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Theory of Nuclear Level Density

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We have compared the level density of a nuclear model deduced from a statistical analysis with the results of the exact counting of the levels of the same model. The tables of levels of Ne^{20} given by Critchfield and Oleksa have been used as a test of the statistical theory. A new derivation of the level density is presented. It starts, as usual, from the independent-particle model. However, it differs from the previous treatments in two respects: (a) the exact states of the nucleons in the central potential are kept throughout the calculations instead of being replaced by a continuous distribution; (b) the effect of mutual interactions of the nucleons of the Majorana, Bartlett, or Heisenberg type is taken into account in the long-range approximation. With these modifications, the statistical theory agrees very well with the exact counting of the levels, both for the total density and for the density of the levels having a given angular momentum. It is shown that the replacement of the nucleon states by a continuous distribution introduced in most previous derivations, and the neglect of the Majorana forces can produce very large errors. An interpretation is presented of the distribution of angular momentum among nuclear levels in terms of rotations of the whole nucleus as a rigid body.

I. INTRODUCTION

THE theory of nuclear level density was initiated by Bethe[1] many years ago, on the basis of the Fermi gas model of the nucleus. It was soon realized, however, that the pure free-particle model which he used was only a very crude approximation. Moreover, it was not clear to what extent the conventional statistical analysis could be applied to systems containing relatively small numbers of particles. Therefore, a more detailed calculation of the potential energy was introduced, and the statistical analysis was replaced by an explicit counting of the levels of the nuclear model.[2–4] Unfortunately, this procedure is so lengthy that it cannot be repeated for many different values of Z and A. In addition, such a calculation is not feasible for heavy nuclei. It will be shown, however, that after proper modification, the statistical theory yields a reliable and accurate counting procedure even for comparatively light nuclei. It will also be shown that nuclear models, which include a dependence of the potential energy on the symmetry of the wave function, can be treated by statistical methods. The validity of the nuclear model, itself, will not be discussed here.

As usual, the independent-particle model is taken as a basis of the statistical theory. After the energy levels of the individual nucleons in the central potential have been determined, the calculation of the level density of the nucleus can be treated as a purely combinatorial problem, the solution of which is expressed exactly by the Darwin-Fowler integral (3). The main step in the statistical theory is the evaluation of the latter integral by the saddle point method. It will be shown by comparison with the calculations of Critchfield and Oleksa,[4] based on an exact counting of the levels, that this approximation does not introduce large errors even when the number of nucleons is not very large. The standard treatment of statistical mechanics, however, includes additional approximations which lose their validity in systems containing small numbers of particles. Among these is the replacement of the individual nucleon states in the central potential by a continuous distribution. This leads to a very simple expression for the level density, but a comparison with Critchfield's results shows that its accuracy is not very satisfactory. The fact that it does not take into account the shell structure of the ground state of the nucleus is a great weakness of the method of continuous distributions.[5] Thus, for example, if the outer shell has a large angular momentum, and is about half-filled, the outer nucleons can be rearranged in a large number of different ways with very little excitation energy. The level density is, then, larger than it would be in a closed shell nucleus of comparable mass.

In nuclear reactions, the transition probabilities vary so rapidly with angular momentum that only the levels having the appropriate angular momenta are excited with reasonably large probabilities. Therefore, it is important to determine the density of the levels with given angular momentum. More generally, one can compute the density of the levels for which any number of constants of the motion take given values. The isotopic-spin formalism is, then, very convenient for representing Z as a constant of the motion similar to an angular momentum. Another application of these calculations, is the treatment of more complex nuclear models. The determination of the densities of levels with given spin, isotopic spin, or symmetry type is, in fact, the key to an estimation of the effects on the level density of Bartlett, Heisenberg, or Majorana forces.

II. THE CONTINUOUS APPROXIMATION

In the following three sections, we shall consider a nuclear model in which the nucleons move indepen-

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[1] H. A. Bethe, Revs. Modern Phys. **9**, 53 (1937).
[2] J. Bardeen and E. Feenberg, Phys. Rev. **54**, 809 (1938).
[3] L. Motz and E. Feenberg, Phys. Rev. **54**, 1055 (1938).
[4] C. Critchfield and S. Oleksa, Phys. Rev. **82**, 243 (1951).
[5] H. Margenau, Phys. Rev. **59**, 627 (1941).

dently in an average central field $V(r)$. For simplicity, we shall assume that V does not depend on the spin and the isotopic spin of the nucleons. The theory can be applied to a *j-j* model with a few obvious modifications. Let $c_0, c_1, \cdots c_K$ be a set of commuting constants of the motion of a single nucleon. Such constants will be, for instance, the projection along the z axis of the orbital angular momentum l_z, or the spin s_z, or the projection along the ζ axis of the isotopic spin τ_ζ. We shall assume that the values C_k of the constants of the motion for the whole system are the sums of the values of the c_k for the individual nucleons. The states of the nucleons in the potential $V(r)$ can be classified according to their values of the c_k. It will be convenient to call a "component" the set of all nucleon states for which the c_k have the same value. Thus, a component α is defined by the specification of the $m_{k,\alpha}$

$$c_0 = m_{0,\alpha}, \quad c_1 = m_{1,\alpha}, \quad \cdots, \quad c_K = m_{K,\alpha}. \tag{1}$$

For instance, if we introduce only one constant of the motion s_z, we have two components: the states with $s_z = \frac{1}{2}$, and the states with $s_z = -\frac{1}{2}$. If l_z is taken as one of the constants of the motion, the number of components becomes infinite, corresponding to $l_z = 0, \pm 1, \pm 2, \cdots$.

