

Bound State Corrections in Two-Body Systems*

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Available expressions for two-body equations contain an interaction kernel which treats particles in intermediate states as free. In situations where the binding is important, such as the calculation of low-energy electrodynamic corrections, a more accurate treatment is necessary. A satisfactory formalism is developed for systems in which an instantaneous interaction is responsible for the binding. The procedure may then be used to evaluate the effects of small retarded perturbations. It consists of summing those binding interactions which occur during the retarded perturbations and which never should have been expanded as "small" effects. The result is expressed in terms of the two-body Green's function of the instantaneously interacting system. This function occurs to describe the propagation of the two particles in the intermediate state. The relative time coordinate does not appear explicitly in the formulas. The method is applied to the calculation of the hyperfine structure of positronium. The infrared divergences which occurred in a previous investigation of this effect are eliminated by the new approach.

I. INTRODUCTION

SEVERAL recent treatments¹⁻⁴ of the two-body system have used the techniques developed by Schwinger⁵ to derive rigorous functional differential equations for the Green's functions of various two-body systems. These equations include an implicitly defined interaction operator. By various techniques of iteration, an expansion of the operator in powers of the coupling constant is developed to the order desired. The equations that have been obtained in this way are called Bethe-Salpeter^{6,7} equations. They are deficient in that all intermediate states that occur during the interaction are expanded about free-particle states by the Born approximation. This fact certainly requires serious examination when the method is applied to strongly interacting bound nucleon systems. The weak electrodynamic coupling, on the other hand, should lead to no difficulties were it not for the fact that photons of arbitrarily long wavelength can occur in radiative corrections. In bound atomic states, therefore, the shape of the wave function will enter significantly into the results of a correct calculation. Methods that neglect this natural cutoff at the atomic radius lead to the familiar infrared catastrophes.

It is the purpose of this paper to correct the equation that describes the bound state of two oppositely charged fermions by taking into account the binding in intermediate states. Because of the weakness of the electric forces, the bound system is essentially non-relativistic. For the calculation of energy displacements of the order of magnitude $\alpha^3 m$ it is therefore sufficient to provide for the binding in those intermediate states in which the relative momentum of the two particles and the virtual energy are small. The procedure begins with an examination of the terms in the interaction operator that comprise the Born expansion of the Coulomb interaction. It is observed that these constitute a formal expansion of the two-particle Coulomb Green's function. The free-particle Green's functions that occur in certain low-order interactions are then replaced by the correct Green's function with the proviso that the terms including the Coulomb energy explicitly are to be omitted from higher-order interactions. The features of electrodynamic systems mentioned earlier simplify this task greatly. It is, for instance, unnecessary to make the correction in states that contain virtual pairs. Since the manifest covariance is lost as soon as the instantaneous Coulomb interaction is separated and given special treatment, there is no particular disadvantage in making further noncovariant distinctions.⁸

In an application, it is of course impossible to calculate with the complete Coulomb Green's function. A separation into high- and low-energy phenomena as measured by the frequency of the virtual quanta leads to the further simplification that the Born approximation to the Green's function is valid for the high-energy effects whereas a nonrelativistic treatment is adequate in the low-energy region. This separation is identical to the one made in the early calculations of the level shift

* A paper that discusses these same ideas, by R. J. Eden, Proc. Roy. Soc. (London) **219**, 516 (1953), appeared as this manuscript was being submitted for publication. It is hoped that the computational details described here will extend the usefulness of the formalism that is developed.

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‡ On leave of absence from Harvard University for the fall semester, 1953-1954.

¹ R. Karplus and A. Klein, Phys. Rev. **87**, 848 (1952), hereafter referred to as KK III. The notation of this paper and of reference 5 is used.

² R. Arnowitt, Phys. Rev. **92**, 1002 (1953).

³ S. Deser and P. C. Martin, Phys. Rev. **90**, 1075 (1953).

⁴ A. Klein, Phys. Rev. **90**, 1101 (1953).

⁵ J. Schwinger, Proc. Nat. Acad. Sci. U.S. **37**, 452, 455 (1951).

⁶ E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951); M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).

⁷ E. E. Salpeter, Phys. Rev. **87**, 328 (1952); G. E. Brown, Proc. Roy. Soc. (London) **A215**, 371 (1952).

⁸ Julian Schwinger and Paul C. Martin have privately described to us a completely covariant treatment of binding in intermediate states.

in an external Coulomb field.⁹ The method of obtaining high-energy corrections is illustrated by an examination and elimination of the infrared catastrophes that were encountered in the corrections to the fine structure of positronium.

II. PERTURBATION THEORY

A perturbation formalism for obtaining the effects of a time-dependent interaction on the energy of a system bound by an instantaneous interaction has been developed by Salpeter.⁷ After some introductory definitions, his results are adapted to the present requirements.

The Green's function for two nonidentical fermions of equal mass with no external field present satisfies the equation¹

$$[F_1(11')F_2(22') - I(121'2')]G(1'2'34) = \delta(13)\delta(24), \quad (2.1)$$

where

$$F_{1,2} = \gamma^{1,2} p_{1,2} + m = -i\gamma^{1,2} \partial_{1,2} + m = [G_0^{1,2}]^{-1}.$$

I is the interaction operator. The two-particle wave function is a solution of the corresponding homogeneous equation. After renormalization of the mass and the coupling constant, the equation for the wave function becomes¹

$$[\bar{F}_1(11')\bar{F}_2(22') - \bar{I}(121'2')]\psi(1'2') = 0, \quad (2.2)$$

or, in relative coordinates,

$$[F_K(x,x') - I_K(x,x')]\varphi_K(x') = 0, \quad (2.3)$$

where

$$F_K(x,x') = \int d^4X' e^{-iK(X-X')} \bar{F}_1(11')\bar{F}_2(22'), \quad (2.4)$$

and I_K arises similarly from \bar{I} . The wave function is an eigenfunction of total energy-momentum K_μ and is, for renormalized charge and coupling,

$$\psi(12) = e^{iKX} \varphi_K(x) \quad [X = \frac{1}{2}(x_1 + x_2), x = x_1 - x_2]. \quad (2.5)$$

The experimental charge and coupling constant appear in all terms of Eqs. (2.3), (2.4), and (2.5).

