were calculated from the equation[4]

$$T=0.693[t_2(\beta_0)-t_2(\beta)]/\ln[n'(\beta)/\alpha'(\beta)], \quad (5)$$

where T is the half-life of the metastable ion. These were found to be 1.9×10^{-6} and 4.8×10^{-6} sec, respectively.

The fact that $C_2H_5PO_2Cl_2^-$ is the parent ion in four of the transitions suggests that the first and sixth transitions of Table II might be explained by the single transition $C_2H_5PO_2Cl_2^- \rightarrow PO_2Cl^- + Cl^- + C_2H_5^+$. Similarly, the third and fourth transitions could be the single transition $C_2H_5PO_2Cl_2^- \rightarrow Cl_2^- + PO_2^- + C_2H_5^+$. If such a transition takes place, it should be possible to observe the positive ion formed in the transition by adjusting the mass spectrometer for negative ions and then reversing the magnetic field. The metastable ion would be accelerated as a negative ion and then would dissociate, and the positive ion thus formed would be bent in the magnetic analyzer in the proper direction for collection. Such positive ions would appear as a broad peak similar to those observed. A search was made for such a peak using the above technique, but no peak was found between mass 1 and mass 170. This is further evidence that the decay schemes given in Table II are correct.

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Multiple Scattering of Relativistic Electrons*

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The Goudsmit-Saunderson sum for the multiple scattering of fully relativistic electrons is evaluated for the conditions of an experiment of Hanson and his collaborators. The results are consistent with those of Molière and Bethe but lie below the experimental points by as much as fifteen percent at large angles.

A METHOD for summing slowly convergent series of Legendre polynomials has been presented recently.[1] We have applied it to the problem of the directional distribution of relativistic electrons which have penetrated a thin foil (of thickness t) of some material.[2]

Molière[3] has given a theory of this directional distribution which makes use of a small angle approximation and is based on a nonrelativistic cross section. Bethe[4] has surmised that Molière's theory could be made exact and relativistic by simply multiplying Molière's results (at large angles where the angular distribution is essentially single scattering) by the ratio of the exact single-scattering cross section to the cross section which Molière used. Hanson and his collaborators[5] have made excellent measurements of the directional distribution of 15.7-Mev electrons emerging from thin gold foils. They compare their results with a curve which is stated to be Molière's theory extrapolated through the region where his small- and large-angle approximations give different results.[6]

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[1] L. V. Spencer, Phys. Rev. **90**, 146 (1953). The relationship between previous multiple-scattering calculations and the present method is treated in detail in that paper.

[2] What is usually calculated for this problem is the directional distribution after the electron has traveled a given *total path length*. [See H. W. Lewis, Phys. Rev. **78**, 527 (1950).] If the foil is thin enough, the total path length which the electrons travel in penetrating the foil will be very nearly equal to the foil thickness. This point is discussed in more detail in reference 14.

[3] G. Molière, Z. Naturforsch. **3A**, 78 (1948).

[4] H. A. Bethe, Phys. Rev. **89**, 1256 (1953). See footnote on page 1259.

[5] Hanson, Lanzl, Lyman, and Scott, Phys. Rev. **84**, 634 (1951).

[6] In view of the importance of the data of Hanson *et al.*, and, in view of some uncertainty regarding the interpretation of their Fig. 3, it seems worth while to present here some discussion of that figure.

The text of reference 5 states that the "theoretical" curve of Fig. 3 represents the "complete Molière theory." The caption of Fig. 3 states that this curve represents "the theory of Molière extrapolated through the region where his small and large angle approximations give different values." Finally, the "theoretical" curve passes exactly through the experimental values for 0° and 30°.

Bethe also comments (reference 4) that Molière's small-angle tables and large-angle formula do not fit accurately together. He derives a different large-angle formula.