In what follows, c_0 will always stand for the number of particles. Thus, $m_{0,\alpha} = 1$ for all components, and for the whole system $C_0 = A$.

We shall denote by ϵ_i the states of a nucleon in the central potential $V(r)$, and by $g_{i,\alpha}$ the statistical weight of ϵ_i in the component α. For simplicity, we shall assume that each of the constants of the motion $c_1, c_2, \cdots c_K$ is similar to the projection on an axis of an angular momentum. Hence, the statistical weights $g_{i,\alpha}$ associated with two components which differ only through the signs of the $m_{k,\alpha}$ are identical.[6]

The central problem in the statistical theory is that of finding the density $\rho(E, M_k)$ of the levels for which the total C_k have given values:

$$C_0 = M_0 = A, \quad C_1 = M_1, \quad \cdots, \quad C_K = M_K. \tag{2}$$

The exact solution to this problem is given by the well-known Darwin-Fowler integral.[7,8]

$$\rho(E, M_k) = (2\pi i)^{-K-2} \int \cdots \int dt dx_0 \cdots dx_K \times \exp(Et - \textstyle\sum_k x_k M_k + \sum_\alpha \Phi_\alpha), \tag{3}$$

where

$$\Phi_\alpha(t, x_k) = \textstyle\sum_i g_{i,\alpha} \log[1 + \exp(\sum_k x_k m_{k,\alpha} - \epsilon_i t)]. \tag{4}$$

In (3) the integrals are taken along contours going from $-i\infty$ to $+i\infty$. It is customary in statistical mechanics to replace the distribution of states ϵ_i by a continuous distribution. Let then $\rho_\alpha(\epsilon)$ be the density of nuclear states in the component α. We can replace (4) by

$$\Phi_\alpha(t, x_k) = \int_0^\infty d\epsilon \rho_\alpha(\epsilon) \log[1 + \exp(\textstyle\sum_k x_k m_{k,\alpha} - \epsilon t)]. \tag{5}$$

We shall first describe the continuous approximation which is based upon the Eqs. (3) and (5).

In the terminology of thermodynamics, the variable t in (3), (4), (5) would be the inverse of the temperature. It is well known that for all excitation energies of interest, a nucleus is a highly degenerate system. Thus, the important values of t in the integral (3) are the large ones. For t large the integrand in (5) falls off very rapidly as ϵ becomes larger than the "Fermi energy of the component α":

$$\epsilon_\alpha = (\textstyle\sum_k x_k m_{k,\alpha}) / t. \tag{6}$$

This permits an asymptotic estimation of the integrals for $\epsilon_\alpha t \gg 1$. According to Sommerfeld,[9,10]

$$\Phi_\alpha = t(\epsilon_\alpha N_\alpha - W_\alpha) + (\pi^2/6t)\rho_\alpha(\epsilon_\alpha), \tag{7}$$

where

$$N_\alpha(\epsilon_\alpha) = \int_0^{\epsilon_\alpha} d\epsilon \rho_\alpha(\epsilon)$$

is the total number of nucleons which can be accommodated in the component α, when all states are filled up to the energy ϵ_α, and

$$W_\alpha(\epsilon_\alpha) = \int_0^{\epsilon_\alpha} d\epsilon \epsilon \rho_\alpha(\epsilon)$$

is the corresponding total energy.

The next step is the evaluation of the integral (3) by the many-variables saddle point method.[7] This yields

$$\rho(E, M_k) = (2\pi)^{-(K+2)/2} D^{-1/2} e^S, \tag{8}$$

where the "entropy" S is the value at the saddle point of

$$S = Et - \textstyle\sum_k x_k M_k + \sum_\alpha \Phi_\alpha,$$

and D is the determinant formed with the values at the saddle point of the second derivatives of $\Phi = \sum_\alpha \Phi_\alpha$ with respect to $t, x_0, x_1, \cdots, x_K$. The coordinates of the saddle point are given by

$$E = -\partial\Phi/\partial t, \quad M_k = \partial\Phi/\partial x_k. \tag{9}$$

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[6] This assumption is introduced in order to decrease the number of terms in the expansions and could be removed easily. It excludes parity, which can be represented, for instance, by a constant of the motion such that $m_\alpha = 0$ for the even nuclear states and $m_\alpha = 1$ for the odd states. The density of the even nuclear levels would then be the sum of the densities of the levels for which $C = 0, \pm 2, \pm 4$, etc. The density of the odd levels would be the sum of the densities of the levels such that $C = \pm 1, \pm 3, \pm 5$, etc. In general, the level density will be a smooth function of C, and the densities of levels of both parities will be equal in the first approximation.

[7] R. H. Fowler, *Statistical Mechanics* (Macmillan Company, New York, 1936).

[8] G. E. Uhlenbeck and C. Van Lier, Physica **4**, 531 (1937).