We next consider such I , or such energy values of our system, for which we can write:

$$\left. \begin{aligned} \bar{I}(1234) &= \bar{I}^c(1234) + \bar{I}'(1234) \\ I_K(x,x') &= I^c(x,x') + I_{K'}(x,x') \end{aligned} \right\}, \quad (2.6)$$

where I^c is one or a number of irreducible interactions which represents the main part of the total interaction (for example, the Coulomb potential in the case of the electron-positron system) and $I_{K'}$ represents small corrections to this. We restrict ourselves to the con-

⁹ H. A. Bethe, Phys. Rev. **72**, 339 (1947); N. M. Kroll and W. E. Lamb, Phys. Rev. **75**, 388 (1949); J. B. French and V. F. Weisskopf, Phys. Rev. **75**, 1240 (1949); R. P. Feynman, Phys. Rev. **74**, 1430 (1948) and **76**, 709 (1949); J. Schwinger, Phys. Rev. **76**, 790 (1949).

sideration of I^c 's local in relative time and in the space time of each particle:

$$I^c(x,x') = -\delta(x-x')\delta(x_0)\delta(\mathbf{r})\gamma_0^1\gamma_0^2. \quad (2.7)$$

Next we rewrite Eq. (2.3):

$$[F_K(x,x') - I^c(x,x')]\varphi_K(x') = I_{K'}(x,x')\varphi_K(x'). \quad (2.3')$$

The unperturbed system is described by the total energy momentum K_μ^c and the wave function $\varphi^c(x)$ in relative coordinates. It satisfies the equation

$$[F_{K^c}(x,x') - I^c(x,x')]\varphi^c(x') = 0, \quad (2.8)$$

whereas the corresponding bound-state Green's function satisfies

$$[\bar{F}_1(11')\bar{F}_2(22') - \bar{I}^c(121'2')]G^c(1'2'34) = \delta(13)\delta(24). \quad (2.9)$$

By following the steps leading to Salpeter's Eq. (13),⁷ one may rearrange both Eqs. (2.1) and (2.8) in such a way that the instantaneous nature of the binding interaction can be explicitly used to simplify the problem in a coordinate system in which the total momentum is zero, $K_\mu = (\mathbf{0}, K_0)$, $K^c = (\mathbf{0}, K_0^c)$. In this manner one obtains the equations involving the Dirac Hamiltonian $H(\mathbf{p})$:

$$\begin{aligned} & [K_0 - H^1(\mathbf{p}) - H^2(-\mathbf{p})]\varphi_K(\mathbf{r}, 0) \\ & + \int \Lambda(\mathbf{r}\mathbf{r}') f(\mathbf{r}') \varphi_K(\mathbf{r}', 0) d\mathbf{r}' \\ & + \int d^4x' d^4x'' \Lambda_K(\mathbf{r}\mathbf{0}, x') \gamma_0^1 \gamma_0^2 I_{K'}(x'x'') \varphi_K(x'') = 0, \end{aligned} \quad (2.10)$$

and

$$\begin{aligned} & [K_0^c - H^1(\mathbf{p}) - H^2(-\mathbf{p})]\varphi^c(\mathbf{r}, 0) \\ & + \int \Lambda(\mathbf{r}\mathbf{r}') f(\mathbf{r}') \varphi^c(\mathbf{r}', 0) d\mathbf{r}' = 0, \end{aligned} \quad (2.11)$$

where

$$\begin{aligned} \Lambda_K(x,x') &= \int d^4X' e^{-iK(X-X')} [G_0^1(11')\gamma_0^1\delta(22') \\ & \quad + G_0^2(22')\gamma_0^2\delta(11')] \\ &= i(2\pi)^{-3} \int d\mathbf{k} e^{i\mathbf{k} \cdot (\mathbf{r}-\mathbf{r}')} \{ e^{-i(E-\frac{1}{2}K_0)|t-t'|} \\ & \quad \times [\Lambda_+^1(\mathbf{k})\theta(t-t') + \Lambda_+^2(-\mathbf{k})\theta(t'-t)] \\ & \quad - e^{-i(E+\frac{1}{2}K_0)|t-t'|} [\Lambda_-^1(\mathbf{k})\theta(t-t') \\ & \quad \quad + \Lambda_-^2(-\mathbf{k})\theta(t-t')] \}, \end{aligned} \quad (2.12)$$

$$[\theta(t) = 1, t > 0; = \frac{1}{2}, t = 0; = 0, t < 0],$$

and

$$\begin{aligned} \Lambda(\mathbf{r}\mathbf{r}') &= -i\Lambda_K(\mathbf{r}\mathbf{t}, \mathbf{r}'\mathbf{t}') = (2\pi)^{-3} \int d\mathbf{k} e^{i\mathbf{k} \cdot (\mathbf{r}-\mathbf{r}')} \\ & \quad \times [\Lambda_+^1(\mathbf{k})\Lambda_+^2(-\mathbf{k}) - \Lambda_-^1(\mathbf{k})\Lambda_-^2(-\mathbf{k})]. \end{aligned} \quad (2.13)$$

In Eq. (2.12) the numbered variables must be expressed in terms of the X and x according to Eq. (2.5). The Λ_{\pm} are the usual projection operators

$$\Lambda_{\pm}^i(\mathbf{k}) = [E(k) \pm H^i(\mathbf{k})] / 2E(k). \quad (2.14)$$

In terms of the function $\chi^C(\mathbf{r})$ which is the solution of the Salpeter's⁷ equation,

$$[K_0^C - H^1(\mathbf{p}) - H^2(-\mathbf{p})]\chi^C(\mathbf{r}) + f(\mathbf{r})\varphi^C(\mathbf{r}, 0) = 0, \quad (2.15)$$

the functions $\varphi^C(x)$ and $\varphi^C(\mathbf{r}, 0)$ are expressible as follows:

$$\begin{aligned} \varphi^C(x) &= -i \int \Lambda_K(x, \mathbf{r}'0) \chi^C(\mathbf{r}') d\mathbf{r}'; \\ \varphi^C(\mathbf{r}, 0) &= \int \Lambda(\mathbf{r}\mathbf{r}') \chi^C(\mathbf{r}') d\mathbf{r}'. \end{aligned} \quad (2.16)$$

The normalization is taken to be

$$\int \chi^{C*}(\mathbf{r}) \varphi^C(\mathbf{r}, 0) d\mathbf{r} = 1. \quad (2.16')$$

Reference to Eq. (2.12) shows that the relative time behavior of the two-particle wave function is described by free-particle Green's functions. It implies that the one particle propagates to the correct relative time in a positive energy state if its time is later or in a negative energy state if its time is earlier. One is interested in the wave function at unequal times because the two particles may experience interactions at different times. The difficulty in the use of Eq. (2.16) occurs because this relationship ignores the existence of a binding interaction of the particle which propagates to the later time with the other particle that has made a transition out of the initial state by undergoing an interaction at the earlier time. Of course, these effects are not omitted completely; they occur, expanded in a Born approximation series, in the interaction operator. The rearrangement to be made and illustrated later will overcome this difficulty.

