to tend to infinity so as to keep N/V a finite and nonzero constant. It is shown that, whatever the system of energy levels, the mean occupation numbers calculated on the basis of the grand canonical ensemble are in the limit L the same as those calculated on the basis of the canonical ensemble. The only requirement is that the mean total number of particles for the grand canonical ensemble be equal to the fixed total number of particles for the canonical ensemble.

It is well known that for an open system of non-interacting quantum-mechanical particles, treated on the basis of the grand canonical ensemble (g.c.e.), the mean occupation number of the jth quantum state of a particle is[1]

$$\bar{\nu}(\bar{N},j)=1/\{\exp[\eta(j)-\alpha]\pm1\}, \quad (1)$$

where $\eta(j)$ is the energy of the jth quantum state divided by kT, the top sign refers to fermions, the bottom signs to bosons, and α is determined by

$$\bar{N}=\textstyle\sum_j \bar{\nu}(\bar{N},j). \quad (2)$$

$\bar{N}$ is the fixed average number of particles in the fixed volume V of the system. The corresponding mean occupation number $\bar{n}(N,j)$ for a closed system, having N particles in a volume V and treated on the basis of a canonical ensemble (c.e.), is similar,[2,3] except that α is replaced by α_j, where

$$\exp(-\alpha_j)\equiv Z_{N+1}\bar{n}(N+1,j)/Z_N\bar{n}(N,j),$$
$$\text{and} \quad (3)$$
$$N=\textstyle\sum_j \bar{n}(N,j).$$

Z_N and Z_{N+1} are the partition functions of the system when it has N or $N+1$ particles in the volume V. The two treatments (g.c.e. and c.e.) will not in general lead to the same results, although, if one assumes $\bar{N}=N$, the difference is usually small. We are not aware, however, of any general investigation into the conditions which must be fulfilled if these two treatments (with $\bar{N}=N$) are to yield *exactly* similar results for the main physical properties of the system. We shall show that *one such condition is that the properties of the system be considered in the "limit L."* This limit is defined by the requirement that N and V be allowed to tend to infinity while their ratio $\rho=N/V$ remains a finite and nonzero constant. The symbol $\sim$ shall denote equality in the limit L.

In the case of bosons a restricted proof of our propo-

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[1] R. C. Tolman, *Statistical Mechanics* (Cambridge University Press, Cambridge, 1938), p. 503.
[2] T. Sakai, Proc. Phys. Math. Soc. Japan 22, 193 (1940).
[3] F. Ansbacher and W. Ehrenberg, Phil. Mag. 40, 626 (1949).

sition is implied in previous work.[4] The restriction resides in the fact that certain assumptions concerning the spectrum of energy levels were made throughout this paper, since they were required for the subject then under discussion. The demonstration that these restrictions are not in fact required for the present purpose is quite straightforward, but will be briefly given here for convenience of exposition. It suffices to appeal to the important result[5]

$$\bar{n}(N+1,\ j) > \bar{n}(N,\ j) \quad \text{(for finite systems)}, \qquad (4)$$

whence[4]

$$1 < \bar{n}(N+1,\ j)/\bar{n}(N,\ j) < [\textstyle\sum_K \bar{n}(N+1,\ K)]/[\textstyle\sum_K \bar{n}(N,\ K)] = 1+1/N \quad \text{(finite systems)}. \qquad (5)$$

It follows that $\bar{n}(N+1,\ j) \sim \bar{n}(N,\ j)$, so that

$$\rho \sim V^{-1} \textstyle\sum_j [1/\{x \exp\eta(j) - 1\}] \qquad (6)$$

must be satisfied by both $x = e^{-\alpha}$ and $x = Z_{N+1}/Z_N$ in the limit L, so that $e^{-\alpha} \sim Z_{N+1}/Z_N$ and $\bar{\nu}(N,j) \sim \bar{n}(N,j)$ (all N, all j).

In the case of fermions the analog of (4) does not appear to be available in the literature. We shall, therefore, indicate a method of establishing it. Let $S(N;r,\lambda)$[6] denote the sum of all those terms in the partition function Z_N of the system which are obtained by allowing distributions of N particles over all the quantum states j, subject to the condition $n(N,\lambda) = r$, where r is a given nonnegative integer (which must be 0 or 1 for fermions). It is known[2,3] that

$$S(N;1,\lambda) = e^{-\eta(\lambda)} S(N-1;0,\lambda).$$

We establish first an auxiliary result. Let

$$X \equiv \frac{S(N;0,\lambda)S(N+1;1,\lambda)}{S(N;1,\lambda)S(N+1;0,\lambda)} = \frac{e^{-\eta(\lambda)} \sum_{\alpha_1} \cdots \sum_{\alpha_N} \sum_{\beta_1} \cdots \sum_{\beta_N} \exp[-\eta(\alpha_1) \cdots - \eta(\alpha_N) - \eta(\beta_1) \cdots - \eta(\beta_N)]}{e^{-\eta(\lambda)} \sum_{\gamma_1} \cdots \sum_{\gamma_{N-1}} \sum_{\delta_1} \cdots \sum_{\delta_{N+1}} \exp[-\eta(\gamma_1) \cdots - \eta(\gamma_{N-1}) - \eta(\delta_1) \cdots - \eta(\delta_{N+1})]}.$$

The summations on the right are subject to the following restrictions: (1) For each term in the sum no two of the α suffixes must be equal, and the same applies to the β, γ, δ suffixes, since an equality of this kind would indicate the presence of two particles in the same quantum state. (2) The λth quantum state must be omitted from all summations, as it is either empty or, if it is occupied, has already been accounted for by the factors $e^{-\eta(\lambda)}$. Thus,

$$X = \{\sum_{a_1} \cdots \sum_{a_{2N}} \exp(-\sum_{p=1}^{2N} \eta(a_p))\} / \{\sum_{b_1} \cdots \sum_{b_{2N}} \exp(-\sum_{q=1}^{2N} \eta(b_q))\}.$$

In this summation, (1) $a_p, b_q \neq \lambda$ for all p and q, and (2) relations of the form $a_j \neq a_k$, $b_j \neq b_k$ $(j \neq k)$ must be imposed. There are $\frac{1}{2}N(N-1)+\frac{1}{2}N(N-1) = N^2-N$ of them for the numerator, and $\frac{1}{2}(N-1)(N-2)+\frac{1}{2}(N+1)N = N^2-N+1$ of them for the denominator. Thus, for each term in the sum of the denominator there exists an equal term in the sum of the numerator, but the latter has additional terms, since its suffixes are subject to a smaller number of restrictions. Hence our auxiliary result,

$$X > 1 \quad \text{(finite system)}. \qquad (7)$$

Since

$$\bar{n}(N,\lambda) = S(N;1,\lambda)/\{S(N;0,\lambda)+S(N;1,\lambda)\},$$

therefore,

$$\frac{1}{\bar{n}(N,\lambda)} - \frac{1}{\bar{n}(N+1,\lambda)} = 1 + \frac{S(N;0,\lambda)}{S(N;1,\lambda)} - 1 - \frac{S(N+1;0,\lambda)}{S(N+1;1,\lambda)} > 0,$$

so that (4) holds in virtue of (7) also in the case of fermions. The argument now continues as for bosons, and the required proposition is established.

Further properties of quantum statistical systems in the limit L will be established in a subsequent communication.

[4] P. T. Landsberg, Proc. Cambridge Phil. Soc. 50, 65 (1954).
[5] A. R. Fraser, Phil. Mag. 42, 165 (1951).
[6] This notation was devised in conjunction with Dr. F. Ansbacher, University of Aberdeen, Aberdeen, Scotland.