[9] A. Sommerfeld, Z. Physik **67**, 1 (1928).

[10] This expression is the beginning of an asymptotic expansion. However, the following terms depend on the derivatives of the $\rho_\alpha(\epsilon)$, and therefore have little meaning in nuclear problems, where the replacement of the discrete energy levels of the nucleons by continuous distributions is quite ambiguous (see Sec. 4).

Substitution of (7) into these equations yields

$$E=\sum_{\alpha}[W_{\alpha}+(\pi^2/6t^2)(\rho_{\alpha}+\epsilon_{\alpha}\rho_{\alpha}')],$$
$$M_k=\sum_{\alpha} m_{k,\alpha}[N_{\alpha}+(\pi^2/6t^2)\rho_{\alpha}], \tag{10}$$

where all functions of ϵ are taken at $\epsilon=\epsilon_{\alpha}$ as defined by (6).

The lowest energy level satisfying the conditions (2) is obtained from (10) by putting $t=\infty$. The energy E^1 and the Fermi energies of this state are given by

$$E^1=\sum_{\alpha} W_{\alpha}(\epsilon_{\alpha}^1),\quad M_k=\sum_{\alpha} m_{k,\alpha}N_{\alpha}(\epsilon_{\alpha}^1). \tag{11}$$

The ground-state energy, on the other hand, is the minimum of E^1 for $M_0=A$ and arbitrary $M_1, \cdots M_K$. It is easily seen that in the ground state of the model $M_1, \cdots M_K$ vanish and that the Fermi energies ϵ_{α} are all equal to a value ϵ^0 defined by

$$\sum_{\alpha} N_{\alpha}(\epsilon^0)=A.$$

We shall now assume that the excitation energy $U^1=E-E^1$, measured above the energy E^1 of the lowest level having the given values of the constants of the motion, is sufficiently small so that $1/t^2$ and the variations $\epsilon_{\alpha}-\epsilon_{\alpha}^1$ can be considered as infinitesimal quantities. Then, from (10) we get in the first approximation

$$U^1=(\pi^2/6t^2)\sum_{\alpha}\rho_{\alpha}(\epsilon_{\alpha}^1).$$

Moreover, if $M_1, M_2, \cdots M_K$ are small, the ϵ_{α}^1 do not differ very much from ϵ^0, and we have

$$\sum_{\alpha}\rho_{\alpha}(\epsilon_{\alpha}^1)\approx\sum_{\alpha}\rho_{\alpha}(\epsilon^0)=1/\delta. \tag{12}$$

Here $1/\delta$ is the total density of states available to a single nucleon at an energy equal to the Fermi energy ϵ^0 of the ground state. It can be easily shown that the difference,

$$\sum_{\alpha}\rho_{\alpha}(\epsilon_{\alpha}^1)-\sum_{\alpha}\rho_{\alpha}(\epsilon^0)=\sum_{\alpha}(\epsilon_{\alpha}^1-\epsilon^0)\rho_{\alpha}'(\epsilon^0),$$

which is neglected in (12) vanishes in the first order in M_k. A short calculation finally yields for the level density in the first approximation

$$\rho(E,M_k)=[(2\pi)^{K\ 2}(\sqrt{}48)U^1\sigma_1\sigma_2\cdots\sigma_K]^{-1}$$
$$\times\exp[\pi(2U^1/3\delta)^{\frac{1}{2}}], \tag{13}$$

where the σ_k are defined by

$$\sigma_k^2=(1/\pi)\langle m_k^2\rangle_{\text{Av}}(6U^1/\delta)^{\frac{1}{2}}. \tag{14}$$

In (14) we have written

$$\langle m_k^2\rangle_{\text{Av}}=\sum_{\alpha} m_{k,\alpha}^2\rho_{\alpha}(\epsilon^0)/\sum_{\alpha}\rho_{\alpha}(\epsilon^0) \tag{15}$$

for the quadratic mean of m_k at the top of the Fermi distribution. A remarkable property of this result is that the level density *depends on the* M_k *only through* U^1.

We must now compute the energy difference E^1-E^0. The relations (11) yield

$$\epsilon_{\alpha}^1-\epsilon^0=\delta\sum_{k=1}^{K} m_{k,\alpha}M_k/\langle m_k^2\rangle_{\text{Av}},$$
$$E^1-E^0=U-U^1=\delta\sum_{k=1}^{K} M_k^2/2\langle m_k^2\rangle_{\text{Av}}. \tag{16}$$

Here we have introduced the excitation energy U measured from the ground state. By substitution of the value of U^1 given by (16) into (13), we get the Gaussian distribution law:

$$\rho(U,M_k)=\rho(U)[(2\pi)^{K/2}\sigma_1\sigma_2\cdots\sigma_K]^{-1}$$
$$\times\exp(-\sum_{1}^{K} M_k^2/2\sigma_k^2), \tag{17}$$

where

$$\rho(U)=(U\sqrt{}48)^{-1}\exp[\pi(2U/3\delta)^{\frac{1}{2}}] \tag{18}$$

is the well-known expression for the total level density.[1,8] In deriving (18) we have introduced the difference U^1-U only into the exponential which is the rapidly varying factor. Similarly, in the expression (14) defining σ_k, U^1 may be replaced by U. It is seen from (17) that σ_k is the *dispersion* of M_k among the levels.