Salpeter⁷ now defines a hybrid wave function

$$\begin{aligned} \varphi^{C'}(x) &= -i \int \Lambda_K(x, \mathbf{r}'0) \chi^C(\mathbf{r}') d\mathbf{r}', \\ \varphi^{C'}(\mathbf{r}, 0) &= \varphi^C(\mathbf{r}, 0), \end{aligned} \quad (2.17)$$

and its adjoint whose time dependence is determined by K , but whose space dependence is determined by K^C . This function has the property that

$$\begin{aligned} \gamma_0^1 \gamma_0^2 [F_K(x, x') - I^C(x, x')] \varphi^{C'}(x') \\ = i(K_0 - K_0^C) \delta(x_0) \chi^C(\mathbf{r}). \end{aligned} \quad (2.18)$$

When Eq. (2.3) is multiplied by $\bar{\varphi}^{C'} = (\varphi^{C'})^* \gamma_0^1 \gamma_0^2$, we

can use the conjugate of Eq. (2.18) to show that

$$\begin{aligned} K_0 - K_0^C = \Delta E = -i \int \bar{\varphi}^{C'}(x) I_{K'}(x, x') \varphi_K(x') d^4x d^4x' \\ \times \int \chi^C(\mathbf{r}) \varphi_K(\mathbf{r}0) d\mathbf{r}. \end{aligned} \quad (2.19)$$

This expression merely serves as the beginning of an iteration procedure because it depends on the wave function φ_K . The small difference between φ_K and $\varphi^{C'}$ can be neglected in lowest order; for greater accuracy φ_K may be calculated from ΔE by solving the equation for $\varphi_K(x) - \varphi^{C'}(x)$ that can be obtained from Eqs. (2.3) and (2.18):

$$\begin{aligned} \varphi_K(x) = \varphi^{C'}(x) + \int G_K(x, x') [\gamma_0^1 \gamma_0^2 I_{K'}(x', x'') \varphi^{C'}(x'') dx'' \\ - i \Delta E \chi^C(\mathbf{r}') \delta(x_0')] d^4x'. \end{aligned} \quad (2.20)$$

Here $G_K(x, x')$ is the Green's function in relative coordinates

$$G_K(x, x') = \int d^4X' e^{-iK(x-X')} G(121'2') \gamma_0^1 \gamma_0^2. \quad (2.21)$$

This Green's function should be understood as a power series in $I_{K'}$ with the Coulomb Green's function as its leading term.

Because the relative time dependence of all the wave functions on the RHS of Eq. (2.20) is known, this expression may be inserted into Eq. (2.10) to yield an equation which has no explicit relative time dependence

$$\begin{aligned} [K_0 - H^1(\mathbf{p}) - H^2(-\mathbf{p})] \varphi_K(\mathbf{r}0) \\ + \int \Lambda(\mathbf{r}\mathbf{r}') f(\mathbf{r}') \varphi_K(\mathbf{r}', 0) d\mathbf{r}' \\ + i \int \mathcal{K}_K(\mathbf{r}\mathbf{r}') \chi^C(\mathbf{r}') d\mathbf{r}' = 0. \end{aligned} \quad (2.22)$$

The kernel \mathcal{K}_K is given by

$$\begin{aligned} \mathcal{K}_K(\mathbf{r}\mathbf{r}') = \\ - \int \Lambda_K(\mathbf{r}0, x^\alpha) \gamma_0^1 \gamma_0^2 I_{K'}(x^\alpha x^\beta) \Lambda_K(x^\beta, \mathbf{r}'0) d^4x^\alpha d^4x^\beta \\ - \int \Lambda_K(\mathbf{r}0, x^\alpha) \gamma_0^1 \gamma_0^2 I_{K'}(x^\alpha x^\beta) G_K(x^\beta x^\gamma) \\ \times I_{K'}(x^\gamma x^\delta) \Lambda_K(x^\delta, \mathbf{r}'0) d^4x^\alpha d^4x^\beta d^4x^\gamma d^4x^\delta \\ + i \Delta E \int \Lambda_K(\mathbf{r}0, x^\alpha) \gamma_0^1 \gamma_0^2 I_{K'}(x^\alpha x^\beta) \\ \times G_K(x^\beta, \mathbf{r}'0) d^4x^\alpha d^4x^\beta. \end{aligned} \quad (2.23)$$

One can look upon Eq. (2.22) as describing the behavior of the compound particle in the nonlocal potential Δf with radiative corrections. The equation is still exact.

An explicit expression for ΔE may be obtained by substituting φ_K , Eq. (2.20), into Eq. (2.19) and solving for ΔE . In electrodynamic problems the energy change and normalization change are usually so small that they can be neglected on the right-hand side. This also implies that $\varphi^{C'} \sim \varphi^C$. We therefore obtain

$$\Delta E = -i \int \bar{\varphi}^C(x) \left[I'_{K^C}(xx') + \int d^4x^\alpha d^4x^\beta I'_{K^C}(xx^\alpha) \right. \\ \left. \times G_{K^C}(x^\alpha x^\beta) \gamma_0^1 \gamma_0^2 I'_{K^C}(x^\beta x') \right] \varphi^C(x') d^4x d^4x', \quad (2.24)$$

$$\Delta E = -i \int \chi^{C*}(\mathbf{r}) \mathcal{K}_{K^C}(\mathbf{r}\mathbf{r}') \chi^C(\mathbf{r}') d\mathbf{r} d\mathbf{r}', \quad (2.25)$$

which is the basis for Eq. (3.6) in KK III.