Numerical verifications made by using Molière's small- and large-angle formulas (9, 3a and b) showed that these formulas join quite well (within a percent or so) in an intermediate range of angles. When the Molière distribution thus obtained was normalized to agree with the experimental value of reference 3 at $\theta=0$, it fell about 35 percent below the experimental value at 30°, rather than exactly on it as indicated in Fig. 3 of reference 5.

The contradiction regarding whether or not Molière's small- and large-angle formulas join smoothly apparently arises because Molière gave *two* large-angle formulas—an accurate one [expressions (9, 3a) and (9, 3b)] and an elegant one [expression (Va)]. Hanson and Bethe apparently refer to Molière's elegant but less accurate expression (Va) in their remarks.

It is believed that Hanson *et al.* proceeded as follows: Molière's small-angle curve was first normalized to agree with the experimental value at zero degrees. These results extend to about 11°. Hanson *et al.* then took an *empirical* formula, representing the relativistic single-scattering law, and fitted to the experimental value at 30°. To establish a relativistic theory of multiple scattering at large angles, this empirical formula was multiplied by

We have calculated directly, using a relativistic cross section,[7] the directional distribution of 15.7-Mev electrons which have penetrated the thinner (18.66 mg/cm^2) of Hanson's gold foils. The results agree quite well with values obtained by applying Bethe's suggested "correction factor" to Molière's results at *all* angles. On the other hand, our calculated values are as much as 15 percent below Hanson's experimental values at larger angles. We are unable to explain satisfactorily this large a discrepancy.

I. A FEW DETAILS OF THE CALCULATION

The cross section which we used in this calculation was the McKinley-Feshbach[8] α^2 approximation, modified by the inclusion of an additional term of the form $\frac{1}{4}A(1-\cos\theta)^2$, which we found adequate to give agreement at 30° and 45° with Feshbach's numerical values. (This modification is not accurate beyond 45°, but this difference does not affect the present application.) Electron-electron collisions were accounted for approximately by increasing the cross section by the factor $(Z+1)/Z$.[9] Bethe[4] has shown that the *form* of the screening cutoff is not significant. We introduced this cutoff by writing everywhere $(1-\cos\theta+2\eta)$ for $(1-\cos\theta)$. The value of the screening cutoff was taken in accordance with Molière's prescription.[10] Thus, putting everything together, our cross section was the following:

$$\sigma(\cos\theta)=\frac{Z(Z+1)e^4}{p^2v^2}(1-\cos\theta+2\eta)^{-2}\times\left\{1+\frac{\pi Z\beta}{137}\left(\frac{1-\cos\theta+2\eta}{2}\right)^{\frac{1}{2}}-\left(\beta^2+\frac{\pi Z\beta}{137}\right)\times\left(\frac{1-\cos\theta+2\eta}{2}\right)+A\left(\frac{1-\cos\theta+2\eta}{2}\right)^2\right\}, \quad (1)$$

where $Z=79$, $\beta=0.9990$, $\eta=0.749\times10^{-6}$, and $A=15.3$.

The expansion of this cross section into spherical harmonics is straightforward. As in reference 1, the coefficients $K_l=N\int_{4\pi}d\Omega[1-P_l(\cos\theta)]\sigma(\cos\theta)$ (N being the number of atoms per unit volume) can be written quite accurately as a sum of Bessel functions of the second kind.[11]

In order to obtain the directional distribution $I(\theta)$ of the particles emerging from the foil, the Goudsmit-Saunderson sum for the scattered intensity,

$$I(\theta)=\sum_{l=0}^{\infty}(l+\tfrac{1}{2})\exp(-tK_l)P_l(\cos\theta), \quad (2)$$

must be evaluated. (Here $I/2\pi$ is probability per unit solid angle.) The summation was carried out by the method of reference 1. In this method, $\exp(-tK_l)$ is first approximated by simple, analytic, *continuous* functions of l, whose Fourier transforms are known analytically. By use of these approximations, the Fourier transform of $\exp(-tK_l)$, i.e., $(2\pi)^{-1}\int_{-\infty}^{\infty}d(l+1/2)\times\cos[x(l+1/2)]\exp(-tK_l)$, is evaluated. Finally, a further integration over this Fourier transform is performed to obtain the $I(\theta)$. We estimate that in our calculation the process of summation was carried out to an accuracy of about 2 percent.