The results obtained for the continuous model can be summarized as follows: *The density of the levels with given values of* $C_1, \cdots C_K$ *is proportional to the total level density* (18) *at an energy* $U-(E^1-E^0)$, *where* E^1 *is the energy of the lowest of all the considered levels, and* E^0 *the energy of the ground state. The constant of proportionality is independent of the values of* C_k, *and can be obtained by normalization of the total density at each energy.*

III. MORE ACCURATE TREATMENT

The calculation of the level density of a highly degenerate gas starts from the assumption that only levels near the top of the Fermi distribution are involved. The replacement of the distribution of the nucleon states by a continuous distribution, on the other hand, requires that the number of nucleon states in the relevant energy interval around the top of the Fermi distribution is large. Moreover, the distribution of these states should be fairly regular so that an equivalent smooth density function could be defined. These conditions are not well satisfied in nuclei, especially in the lighter ones. Actually, the number of relevant energy values of the nucleon states is usually rather small, and some of the states are highly degenerate due to spin, isotopic spin, and angular momentum. The density functions are then not well defined. A more accurate treatment should start from the exact definition of Φ_{α} given in (4).

Let us assume, for example, that in the ground state of the nucleus, the states $\epsilon_1, \epsilon_2, \cdots\epsilon_{n-1}$ are filled, that ϵ_n is partially filled, and that $\epsilon_{n+1}, \epsilon_{n+2}\cdots$ are empty. We have seen in the preceding section that as $t\rightarrow\infty$, the value of ϵ_{α} at the saddle point tends to ϵ^0, the energy of the top of the Fermi distribution. In other words $x_0 \to t\epsilon^0$, and the x_k remain finite. Similarly, in the present case we can expect that x_0 will have the limit $t\epsilon_n$ for large t. Hence, let us write

$$x_0 = t\epsilon_n + a + \delta x_0, \tag{19}$$

where a is a constant which must be adjusted in such a way that

$$\delta x_0 \to 0 \quad \text{as} \quad t \to \infty .$$

We have also seen in the continuous treatment that the x_k ($k=1, \cdots K$) at the saddle point are proportional to M_k. Consequently, for a certain range of values of M_k, we can consider the x_k as small quantities. For convenience, we shall write $x_k = \delta x_k$ for $k=1, \cdots K$. Thus,

$$\sum_k x_k m_{k,\alpha} = t\epsilon_n + a + \sum_{k=0}^{K} \delta x_k m_{k,\alpha}.$$

The method we shall use here consists in treating exactly the dependence of Φ_α on t and a, and in using a power series expansion for the dependence on the δx_k. Thus,

$$\Phi_\alpha = \sum_{i,p} \frac{g_{i,\alpha}}{p!} \Big[\sum_{k=0}^{K} \delta x_k m_{k,\alpha}\Big]^p f^{(p)}[t(\epsilon_n - \epsilon_i) + a], \tag{20}$$

where

$$\begin{aligned} f(x) &= \log(1+e^x), \\ f'(x) &= 1/(1+e^{-x}), \\ f''(x) &= \tfrac{1}{4} \operatorname{sech}^2(x/2), \\ f^{(3)}(x) &= -\tfrac{1}{4} \tanh(x/2) \operatorname{sech}^2(x/2), \text{ etc.} \end{aligned}$$

In the sum (20), the main contributions at low excitation energy come from the state ϵ_n and the states which are close to it. This follows from the rapid decrease of the functions $f^{(p)}$ with $|\epsilon_n - \epsilon_i|$ for large t. The terms involving f or f' do not decrease with $|\epsilon_n - \epsilon_i|$ when $\epsilon_i < \epsilon_n$. These terms, however, can be written

$$\begin{aligned} f[t(\epsilon_n - \epsilon_i) + a] &= t(\epsilon_n - \epsilon_i) + a + f[-t(\epsilon_n - \epsilon_i) - a], \\ f'[t(\epsilon_n - \epsilon_i) + a] &= 1 - f'[-t(\epsilon_n - \epsilon_i) - a], \end{aligned}$$

where the remaining functions f or f' now have the proper behavior. It is convenient to introduce the notation

$$\begin{aligned} f_i^{(p)} &= f^{(p)}[t(\epsilon_n - \epsilon_i) + a] && \text{for } \epsilon_i \geq \epsilon_n, \\ f_i^{(p)} &= (-)^p f^{(p)}[-t(\epsilon_n - \epsilon_i) - a] && \text{for } \epsilon_i < \epsilon_n. \end{aligned} \tag{21}$$

