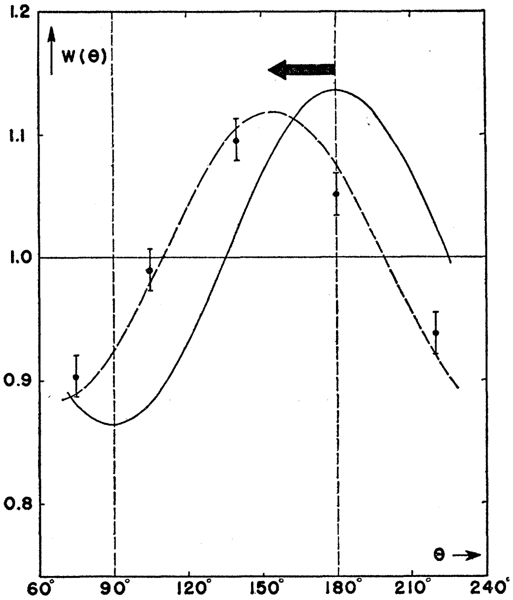

FIG. 2. Angular shift of the directional correlation function $W(\theta)$ of the Pb^{204} $\gamma-\gamma$ cascade in an external magnetic field H. The solid line represents the zero-field correlation; the dotted line is the theoretical curve for $g=+0.07$ and $H=4300$ oersteds. The arrow indicates the classical precession angle of a magnetic dipole.

over shown by simultaneous irradiation of natural Tl and of Tl enriched in Tl^{203} with deuterons, that the 65-min activity is produced from Tl^{203} and therefore cannot be Pb^{205}.

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[1] K. Alder, Helv. Phys. Acta 25, 235 (1952).
[2] L. C. Biedenharn and M. E. Rose, Revs. Modern Phys. 25, 729 (1953).
[3] Aeppli, Albers-Schönberg, Frauenfelder, and Scherrer, Helv. Phys. Acta 25, 339 (1953).
[4] Sunyar, Alburger, Friedlander, Goldhaber, and Scharff-Goldhaber, Phys. Rev. 79, 181 (1950).
[5] Frauenfelder, Lawson, Jentschke, and DePasquali, Phys. Rev. 92, 1241 (1953).
[6] Frauenfelder, Lawson, Jentschke, and DePasquali, Phys. Rev. 92, 513 (1953).
[7] A. Abragam and R. V. Pound, Phys. Rev. 92, 943 (1953).
[8] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 27, No. 16, 117 (1953).
[9] K. W. Ford, Phys. Rev. 90, 29 (1953).
[10] P. Stähelin and P. Preiswerk, Nuovo cimento 10, 1219 (1953).
[11] A. De-Shalit and M. Goldhaber, Phys. Rev. 92, 1211 (1953).

Scattering of High-Energy Electrons by Heavy Nuclei*

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THE elastic scattering of high-energy electrons is a tool which can be used to obtain information on the radius and charge distribution of nuclei. There have been several numerical calculations of this process.[1,2] Although the numerical method will undoubtedly remain the most accurate, it may be thought that an analytical solution, even if approximate, would help in the understanding of the physical happenings and would show more clearly the dependence on the various parameters involved. The Born approximation is such an analytical method; however, it cannot be trusted for such heavy nuclei as gold or lead, because $Ze^2/\hbar c$ is too large.

The purpose of this letter is to report on some results obtained with the WKB method. This method is applicable if the potential varies slowly over distances of the order of the electron wavelength. This is the case if $kR \gg 1$, where R is the radius of the nucleus and $k=\lambdabar^{-1}$ is the electron wave number. At the energies at which the experiments of Hofstadter, Fechter, and McIntyre[3] have been done ($\sim$125 Mev), kR is about 5 for heavy nuclei. This is about the lower limit of the energy region where the WKB method can be considered valid.

We start from the Dirac equation and neglect the mass of the electron. In that case the phase shift η depends only on the angular momentum j, and not on the parity.[4] We apply the WKB method in the form given by Bessey and Uhlenbeck,[5] with the result

$$\eta_j=\lim_{r\to\infty}\left\{\int_{r_0}^{r} Q(r')dr'-kr-\alpha\ln 2kr+l\frac{\pi}{2}\right\}, \tag{1}$$

where $Q(r)=[(k-V(r)/\hbar c)^2-l^2/r^2]^{\frac{1}{2}}$; $V(r)$ is the potential energy; r_0 is the turning point, i.e., $Q(r_0)=0$; $\alpha=Ze^2/\hbar c$; and l is *defined* as $j+\frac{1}{2}$.