TABLE I. Intensity *versus* angle.

Multiple scattering angle (degrees)	Present theory[a]	Molière[a,b]	Molière times σ(exact)/σ(Molière)[c]	Bethe's asymptotic (R) approximation[d]
0	923	939	923	
0.554	871	892	885	
1.107	742	772	772	
1.661	574	599	604	
2.22	407	420	427	
2.77	267	273	279	
3.32	165	169	174	
3.88	100	100	104	
4.43	61.2	59.6	62.4	
4.99	36.9	36.0	38.0	
5.54	22.8	22.2	23.6	
6.09	14.3	13.9	14.9	28.9
6.65	9.30	8.88	9.57	
7.20	6.33	5.79	6.29	7.75
7.76	4.40	3.91	4.28	
8.31	3.15	2.75	3.02	3.35
8.86	2.28	2.01	2.22	
9.42	1.69	1.51	1.68	1.76
9.97	1.29	1.16	1.30	
10.53	1.05	0.902	1.02	1.02
11.08	0.800	0.713	0.810	
12.47	0.475	0.417	0.482	
13.85	0.303	0.261	0.306	
15.24	0.203	0.172	0.204	
16.62	0.143	0.118	0.143	
19.39	0.0762	0.0615	0.0758	
22.16	0.0454	0.0352	0.0449	
24.93	0.0287	0.0216	0.0282	
27.70	0.0189	0.0140	0.0189	

[a] The integral is normalized to unity when the angle is expressed in radians.
[b] Nonrelativistic, small-angle.
[c] Renormalized by a factor which makes the $\theta=0$ value agree with column 2.
[d] Renormalized by the same factor as that used to renormalize column 4.

Molière's elegant expression (Va). The resulting "theoretical" curve passed through the experimental value at 30°, since the experimental value had been normalized in a similar manner. Since the large-angle and small-angle theoretical curves did not join smoothly at 11°, a smooth junction was constructed.

Thanks are due Dr. Hanson for checking the accuracy of these statements.

[7] H. Feshbach, Phys. Rev. **88**, 295 (1952).

[8] Quoted in reference 7.

[9] The electron-electron collisions should yield corrections to our results of the order of $1/Z\approx1$ percent in Au. More accurate estimates based on a calculation from U. Fano [Phys. Rev. **93**, 117 (1954)] show that the corrections are about 2 percent.

[10] G. Molière, Z. Naturforsch. **2A**, 133 (1947). Notice that $\eta=\chi_\alpha^2/4$.

[11] We found these functions easy to work with because (1) they are tabulated and (2) they have extremely simple recursion relationships. It would no doubt have been possible to represent the K_l as powers times logarithms, as Molière and Bethe did.