The distinction $\epsilon_i \gtreqless \epsilon_n$ is, of course, relevant only for $p = 0, 1$, since

$$f^{(p)}(-x) = (-)^p f^{(p)}(x) \quad \text{for } p \geq 2.$$

If we assume, as above, that $C_1 \cdots C_K$ are such that the numbers $g_{i,\alpha}$ of states with given values of the constants of the motion depend only on the absolute magnitudes of these values and not on their signs, then only terms containing even powers of $\delta x_1, \cdots \delta x_K$ appear in (20).[11] Thus, we get an expansion of the type

$$\begin{aligned} \Phi = {} & (t\epsilon_n + a + \delta x_0) N_{n-1} - tW_{n-1} + F + F'\delta x_0 \\ & + \tfrac{1}{2} \sum_{0}^{K} F_k''\delta x_k^2 + \tfrac{1}{6} F^{(3)} \delta x_0^3 + \tfrac{1}{2} \sum_{1}^{K} F_k^{(3)} \delta x_k^2 \delta x_0 \\ & + \frac{1}{24} \sum_{0}^{K} F_{kk}^{(4)} \delta x_k^4 + \tfrac{1}{8} \sum_{k \neq l} F_{kl}^{(4)} \delta x_k^2 \delta x_l^2 + \cdots, \end{aligned} \tag{22}$$

where N_{n-1} is the number of nucleon states of energy less than ϵ_n, W_{n-1} is the corresponding total energy, and the coefficients F are given by

$$\begin{aligned} F^{(p)} &= \sum_{i,\alpha} g_{i,\alpha} f_i^{(p)}, \\ F_k^{(p)} &= \sum_{i,\alpha} g_{i,\alpha} m_{k,\alpha}^2 f_i^{(p)}, \\ F_{kl}^{(p)} &= \sum_{i,\alpha} g_{i,\alpha} m_{k,\alpha}^2 m_{l,\alpha}^2 f_i^{(p)}, \text{ etc.} \end{aligned}$$

These coefficients are expressed by elementary functions in terms of the parameter t, and are easily computed, especially since at small excitation energy, only the terms associated with the state ϵ_n and its close neighbors contribute appreciably. We shall further need the following coefficients:

$$\begin{aligned} G^{(p)} &= \sum_{i,\alpha} (\epsilon_i - \epsilon_n) g_{i,\alpha} f_i^{(p)}, \\ G_k^{(p)} &= \sum_{i,\alpha} (\epsilon_i - \epsilon_n) g_{i,\alpha} m_{k,\alpha}^2 f_i^{(p)}, \text{ etc.} \end{aligned}$$

and

$$H^{(p)} = \sum_{i,\alpha} (\epsilon_i - \epsilon_n)^2 g_{i,\alpha} f_i^{(p)}, \text{ etc.}$$

The latter coefficients are related to the coefficients F by

$$\partial F_k^{(p)}/\partial t = -G_k^{(p+1)}, \quad \partial G_k^{(p)}/\partial t = -H_k^{(p+1)}, \text{ etc.}$$

Equations (9) defining the saddle point read, now,

$$A = N_{n-1} + F' + F''\delta x_0 + \tfrac{1}{2} \sum_{k=0}^{K} F_k^{(3)} \delta x_k^2 + \cdots, \tag{23}$$

$$M_k = F_k''\delta x_k + F_k^{(3)} \delta x_k \delta x_0 + \cdots. \tag{24}$$

The equation in E can be written after some simple transformations

$$U = G' + G''\delta x_0 + \tfrac{1}{2} \sum_{k=0}^{K} G_k^{(3)} \delta x_k^2 + \cdots, \tag{25}$$

where U is the excitation energy measured above the ground state

$$E^0 = W_{n-1} + \epsilon_n (A - N_{n-1}).$$

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[11] When S_z, T_ζ, Y_ζ are introduced at the same time, the three corresponding m values for a single nucleon are related by $m_S m_T m_Y = \frac{1}{8}$. From this correlation, it follows that the expansion (20) will contain additional terms such as, for example, a term in $\delta x_S \delta x_T \delta x_Y$, which will yield terms in $S_z T_\zeta Y_\zeta$ in the final level density formula. This minor correction will be omitted.

The limit of F' for large t is given by the contribution of the level ϵ_n, which is the only term independent of t. Thus, we see from (23) that for large t, δx_0 is proportional to $A-N_{n-1}-F'(t=\infty)$. Consequently, the condition that δx_0 should go to zero for large t reads

$$A-N_{n-1}=F'(t=\infty)=f_n'\sum_\alpha g_{n,\alpha}.$$

This condition defines a. As $f_n'=1/(1+e^{-a})$, we have

$$e^a=[A-N_{n-1}]/[\sum_\alpha g_{n,\alpha}-(A-N_{n-1})]. \qquad (26)$$

The quantity $A-N_{n-1}$ is the number of nucleons of energy ϵ_n in the ground state, and $\sum_\alpha g_{n,\alpha}$ is the total number of nucleon states of energy ϵ_n. Thus, e^a is equal to the number of nucleons occupying the level ϵ_n in the ground-state configuration divided by the number of holes in that same level.

We must now solve Eqs. (23), (24), and (25) giving t, δx_0, δx_k in terms of U, A, and M_k, and compute the entropy $S=Et-\sum_k x_kM_k-\Phi$, and the determinant D. The calculation is quite straightforward, and the resulting value of the entropy can be written

$$S=S_0+S_1-\sum_1^K M_k^2/2\sigma_k^2-\sum_1^K\sum_1^K B_{kl}M_k^2M_l^2, \qquad (27)$$

where the two first terms depend only on U and give the general level density, the third term gives the Gaussian law already obtained in the continuous theory and the fourth term gives the deviations from this simple law.