For instance, with a uniform charge distribution, of external radius R, the phase shift is

$$\begin{aligned}\eta_j=&\alpha(1-x^2)^{\frac{1}{2}}+\tfrac{1}{3}\alpha(1-x^2)^{\frac{3}{2}}-\alpha\ln[1+(1-x^2)^{\frac{1}{2}}]-\alpha\ln kR\\&+(\alpha^2/2kR)[x^{-1}\sin^{-1}x-(1-x^2)^{\frac{1}{2}}+\tfrac{2}{3}(1-x^2)^{\frac{1}{2}}x^2(4-x^2)],\\&\qquad\text{for } l<kR; \qquad (2)\\ \eta_j=&\eta_j{}^c,\quad \text{for } l>kR,\end{aligned}$$

where $x=1/kR$, and $\eta_j{}^c$ is the Coulomb phase shift. In going from (1) to (2), we kept only the first two terms of the expansion in powers of the small parameter α/kR.

The cross section is conveniently written in the following form:

$$d\sigma/d\Omega=\sec^2\tfrac{1}{2}\theta\,|f(\theta)|^2,$$
$$f(\theta)=f^c(\theta)+(2ik)^{-1}\textstyle\sum_j l[\exp(2i\eta_j)-\exp(2i\eta_j{}^c)]\times[P_l(\cos\theta)+P_{l-1}(\cos\theta)], \tag{4}$$

where $f^c(\theta)$ is the Coulomb scattering amplitude which has been calculated, at high energies, by Feshbach,[6] and more recently by Yennie *et al.*[2]

The number of terms in the summation in (4) is of order kR. Doing this sum exactly, but using the WKB phase shifts, we obtain the results shown in Figs. 1, 2, 3. It should be noted that the cross section depends only on the combination kR, except for a factor $1/k^2$. In Fig. 1, it is seen that the result compares favorably with the numerical one of Yennie *et al.* There is no agreement between this calculation and the Born approximation. The maxima

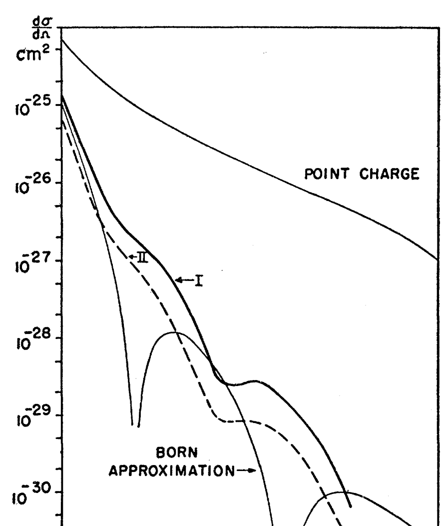

FIG. 1. The differential scattering cross section for a uniform charge distribution. Curve I is the result of the WKB method for $Z=80$, with $kR=5$, $R=1.4A^{\frac{1}{3}}\times10^{-13}$ cm. Curve II is the result obtained by Yennie *et al.* for $Z=79$, $kR=5.4$, $R=1.22A^{\frac{1}{3}}\times10^{-13}$ cm.

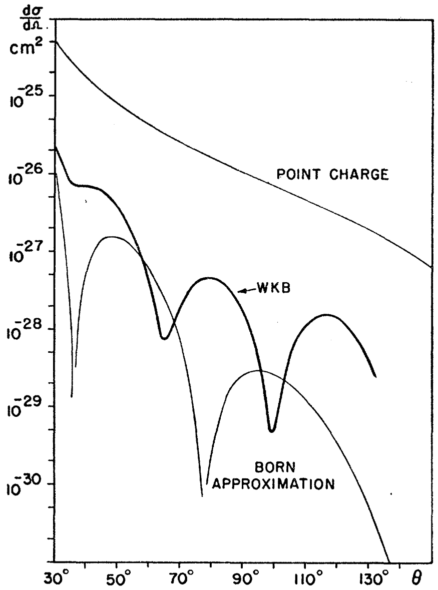

FIG. 2. The differential scattering cross section for a shell distribution with $Z=80$, $kR=5$, and $R=(3/5)^{\frac{1}{2}}\times 1.4A^{\frac{1}{3}}\times 10^{-13}$ cm.

and minima of the Born approximation are greatly smoothed out, especially the first one, and shifted to smaller angles.

However, for reasons given at the beginning of this letter, it would be more interesting to do the summation by an analytical method. We are at the moment working on such a method, involving the replacement of the sum over j by an integral and $P_l(\cos\theta)+P_{l-1}(\cos\theta)$ by $2J_0(2l\sin\frac{1}{2}\theta)$. It then becomes evident that most of the $f^c(\theta)$ part cancels against the Σ part, leaving a result much smaller than the Coulomb amplitude. The method also shows that the Born approximation is only one of several terms which must be taken into account, and not always the largest. By including a sufficient number of corrections to our integral, it is possible to reproduce exactly the curves of Figs. 1–3. However, greater accuracy in this procedure is obtained only at