III. PRESCRIPTION FOR BOUND STATE CORRECTIONS

We turn next to the rearrangement that must be carried out in the interaction $I'(1234)$ which we consider given as a power series expansion in the coupling constant. Let us consider a term T which arises from a particular virtual process P . P is restricted in that it must include no interaction I^C between the two particles (it may contain this interaction as a self-interaction of one of the particles); P is irreducible⁶ because it occurs in $I_{K'}$. We shall further distinguish with a subscript on P and T among the different possible orderings in time of the elementary interactions of which P is composed. For an n th order interaction $P_\alpha(n)$ there will then be $n-1$ time intervals or, in the language of conventional perturbation theory, $n-1$ intermediate states. In each of these the particles are described by noninteracting Green's functions. Now consider in addition the collection P_α^C of all those processes that differ from P_α only in that interactions I^C take place during time intervals in which no more than two par-

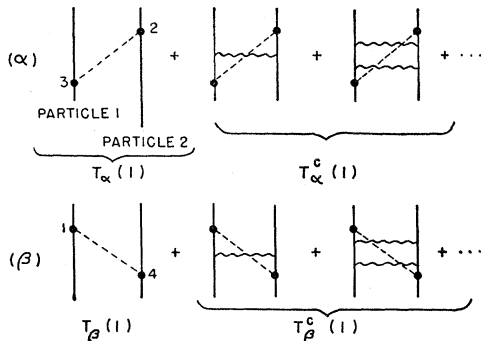


FIG. 1. Coulomb corrections to the exchange of a transverse quantum. Solid lines refer to electrons, broken lines to quanta, and wavy lines to the instantaneous Coulomb interaction.

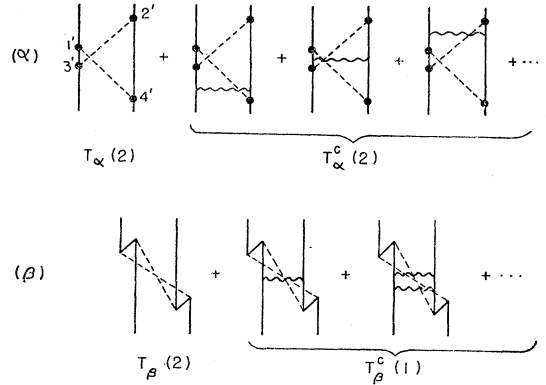


FIG. 2. Coulomb corrections to the exchange of two transverse quanta; pairs are present in some intermediate states.

ticles are present. As described in the introduction, the sum,

$$\tau_\alpha = T_\alpha + T_\alpha^C, \quad (3.1)$$

of the corresponding terms then represents the contribution to the interaction operator of a P_α whose two-particle intermediate states are states of the unperturbed system governed by Eq. (2.8). Of course Eq. (3.1) must be understood in a formal sense only, because the infinite sum on the RHS will usually not converge under the circumstances we contemplate.

In a practical case, Eq. (3.1) is recognized to describe the expansion of the Green's function G^C in a Born approximation about the noninteracting $G_0^1 G_0^2$. In order to accomplish the summation, one must replace the free Green's functions in T_α by the functions G^C that describe the propagation of the system in each time interval. This will now be done for the P 's illustrated in the Feynman diagrams Figs. 1 and 2. The T 's and τ 's are given in Eqs. (3.2) to (3.9):¹⁰

$$T_\alpha(1) = ie^2 \sum_{i=1}^2 \gamma_i^2 \mathcal{G}_+(23) \gamma_i^1 \delta(13) \delta(24), \quad t_3 < t_2; \quad (3.2)$$

$$\tau_\alpha(1) = ie^2 \sum_{i=1}^2 \gamma_i^2 F_1(11') G^C(1'234') F_2(4'4) \gamma_i^1 \mathcal{G}_+(23); \quad (3.3)$$

$$T_\beta(1) = ie^2 \sum_{i=1}^2 \gamma_i^1 \mathcal{G}_+(14) \gamma_i^2 \delta(13) \delta(24), \quad t_4 < t_1; \quad (3.4)$$

$$\tau_\beta(1) = ie^2 \sum_{i=1}^2 \gamma_i^1 F_2(22') G^C(12'3'4) \times F_1(3'3) \gamma_i^2 \mathcal{G}_+(14); \quad (3.5)$$

$$T_\alpha(2) = (ie^2)^2 \sum_{i,j=1}^2 \gamma_i^2 \gamma_j^1 \delta(11') G_0^1(1'3') G_0^2(24) \times \gamma_j^2 \gamma_i^1 \delta(3'3) \mathcal{G}_+(1'4) \mathcal{G}_+(3'2), \quad t_4 < t_3' < t_1' < t_2; \quad (3.6)$$

¹⁰ The $\sum_{i=1,2}$ represents the sum over transverse components with respect to the Fourier transform momentum of the photon Green's functions.

$$\begin{aligned}
 \tau_\alpha(2) &= (ie^2)^2 \sum_{i,j=1}^2 \gamma_i^2 F_1(11'') G^c(1''21'2'') \\
 &\quad \times \gamma_j^2 F_2(2''2') G^c(1'2'3'4') F_2(4'4'') \\
 &\quad \times \gamma_i^2 G^c(3'4''3'4) F_1(3'3) \\
 &\quad \times \gamma_j^2 \mathcal{G}_+(1'4) \mathcal{G}_+(3'2); \quad (3.7) \\
 T_\beta(2) &= (ie^2)^2 \sum_{i,j=1}^2 \gamma_i^2 G_0^1(11') \gamma_j^2 \delta(1'3) \delta(24') \\
 &\quad \times \gamma_j^2 G_0^2(4'4) \gamma_i^2 \mathcal{G}_+(14) \mathcal{G}_+(1'4'), \\
 &\quad t_4' < t_4 < t_1 < t_1'; \quad (3.8)
 \end{aligned}$$

$$\begin{aligned}
 \tau_\beta(2) &= (ie^2)^2 \sum_{i,j=1}^2 \gamma_i^2 G_0^1(11') \gamma_j^2 F_2(22') G^c(1'2'3'4') \\
 &\quad \times F_1(3'3) \gamma_j^2 G_0^2(4'4) \gamma_i^2 \mathcal{G}_+(14) \mathcal{G}_+(1'4'). \quad (3.9)
 \end{aligned}$$

In these expressions it was unnecessary to refer explicitly to the times of all the successive events because the Coulomb Green's function at nonvanishing relative times is related to its values at zero relative time by free-particle Green's functions. The free-particle operators F_i applied to it therefore lead to the delta functions in relative time that reduce the above expressions to the result referring to the time intervals one would expect from the discussion leading to Eq. (3.1). This property of G^c is evident when its defining equation is rewritten symmetrically:

$$\begin{aligned}
 G^c(1234) &= G^0(1234) + G^0(121'2') [I^c(1'2'3'4') \\
 &\quad + I^c(1'2'56) G^c(5678) I^c(783'4')] G^0(3'4'34). \quad (3.10)
 \end{aligned}$$

It arises from the instantaneous character of the binding interaction I^c .