The term S_0 is a constant equal to the value of S in the limit $t\rightarrow\infty$, $M_k=0$; it is given by

$$S_0=a(N_{n-1}-A)+f_n\sum_\alpha g_{n,\alpha}.$$

Upon introducing the values of f_n and of a this becomes

$$S_0=(m+n)\log(m+n)-m\log m-n\log n, \qquad (28)$$

where m and n stand for the number of particles and of holes in the level ϵ_n in the ground-state configuration. In what follows, we shall give the first approximations only of the various quantities occurring in the level density formula. More complete expansions are given in the appendix.

The term S_1 is given by

$$S_1=Ut+\bar{F}-\bar{F}'^2/2F'', \qquad (29)$$

where

$$\bar{F}=\sum_\alpha\sum_{i\neq n} g_{i,\alpha}f_i, \quad \bar{F}'=\sum_\alpha\sum_{i\neq n} g_{i,\alpha}f_i'$$

are identical with F and F' except that the term coming from ϵ_n is omitted. The ratio $\bar{F}'/F''$ is a small quantity at low energy; it vanishes in the limit of $U\rightarrow 0$. Thus, terms proportional to the successive powers of $\bar{F}'/F''$ are correction terms of increasing order. The dependence of the excitation energy on t is given by

$$U=G'-G''(\bar{F}'/F''), \qquad (30)$$

and the dispersion coefficients by

$$\sigma_k^2=F_k''-F_k^{(3)}(\bar{F}'/F''). \qquad (31)$$

The coefficients B_{kl} of the fourth-order terms are very small in practical cases; their expression is given in the appendix.

The determinant D is readily computed because the off-diagonal terms are much smaller than the diagonal terms. One finds in the first approximation:

$$D=(1-G''^2/H''F'')(H''-H^{(3)}\bar{F}'/F'')\prod_{k=0}^K\sigma_k^2. \qquad (32)$$

The preceding derivation does not apply when the outer shell is complete in the ground state, since in this case (26) gives $a=+\infty$ or $a=-\infty$ depending on whether ϵ_n is the last filled level, or the first empty one. Let ϵ_n be the first empty level. It is natural, then, to replace (19) by

$$x_0=t\epsilon^0+a+\delta x_0,$$

where ϵ^0 is some energy between ϵ_{n-1} and ϵ_n, to be determined by the condition that $\delta x_0\rightarrow 0$ as $t\rightarrow\infty$. We have $A=N_{n-1}$, and the relation (23) giving δx_0 becomes

$$0=F'+F''\delta x_0+\cdots.$$

In the limit of large t we have

$$F'\approx\sum_\alpha g_{n,\alpha}\exp[-t(\epsilon_n-\epsilon^0)-a]-g_{n-1,\alpha}\exp[-t(\epsilon^0-\epsilon_{n-1})+a],$$

$$F''\approx\sum_\alpha g_{n,\alpha}\exp[-t(\epsilon_n-\epsilon^0)-a]+g_{n-1,\alpha}\exp[-t(\epsilon^0-\epsilon_{n-1})+a].$$

Consequently, if we choose ϵ^0 and a according to

$$\epsilon^0=(\epsilon_n+\epsilon_{n-1})/2, \quad e^{2a}=\sum_\alpha g_{n,\alpha}/\sum_\alpha g_{n-1,\alpha},$$

F' goes to zero much faster than F'', and δx_0 goes to zero. In conclusion, ϵ^0 has to be taken halfway between ϵ_n and ϵ_{n-1}, and e^{2a} is the ratio of the weight of ϵ_n to the weight of ϵ_{n-1}. With this modification, all the formulas given above apply, provided that in the definition (21) of the $f_i^{(p)}$ ϵ_n is replaced by ϵ^0. Of course, in the present case, all the functions F go to zero as $U\rightarrow 0$. In particular, $S_0=0$.

We shall finally mention the modifications which have to be made, if one wishes to calculate the number $N(E,M_k)$ of levels of energy less than, or equal to, E.

This quantity is immediately derived from (3):

$$N(E,M_k)=(2\pi i)^{-K-2}\int\cdots\int(dt/t)dx_0dx_1\cdots dx_K \times\exp(Et-\sum_0^K x_kM_k+\Phi).$$

This integral can be evaluated by the saddle point method, and the calculation is identical with that of the level density, except that Φ has to be replaced by

$\Phi-\log t$. The result is

$$N(U,M_k)=(2\pi)^{-(K+2)/2}D^{-\frac{1}{2}}t^{-1}e^{S}, \qquad (33)$$

where S is identical with the expression (27). However, the relation (30) between energy and temperature is replaced by[12]

$$U=1/t+G'-G''(\bar{F}'/F''); \qquad (34)$$

and in the calculation of D, H'' has to be replaced by $H''+1/t^2$.