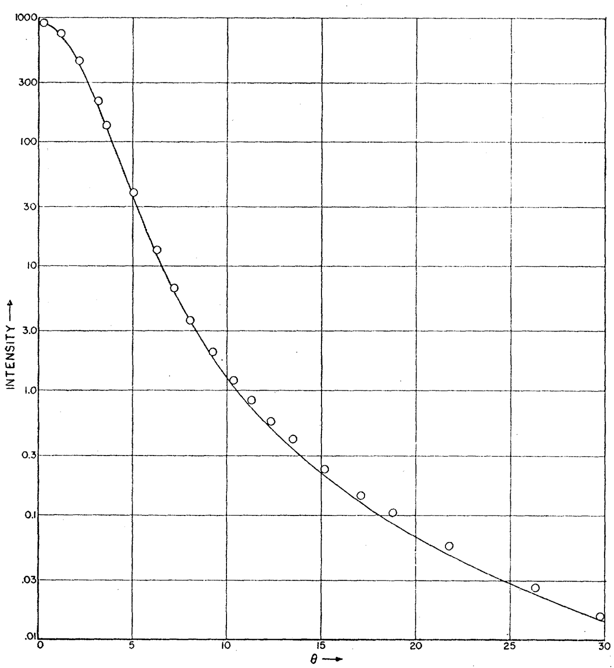

FIG. 1. Comparison between theoretical (solid line) and experimental angular distributions of multiply scattered electrons. The experimental points were normalized to agree with the theory at $\theta=0$.

II. DISCUSSION

The results of this calculation are compared with other theoretical results in Table I and with Hanson's experimental values in Fig. 1.

In column 2 of Table I our values are given as a function of the angle of deflection. Column 3 gives the Molière theory (small-angle approximation, nonrelativistic) results for the same situation. (Both our results and Molière's have been normalized so that the integral over all angles yields unity.) In column 4 are values obtained by multiplying Molière's values of column 3 by the ratio of the exact cross section [Eq. (1)] to Molière's cross section, and then normalizing to obtain agreement with our value at zero angle. Finally, in column 5 are a few values calculated by using Bethe's asymptotic representation of Molière's results, renormalized by the same factor as column 4.

There is good agreement between columns 2 and 4 of Table I, showing the validity of Bethe's surmise. The largest discrepancy, which is about 4 percent, occurs at angles small enough so that the distribution is predominantly determined by highly multiple scattering, but large enough so that the ratio $[\sigma(\text{exact})/\sigma(\text{Molière})]$ is appreciable. (For thicker foils this discrepancy would be greater.)

On the other hand, the gap between the results in columns 5 and 3 at angles where columns 3 and 2 are not in close agreement indicates that it is not sufficient to modify with a relativistic correction factor *only* the results obtained from an asymptotic formula such as Bethe's R.

The experimental values plotted in Fig. 1 were very kindly sent to us by Dr. Hanson. They consisted of a set of measurements from 0° to 6° made with a very small detector aperture and another set of measurements from 5° to 30° made with a much larger aperture. Dr. Hanson has indicated that, because of background, the small-aperture point at 6° and the large-aperture points beyond 20° are in some doubt. We applied an aperture correction to the large-aperture values[12] and joined them with the points at small angles, neglecting the point at 6°, which was out of line.

The discrepancy of about 15 percent at large angles has been rather puzzling to us. It persists at angles smaller than 20°, where the Hanson data should be unaffected by background. Further, most of the effects neglected in the calculation (and thoroughly discussed by Lyman, Hanson, and Scott)[13] are in the direction of increasing the discrepancy. Two effects not discussed by Lyman, Hanson, and Scott are the increased path length which accompanies a large deflection, and the error made by writing (Z^2+Z) instead of treating the electron-electron collisions exactly. Both of these effects can be shown to modify the distribution by no more than one or two percent.[9,14]

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[12] This correction, which is the ratio of the integral $\int_{\text{aperture}} d\theta I(\theta)$ to the product of the median value and the acceptance angle of the aperture, amounted to about 4 percent at 6°.

[13] Lyman, Hanson, and Scott, Phys. Rev. 84, 626 (1951).

[14] The main part of the electron distribution lies at such small angles ($\sim 3°$) that the difference between foil thickness and path length is negligible. At large angles the distribution is determined by (1) a single large deflection which takes place while the electron has still not been deflected appreciably from its original direction and (2) a multiple-scattering "smear" which is essentially Gaussian since the probability of a second large collision is prohibitively small. The increased path length caused by the large deflection broadens somewhat the superposed small-angle multiple-scattering distribution. Estimates of the size of this effect were obtained by folding Gaussians over the Rutherford angular distribution.