In this manner the entire interaction operator \bar{I}' is rewritten as an operator \mathcal{G}' from which a relative coordinate transform \mathcal{K}' can be obtained by Eq. (2.4). Feynman diagrams may still be used to list the interaction terms; one must take those diagrams that contain no I^c and those that contain it only in intermediate states in which pairs created by a lower-order contribution to \mathcal{G}' are present. The Fermion lines are then represented by interacting Green's functions as in Eqs. (3.2)–(3.9) and the quantum lines by the quantum Green's functions.

One encounters a difficulty with the above prescription. This will now be resolved. It has to do with those processes P_α^c that contain only interactions I^c between the two particles. These represent the self-energy of each particle in the presence of the other. The corresponding T_α is not contained in I' but has been included in $F_1 F_2$, Eq. (2.1), to facilitate renormalization. One may "borrow" the necessary term and carry out the rearrangement—in other words, rewrite Eq. (3.1):

$$T_\alpha^c = (T_\alpha + T_\alpha^c) - T_\alpha = \tau_\alpha - T_\alpha. \quad (3.1')$$

One must observe the customary precautions for han-

dling the difference between the two usually divergent quantities in the right-hand side.

The effective interactions appearing in the second and third terms of Eq. (2.23) present no new problems. For convenience the Green's function appearing there may be expanded in the Born approximation series. The resulting expression is then very similar to I' itself; all terms in it, however, are reducible. Now, the irreducibility mentioned earlier played no role in the discussion. The prescription can therefore be applied without further argument.

IV. APPLICATION TO THE HYPERFINE STRUCTURE OF POSITRONIUM

As a test for the technique developed in Sec. III, Part IV of KK III was re-examined critically. In this part, infrared divergences which arise in the calculation of the hfs of positronium from the consideration of terms of the direct interaction are shown to cancel when the photon momenta are cut off at a small value. We have recalculated these effects and confirmed the numerical results of KK III by showing that the low-energy terms that were omitted actually do vanish.

To illustrate the formalism, we shall carry out the calculation as it relates to the contributions from a single transverse quantum. The perturbation considered is therefore the one stated in Eqs. (3.3) and (3.5) and illustrated in Fig. 1. It should be pointed out that this includes both I_{K1B} and the Coulomb-transverse terms, of $I_{K2B}^{(2)}$.¹ The explicit second-order terms, which we shall not discuss here, involve two transverse quanta. The energy change to be investigated is that derived from Eq. (3.3) by taking the center-of-mass transform,

$$\begin{aligned}
 \mathcal{T}_K(x, x') &= ie^2 \int e^{-iK(X-X')} d^4 X' d^4 x_1' d^4 x_2' d^4 x_3' d^4 x_4' \\
 &\quad \times \sum_{i=1}^2 \gamma_i^2 F_1(11') \delta(22') G^c(1'2'3'4') \delta(3'3) \\
 &\quad \times F_2(4'4) \gamma_i^2 \mathcal{G}_+(23) \theta(t_2 - t_1) \theta(t_4 - t_3) \theta(t_2 - t_3), \quad (4.1)
 \end{aligned}$$

and introducing the functions $\theta(t_2 - t_1)$ and $\theta(t_4 - t_3)$ to exclude the contributions from processes of unwanted time ordering. The positive energy part of the associated function $\mathcal{K}_K(\mathbf{r}\mathbf{r}')$, Eq. (2.23), whose matrix element between equal-times wave functions gives the energy change, Eq. (2.25), is

$$\begin{aligned}
 \mathcal{K}_K(\mathbf{r}\mathbf{r}') &= -i\alpha(2\pi)^{-9} \int d\mathbf{k} d\mathbf{p} d\mathbf{p}' dK_0' dx dx' \\
 &\quad \times k^{-1} (k + K_0' - K_0)^{-1} e^{i\mathbf{p} \cdot (\mathbf{r}-\mathbf{x})} e^{-i\mathbf{p}' \cdot (\mathbf{r}'-\mathbf{x}')} \\
 &\quad \times e^{-\frac{1}{2}i\mathbf{k} \cdot (\mathbf{x}+\mathbf{x}')} \sum_{i=1}^2 \alpha_i^2 \Lambda_+^{-1}(\mathbf{p}) \\
 &\quad \times G^c_{-k, K_0'}(\mathbf{x}0, \mathbf{x}'0) \Lambda_+^{-2}(-\mathbf{p}') \alpha_i^2, \quad (4.2)
 \end{aligned}$$

which, of course, depends only on the Coulomb Green's function between states at equal time. The integrations

over \mathbf{x} and \mathbf{x}' define a Fourier transform of this Green's function. The momenta involved are \mathbf{p} and \mathbf{p}' of the order of the Bohr momentum αm , and \mathbf{k} , the photon momentum. Clearly, the function G^C may be approximated nonrelativistically for values of $k \lesssim \alpha m$, whereas a free-particle expansion is valid for $k > \alpha m$. This separation according to $k \gtrless k_m \approx \alpha m$ will then permit us to take into account the effects of the binding where necessary and to handle the high-energy contributions correctly. We therefore define

$$\mathcal{K}_{K>}(\mathbf{r}\mathbf{r}') = \int_{k>k_m} d\mathbf{k} \dots; \quad \mathcal{K}_{K<} = \int_{k<k_m} d\mathbf{k} \dots \quad (4.3)$$