As an example, let us compute the total number of levels of excitation energy $U=0$ in a system of A nucleons. The exact value is given by the binomial coefficient $\binom{m+n}{m}$, where m and n are the numbers of particles and of holes in the level ϵ_n in the ground-state configuration. On the other hand, we apply (33) taking $K=0$. In the limit $U\rightarrow 0$ we have

$$S=S_0+t(1/t+G'+\cdots)\rightarrow S_0+1,$$
$$D=(1/t^2+H''+\cdots)\sigma_0^2\rightarrow(1/t^2)F''(t=\infty).$$

From the definition of F'' we have

$$F''(t=\infty)=\sum_\alpha g_{n,\alpha}f_n''$$
$$=\tfrac{1}{4}\sum_\alpha g_{n,\alpha}\,\mathrm{sech}^2(a/2)=mn/(m+n),$$

where the value (26) for a has been used. Finally we get, after substitution of (28) for S_0,

$$N(0)=[e/(2\pi)^{\frac{1}{2}}]\times\text{Stirling approximation for }\binom{m+n}{m}$$
$$=1.084\times\text{Stirling approximation for }\binom{m+n}{m}.$$

It is seen that even in this extreme case the error introduced by the saddle point method is very small. The integration with respect to x_0 has had the effect of replacing the binomial coefficient by its Stirling approximation, whereas the integration with respect to t has introduced an error of the order of 8 percent.

IV. ANGULAR MOMENTA

In this section, we shall study the distribution of angular momenta among the nuclear levels.

The present experimental knowledge of the spins of the nuclear-excited states is very scarce except for a few low-lying states to which the statistical theory does not apply. Therefore, it is not possible to verify directly the theoretical predictions. It is, however, interesting to check the accuracy of the statistical treatment by comparing its results with those obtained by a direct counting of the levels performed on the same model. Thus, the discussion of the validity of the model itself will be left entirely aside. Critchfield and Oleksa[4] have given a complete account of the levels of energy less than 25 Mev for the nuclei with $A=20$, which permits an excellent test of the statistical method. Their model actually includes Majorana forces, but it will be shown in the next section that these forces do not change appreciably the distribution of angular momenta.

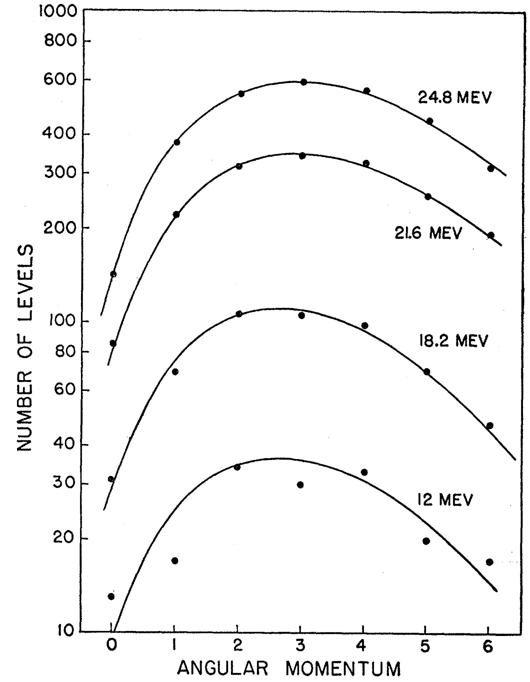

FIG. 1. Number of levels with a given angular momentum and an energy less than a given value. The dots represent the results of Critchfield and Oleksa corresponding to four different energies. The curves represent the function (36) with parameters adjusted for best fit at each energy.

The number N_J of the levels with total angular momentum J and energy less than E is given in terms of the number $N(M)$ of the levels with angular momentum M along the z axis by

$$N_J=N(J)-N(J+1). \qquad (35)$$

The dependence on M of the number of the levels with the angular momentum M along the z axis is given in the first approximation by a Gaussian law $N(M)\sim\exp(-M^2/2\sigma^2)$ [see (17) or (27) and (33)]. Hence, we have for the dependence on J of the number of levels with angular momentum J and energy up to a given value

$$N_J\sim(2J+1)\exp[-(J+\tfrac{1}{2})^2/2\sigma^2]. \qquad (36)$$

First, we shall verify that Critchfield's results obey this law.

The numbers[13] of levels of given angular momentum and of energy less than a given value have been plotted on Fig. 1 as functions of the angular momentum for four different energies together with the curves defined by (36). The parameter σ in (36) and the proportionality

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[12] Note here that the relation between energy and temperature depends on the integral which is computed.

[13] These numbers have been taken from Table IV in Critchfield and Oleksa (see reference 4).

factor have been adjusted each time for best fit. It is seen that the agreement is excellent as soon as the number of levels is sufficiently large. The errors are, in fact, of the order of magnitude of random fluctuations. The most probable angular momentum $J_m = \sigma - \frac{1}{2}$ exhibits a slow increase with the energy. Actually, the curves on Fig. 1 correspond to $J_m = 3.0$ for 24.8 and 21.6 Mev, and to $J_m = 2.82$ for 18.2 and 12 Mev. Below 12 Mev, J_m is not well defined.