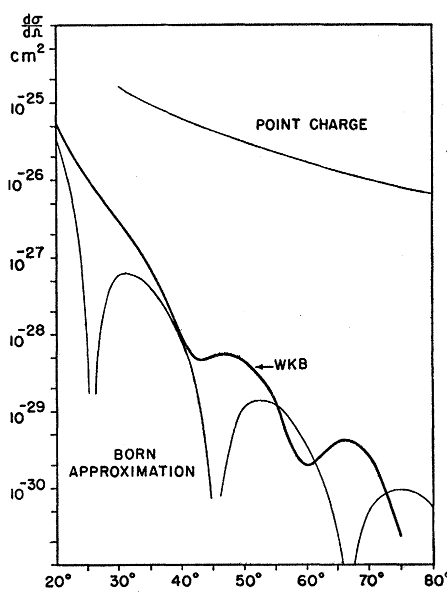

FIG. 3. The differential scattering cross section for a uniform distribution with $Z=80$, $kR=10$, $R=1.4A^{\frac{1}{3}}\times 10^{-13}$ cm.

the expense of analytical simplicity. The details of this work will be published later.

This problem was suggested by Professor H. A. Bethe and the author is greatly indebted to him for continued guidance throughout this work.

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[1] Louis K. Acheson, Jr., Phys. Rev. **82**, 488 (1951); L. R. B. Elton, Proc. Phys. Soc. (London) **A63**, 1115 (1950); G. Parzen, Phys. Rev. **80**, 355 (1950). The values of the phase shifts given in the latter paper disagree with the results of the WKB method, discussed below. The author has recalculated one of these phase shifts, using Parzen's method, with a result that agrees with the WKB value, and not with Parzen's result. Parzen has since found a numerical error in his work (private communication).

[2] Yennie, Wilson, and Ravenhall, Phys. Rev. **92**, 1325 (1953). We would like to thank Dr. Yennie for sending us his results for the Coulomb scattering amplitude.

[3] Hofstadter, Fechter, and McIntyre, Phys. Rev. **91**, 422 (1953).

[4] Herman Feshbach, Phys. Rev. **84**, 1206 (1951).

[5] R. J. Bessey, Ph.D. thesis, University of Michigan, 1942. For a brief report of this work, see R. H. Good, Phys. Rev. **90**, 131 (1953).

[6] Herman Feshbach, Phys. Rev. **88**, 295 (1952).

The Sign of the Phase Shift in the Elastic Scattering of Electrons*

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IN the phase-shift analysis of the scattering of electrons from a nuclear charge distribution, the total phase shift can be expressed as the sum of two terms:

$$\eta_l=\eta_l^c+\delta_l. \tag{1}$$

η_l^c is the phase shift of the lth partial wave for a pure Coulomb field and δ_l is the additional phase shift due to the modification of the Coulomb field inside the nucleus. It is the purpose of this note to point out an error in the literature,[1] in which it is asserted that for large l, δ_l approaches zero through positive values. It will be shown for all reasonable charge distributions inside the nucleus that $\delta_l<0$. For simplicity we neglect the rest energy of the electron compared to the total energy; with a slight modification the proof can be carried through in the more general case. In dimensionless form, the equations giving the radial wave functions are

$$\begin{aligned}(d/dx)F_{l,v}+[(l+1)/x]F_{l,v}-(1-v)G_{l,v}&=0,\\(d/dx)G_{l,v}-[(l+1)/x]G_{l,v}+(1-v)F_{l,v}&=0,\end{aligned} \tag{2}$$

with

$$x=Er/\hbar c,\quad v=V/E.$$

For large x, $G_{l,v}$ has the asymptotic form

$$G_{l,v}\sim\sin(x+\gamma\ln 2x-\tfrac{1}{2}l\pi-\eta_l^c+\delta_{l,v}). \tag{3}$$

According to Elton,[2] the difference in the phase shifts due to two different potentials v and v' is given by

$$\sin(\delta_{l,v'}-\delta_{l,v})=-\int_0^\infty(v'-v)(F_{l,v'}F_{l,v}+G_{l,v'}G_{l,v})dx. \tag{4}$$

Our proof is based on the fact that the potential can be varied continuously from the pure Coulomb potential to the final value in such a way that for each small change in v the corresponding change in δ_l is negative. For example, we may define a one-parameter family of potentials:

$$v_\epsilon(x)=\epsilon^{-1}v(x/\epsilon),\quad 0\leqslant\epsilon\leqslant 1; \tag{5}$$

(v_ϵ is the same function of x as v is for an energy ϵE). As ϵ approaches zero, v_ϵ approaches the pure Coulomb potential; at the other limit v_ϵ approaches the potential given by the charge distribution. Explicitly:

$$v_\epsilon=-(\gamma/x)\int_0^{x/\epsilon}\rho x^2dx-(\gamma/\epsilon)\int_{x/\epsilon}^\infty\rho x dx, \tag{6}$$

with

$$\gamma=Ze^2/\hbar c,\quad \int_0^\infty\rho x^2dx=1. \tag{7}$$