The appropriate expressions for the Green's functions are the expansion about free-particle states:

$$\begin{aligned} \Lambda_+^{-1}(\mathbf{p}) \int G^C_{-k, K_0'}(\mathbf{x}0, \mathbf{x}'0) \Lambda_+^{-2}(-\mathbf{p}') e^{-i\mathbf{p} \cdot \mathbf{x}} \\ \times e^{i\mathbf{p}' \cdot \mathbf{x}'} e^{-\frac{1}{2}i\mathbf{k} \cdot (\mathbf{x}+\mathbf{x}')} d\mathbf{x}d\mathbf{x}' \\ \rightarrow (2\pi)^3 i \frac{\Lambda_+^{-1}(\mathbf{p}) \delta(\mathbf{p}-\mathbf{p}'+\mathbf{k}) \Lambda_+^{-2}(-\mathbf{p}')}{E(p)+E(p')-K_0'} \\ + 4\pi i \alpha \Lambda_+^{-1}(\mathbf{p}) \frac{\Lambda_+^{-2}(-\mathbf{p}-\mathbf{k})}{E(p)+E(\mathbf{p}+\mathbf{k})-K_0'} \\ \times (\mathbf{p}-\mathbf{p}'+\mathbf{k})^{-2} \frac{\Lambda_+^{-1}(\mathbf{p}'-\mathbf{k})}{E(p')+E(\mathbf{p}'-\mathbf{k})-K_0'} \Lambda_+^{-2}(-\mathbf{p}'), \end{aligned} \quad (4.4)$$

and the sum over nonrelativistic Pauli wave functions, $\varphi_n(\mathbf{x})$:¹¹

$$\begin{aligned} G^C_{-k, K_0'}(\mathbf{x}, 0, \mathbf{x}'0) \\ = i \sum_n \left(1 + \frac{\alpha^1 \cdot (\bar{\mathbf{p}} - \frac{1}{2}\mathbf{k})}{2m} \right) \left(1 - \frac{\alpha^2 \cdot (\bar{\mathbf{p}} + \frac{1}{2}\mathbf{k})}{2m} \right) \varphi_{-k, n}(\mathbf{x}) \\ \times \varphi_{-k, n}^*(\mathbf{x}') \left(1 + \frac{\alpha^1 \cdot (\bar{\mathbf{p}}' - \frac{1}{2}\mathbf{k})}{2m} \right) \\ \times \left(1 - \frac{\alpha^2 \cdot (\bar{\mathbf{p}}' + \frac{1}{2}\mathbf{k})}{2m} \right) \frac{1}{K_{n'} - K_0'}, \end{aligned} \quad (4.5)$$

where $i\bar{\mathbf{p}}$ and $i\bar{\mathbf{p}}'$ stand for the gradients with respect to \mathbf{x} and \mathbf{x}' . The presence of the photon momentum arises from the fact that the system described by the Green's function has a total momentum $-k$ because of recoil. The wave equation satisfied by these wave functions is

$$\begin{aligned} [E(\bar{\mathbf{p}} + \frac{1}{2}\mathbf{k}) + E(\bar{\mathbf{p}} - \frac{1}{2}\mathbf{k}) - \alpha/|\mathbf{x}|] \varphi_{-k, n}(\mathbf{x}) \\ = K_{n'} \varphi_{-k, n}(\mathbf{x}), \end{aligned} \quad (4.6)$$

¹¹ R. J. Eden, Proc. Roy. Soc. (London) A215, 133 (1952), has used similar Green's functions of compound particles without, however, explicitly referring to the structure of the system.

or

$$\begin{aligned} \left[2m + \frac{\bar{p}^2}{2m} + \frac{k^2}{4m} - \frac{\alpha}{|\mathbf{x}|} \right] \varphi_{-k, n}(\mathbf{x}) \\ = \left[\mathcal{H}(\bar{p}) + \frac{k^2}{4m} \right] \varphi_{-k, n}(\mathbf{x}) = K_{n'} \varphi_{-k, n}(\mathbf{x}), \end{aligned} \quad (4.7)$$

in a nonrelativistic expansion. \mathcal{H} is the hydrogenic Hamiltonian with reduced mass $\frac{1}{2}m$ and eigenvalues $2m - \alpha^2 m / 4n^2$. The only other quantity that enters the perturbation energy, Eq. (2.25), is the ground-state wave function $\chi^C(\mathbf{r})$, which also may be approximated in terms of a Pauli wave function, $\varphi_0(\mathbf{r})$:

$$\begin{aligned} \chi^C(\mathbf{r}) \cong \left(1 + \frac{\alpha^1 \cdot \mathbf{p}}{2m} \right) \left(1 - \frac{\alpha^2 \cdot \mathbf{p}}{2m} \right) \varphi_0(\mathbf{r}) \\ = (2\pi)^{-3} \int d\mathbf{p} e^{i\mathbf{p} \cdot \mathbf{r}} \left(1 + \frac{\alpha^1 \cdot \mathbf{p}}{2m} \right) \left(1 - \frac{\alpha^2 \cdot \mathbf{p}}{2m} \right) \varphi_0(\mathbf{p}). \end{aligned} \quad (4.8)$$

The high-energy contribution,

$$\Delta E_{>} = -i \int \chi^{C*}(\mathbf{r}) \mathcal{K}_{K>}(\mathbf{r}\mathbf{r}') \chi^C(\mathbf{r}') d\mathbf{r}d\mathbf{r}', \quad (4.9)$$

can be calculated by the methods of KK III. The spin matrix element, denoted by $\langle \rangle$, from the first term of Eq. (4.4) gives

$$\begin{aligned} \sum_{i=1}^2 \left\langle \left(1 + \frac{\alpha^1 \cdot \mathbf{p}}{2m} \right) \left(1 - \frac{\alpha^2 \cdot \mathbf{p}}{2m} \right) \Lambda_+^{-1}(\mathbf{p}) \alpha_i^1 \alpha_i^2 \right. \\ \left. \times \Lambda_+^{-2}(-\mathbf{p}') \left(1 + \frac{\alpha^1 \cdot \mathbf{p}'}{2m} \right) \left(1 - \frac{\alpha^2 \cdot \mathbf{p}'}{2m} \right) \right\rangle \\ \rightarrow [E(p) + m][E(p') + m] \langle \boldsymbol{\sigma}^1 \times (\mathbf{p} - \mathbf{p}') \\ \cdot \boldsymbol{\sigma}^2 \times (\mathbf{p} - \mathbf{p}') \rangle / 16m^2 E(p) E(p') \\ \rightarrow [E(p) + m][E(p') + m] \\ \times \frac{2}{3} k^2 \boldsymbol{\sigma}^1 \cdot \boldsymbol{\sigma}^2 / 16m^2 E(p) E(p'), \end{aligned} \quad (4.10)$$

where we have made use of momentum conservation and spherical symmetry. The resulting expression can be reduced to the first terms of Eq. (4.8), KK III,

$$\begin{aligned} \frac{\alpha}{6m^2} \frac{\langle \boldsymbol{\sigma}^1 \cdot \boldsymbol{\sigma}^2 \rangle}{(2\pi)^5} \int_{k>k_m} d\mathbf{k} d\mathbf{p} d\mathbf{p}' \varphi_0^*(\mathbf{p}) \\ \times \frac{[E(p) + m][E(p') + m]}{4E(p)E(p')} \frac{k\delta(\mathbf{p} - \mathbf{p}' + \mathbf{k})}{k + E(p) + E(p') - K_0} \varphi_0(\mathbf{p}'), \end{aligned} \quad 4.11$$

by carrying out the contour integral over K_0' . The factor of one-half of this contribution compared with the formula in KK III is accounted for by the existence of an equal energy change due to the τ_β in Eq. (3.5). By the approximations described in KK III the result,

accurate to α (hfs) under the assumption that $k_m \sim \alpha m$, is obtained:

$$\frac{\alpha \langle \sigma^1 \cdot \sigma^2 \rangle}{6m^2 (2\pi)^2} \int_{k > k_m} \varphi_0(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} \varphi_0(\mathbf{r}) d\mathbf{r} d\mathbf{k} - (2\alpha^2/3m^2) \langle \sigma^1 \cdot \sigma^2 \rangle |\varphi_0(0)|^2 \{1 + \ln(m/k_m)\}. \quad (4.12)$$

For the second term in Eq. (4.4), it is sufficient to set

$$\chi^c(\mathbf{r}) \cong \varphi_0(0). \quad (4.13)$$

The resulting spin matrix elements are of the form

$$\sum_{i=1}^2 \langle \alpha_i^2 \Lambda_+^2(-\mathbf{k}) \Lambda_+^1(-\mathbf{k}) \alpha_i^1 \rangle \rightarrow \frac{2}{3} k^2 \langle \sigma^1 \cdot \sigma^2 \rangle / 4E^2. \quad (4.14)$$

The integration over K_0' may be carried out as before and yields the expression

$$\frac{\alpha^2 \langle \sigma^1 \cdot \sigma^2 \rangle}{3m^2 2\pi} |\varphi_0(0)|^2 \int_{k > k_m} \frac{d\mathbf{k} m^2}{k E^2 (E+k-m)^2}, \quad (4.15)$$

which corresponds to a part of Eq. (4.13), KK III. The integration of k gives:

$$(2\alpha^2/3m^2) \langle \sigma^1 \cdot \sigma^2 \rangle |\varphi_0(0)|^2 \left\{ -\frac{1}{4}\pi + \ln(m/k_m) \right\}, \quad (4.16)$$

which is added to Eq. (4.12) for $\Delta E_>$,

$$\Delta E_> = (\alpha/24m^2\pi^2) \langle \sigma^1 \cdot \sigma^2 \rangle \times \int_{k > k_m} \varphi_0^*(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} \varphi_0(\mathbf{r}) d\mathbf{r} d\mathbf{k} - (2\alpha^2/3m^2) \langle \sigma^1 \cdot \sigma^2 \rangle |\varphi_0(0)|^2 (1 + \frac{1}{4}\pi). \quad (4.17)$$

The low-energy contribution,

$$\Delta E_{<} = -i \int \chi^{c*}(\mathbf{r}) \mathcal{K}_{K_0}(\mathbf{r}\mathbf{r}') \chi^c(\mathbf{r}') d\mathbf{r} d\mathbf{r}', \quad (4.18)$$

is evaluated by using Eqs. (4.5) and (4.8) to simplify the expression. The spin matrix elements that occur are

$$\left\langle \left(1 + \frac{\alpha^1 \cdot \mathbf{p}}{2m} \right) \left(1 - \frac{\alpha^2 \cdot \mathbf{p}}{2m} \right) \Lambda_+^1(\mathbf{p}) \alpha_j^2 \left(1 + \frac{\alpha^1 \cdot \mathbf{p}}{2m} \right) \times \left(1 - \frac{\alpha^2 \cdot (\mathbf{p} + \mathbf{k})}{2m} \right) \right\rangle \rightarrow i \frac{(\sigma^2 \times \mathbf{k})_j}{2m} \left\{ 1 + \frac{E(\mathbf{p}) - m}{2m} + \frac{(E(\mathbf{p}) - m) p^2}{8m^2 E(\mathbf{p})} \right\} \rightarrow i (\sigma^2 \times \mathbf{k})_j / 2m, \quad (4.19)$$

and

$$\left\langle \left(1 + \frac{\alpha^1 \cdot (\mathbf{p}' - \mathbf{k})}{2m} \right) \left(1 - \frac{\alpha^2 \cdot \mathbf{p}'}{2m} \right) \alpha_j^1 \Lambda_+^2(-\mathbf{p}') \times \left(1 + \frac{\alpha^1 \cdot \mathbf{p}'}{2m} \right) \left(1 - \frac{\alpha^2 \cdot \mathbf{p}'}{2m} \right) \right\rangle \rightarrow -i (\sigma^1 \times \mathbf{k})_j / 2m, \quad (4.20)$$

where the omitted terms are quadratic in the momenta and therefore of order α^2 (hfs). Thus,

$$\Delta E_{<} = (\alpha/16m^2\pi^2) \sum_n \int_{k < k_m} d\mathbf{k} e^{-\frac{1}{2}i\mathbf{k} \cdot (\mathbf{r} + \mathbf{r}')} k^{-1} d\mathbf{r} d\mathbf{r}' \times (k + K_{n'} - K_0)^{-1} \sum_{i=1}^2 \varphi_0^*(\mathbf{r}) (\sigma^2 \times \mathbf{k})_i \times \varphi_{-k, n}(\mathbf{r}) \varphi_{-k, n}^*(\mathbf{r}') (\sigma^1 \times \mathbf{k})_i \varphi_0(\mathbf{r}'). \quad (4.21)$$

In order to eliminate the wave functions of the excited states occurring in Eq. (4.22), we may replace $K_{n'}$ by $\mathcal{J}C + k^2/4m$, Eq. (4.7), acting on $\varphi_{-k, n}(\mathbf{r})$. The sum over n now yields a δ function between r and r' :

$$\Delta E_{<} = (\alpha/16m^2\pi^2) \int_{k < k_m} d\mathbf{k} \varphi_0^*(\mathbf{r}) (\sigma^2 \times \mathbf{k}) \cdot (\sigma^1 \times \mathbf{k}) \times e^{-\frac{1}{2}i\mathbf{k} \cdot \mathbf{r}} k^{-1} (k + \mathcal{J}C - K_0 + k^2/4m)^{-1} \times e^{-\frac{1}{2}i\mathbf{k} \cdot \mathbf{r}} \varphi_0(\mathbf{r}) d\mathbf{r}. \quad (4.22)$$

This expression may be evaluated approximately if one recognizes that the energy level differences in the denominator are of order $\alpha^2 m$ whereas the momentum k is of order αm . It is therefore sufficiently accurate to retain only the first two terms in the expansion

$$k^{-1} (k + \mathcal{J}C - K_0 + k^2/4m)^{-1} = k^{-2} - k^{-3} (\mathcal{J}C - K_0 + k^2/4m) + k^{-3} (\mathcal{J}C - K_0 + k^2/4m)^2 (k + \mathcal{J}C - K_0 + k^2/4m)^{-1}. \quad (4.23)$$

The spherical symmetry of the ground state implies that one may average over angles with the result

$$\langle \sigma^2 \times \mathbf{k} \cdot \sigma^1 \times \mathbf{k} \rangle \rightarrow \frac{2}{3} k^2 \langle \sigma^1 \cdot \sigma^2 \rangle. \quad (4.24)$$

With these two observations, the first term of Eq. (4.23) inserted into Eq. (4.22) gives

$$(\alpha/24m^2\pi^2) \langle \sigma^1 \cdot \sigma^2 \rangle \int_{k < k_m} \varphi_0^*(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} \varphi_0(\mathbf{r}) d\mathbf{r} d\mathbf{k}, \quad (4.25)$$

whereas the second one gives

$$-(\alpha/24m^2\pi^2) \langle \sigma^1 \cdot \sigma^2 \rangle \int_{k < k_m} \varphi_0^*(\mathbf{r}) e^{-\frac{1}{2}i\mathbf{k} \cdot \mathbf{r}} k^{-1} \times [\mathcal{J}C(p) - K_0 + k^2/4m] e^{-\frac{1}{2}i\mathbf{k} \cdot \mathbf{r}} \varphi_0(\mathbf{r}) d\mathbf{r} d\mathbf{k}. \quad (4.26)$$

Now,

$$(\mathcal{J}C(p) - K_0 + k^2/4m) e^{-\frac{1}{2}i\mathbf{k} \cdot \mathbf{r}} \varphi_0(\mathbf{r}) = [\mathcal{J}C(\mathbf{p} + \frac{1}{2}\mathbf{k}) - K_0 - \mathbf{p} \cdot \mathbf{k}/m] e^{-\frac{1}{2}i\mathbf{k} \cdot \mathbf{r}} \varphi_0(\mathbf{r}) = -(\mathbf{p} \cdot \mathbf{k}/m) e^{-\frac{1}{2}i\mathbf{k} \cdot \mathbf{r}} \varphi_0(\mathbf{r}) + e^{-\frac{1}{2}i\mathbf{k} \cdot \mathbf{r}} [\mathcal{J}C(p) - K_0] \varphi_0(\mathbf{r}) = -(\mathbf{p} \cdot \mathbf{k}/m) e^{-\frac{1}{2}i\mathbf{k} \cdot \mathbf{r}} \varphi_0(\mathbf{r}). \quad (4.27)$$

Since, furthermore, the ground-state wave function is

real, the integrand of Eq. (4.26) may be rewritten

$$\begin{aligned}
 & -(\varphi_0(\mathbf{r})e^{-\frac{1}{2}i\mathbf{k}\cdot\mathbf{r}})\frac{\mathbf{k}\cdot\mathbf{p}}{m}(e^{-\frac{1}{2}i\mathbf{k}\cdot\mathbf{r}}\varphi_0(\mathbf{r}))k^{-1} \\
 & = -\frac{1}{2}\frac{\mathbf{k}\cdot\mathbf{p}}{m}[e^{-i\mathbf{k}\cdot\mathbf{r}}(\varphi_0(\mathbf{r}))^2]k^{-1}, \quad (4.28)
 \end{aligned}$$

which becomes a vanishing surface integral when integrated over \mathbf{r} . Consequently $\Delta E_<$ is equal to the expression given in Eq. (4.25). When this simple result is added to the high-energy contribution, Eq. (4.17), the total energy change from the perturbation Eq. (4.1) is equal to

$$\begin{aligned}
 \Delta E &= (\alpha/24m^2\pi^2)\langle\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2\rangle\int_{\text{all } k}\varphi_0^*(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}\varphi_0(\mathbf{r})d\mathbf{r}d\mathbf{k} \\
 & \quad - (2\alpha^2/3m^2)\langle\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2\rangle|\varphi_0(0)|^2(1+\frac{1}{4}\pi) \quad (4.29) \\
 & = (\pi\alpha/3m^2)\langle\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2\rangle|\varphi_0(0)|^2\{1-(2\alpha/\pi)(1+\frac{1}{4}\pi)\}.
 \end{aligned}$$

It is clear that this result is precisely the one that would have been obtained for these terms by the use of

artificial cutoffs at small momenta for the divergent contributions. It is a consequence of the fact that the low-energy correction Eq. (4.25) contained no corrections of relative order α to the hyperfine structure. The same procedure, with a similar result, can be applied to the corrections coming from the exchange of two transverse quanta in Eq. (4.11), KK III. The remaining contributions present no difficulties at low energies and need not be examined.

We may point out now that the ideas discussed here are just the ones that were used to evaluate the Lamb shift in hydrogen. The formalism therefore provides the correct connection for joining a nonrelativistic perturbation result such as Eq. (4.21) to a relativistic treatment of the radiation interaction that permits the removal of divergences. The method is being applied to situations where the low-energy contributions that depend on the bound state will not vanish.

We have enjoyed conversations about this subject with Julian Schwinger, Abraham Klein, Malvin Ruderman, and Paul Martin.