For a discussion of the values of σ, the detailed features of the model have to be introduced. Critchfield and Oleksa assume that the nucleons move in a 16.8-Mev deep potential hole of radius $R = 1.45A^{\frac{1}{3}} \times 10^{-13}$ cm. The corresponding nucleon states are given in Table I, together with the statistical weights $\sum_\alpha g_{i,\alpha}$ and the sums of m^2. The latter sums include the contribution of the spin. The parameters δ and $\langle m^2 \rangle_{\text{Av}}$ of the continuous theory have to be derived from these data. For this purpose, it is convenient to first introduce the sums

$$N(\epsilon) = \sum_\alpha N_\alpha(\epsilon), \quad \sum_\alpha m_\alpha^2 N_\alpha(\epsilon), \tag{37}$$

where $N_\alpha(\epsilon)$ is the number of states of energy less than or equal to ϵ in the component α. The quantities $1/\delta$ and $\langle m^2 \rangle_{\text{Av}}/\delta$ are then the derivatives of the expressions (37) with respect to ϵ at $\epsilon = \epsilon^0$. The sums (37) can be deduced from the data of Table I when ϵ is equal to the energy ϵ_i of one of the nucleon states. For other values of ϵ, an interpolation procedure should be used. Moreover, there is some arbitrariness in the definition of the sums (37) even at the energies ϵ_i. A procedure which yields smooth plots consists in taking the sum of the contributions of the states below ϵ_i, plus half of the contribution of the state ϵ_i. Thus, we get five points for each function, through which we have to draw continuous curves (Fig. 2). The Fermi energy ϵ^0 is the value of ϵ at which $N(\epsilon)$ is equal to $A = 20$. The values of $1/\delta$ and $\langle m^2 \rangle_{\text{Av}}/\delta$ are then given by the slopes of the two curves at ϵ^0. Clearly, the definition of the latter quantities is not very precise, and this shows one of the weaknesses of the continuous approximation for light nuclei. Measurement of the slopes on Fig. 2 yields[14]

$$1/\delta = 1.40 \text{ Mev}^{-1}, \quad \langle m^2 \rangle_{\text{Av}}/\delta = 2.36 \text{ Mev}^{-1}. \tag{38}$$

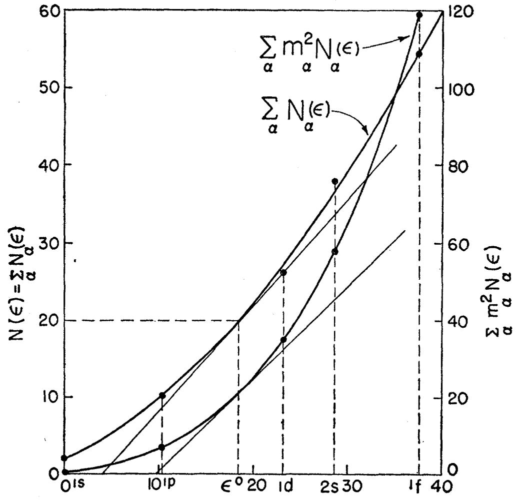

FIG. 2. Determination of the parameters of the continuous theory from the nucleon states. The dots indicate the values of the functions $N(\epsilon)$ and $\Sigma_\alpha m_\alpha^2 N_\alpha(\epsilon)$ at the energies of the five nucleon states given in Table I. The curves are interpolated between these points.

With these values, the relation (14) gives for the most probable angular momentum J_m

$$J_m = 1.24 U^{\frac{1}{4}} - \tfrac{1}{2}.$$

The plot of this function on Fig. 3 (curve a) shows that the values of J_m obtained by the present approximation are significantly smaller than the values deduced from the curves of Fig. 1.[15] The reason for this is easy to understand. The nuclei with $A = 20$ have an incomplete d shell with four particles. This is a very favorable situation for producing large angular momenta. An almost empty, or almost filled shell, or an outer shell of small angular momentum, would, on the contrary, yield exceptionally small angular momenta. The continuous theory, which ignores these peculiarities of the nuclei, must deviate one way or the other, depending on the ground state configuration. However, it predicts correctly the order of magnitude of J_m.[16]

We shall now discuss the values of J_m given by the theory of Sec. 3. This theory introduces directly the nucleon states described in Table I. Thus, the functions F, G, $\cdots$ become sums of five terms which are easily computed numerically. The energy ϵ_n is that of the $1d$ state. The outer shell has 4 particles and 16 holes. Hence, $e^a = \frac{1}{4}$. In the limit $U = 0$, $\bar{F}'$ vanishes. Therefore, the limit of σ is simply given by the contribution of the $1d$ state to F_J''. Thus,

$$\sigma^2(0) = \sum_\alpha m_\alpha^2 g_{n,\alpha} (e^{a/2} + e^{-a/2})^{-2}.$$

For the present nuclei, $\sigma^2(0) = 7.2$ and $J_m(0) = 2.3$. The values at higher excitation energies are easily computed by means of the expansions given in the appendix. The correction terms in σ^2 have three different origins and it is necessary to take the first few terms in the expansions of each kind of correction. These expansions are rapidly convergent. The final results are represented by the curve c on Fig. 3. They are in excellent agreement with the values corresponding to the four energies con-

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[14] Modifications of the procedure used in plotting the curves of Fig. 1, do not usually alter these values by more than 10 percent.

[15] Everywhere in the present discussion we have neglected the fact that the most probable angular momentum among the levels of energy less than or equal to E, is not exactly the same as the corresponding quantity for the levels of energy equal to E. However, the difference between the two quantities is very small because of the rapid increase of the level density with energy, and the slow variation of J_m.

[16] It has often been stated incorrectly in the literature that the statistical theory predicts much too many levels with very large angular momentum.

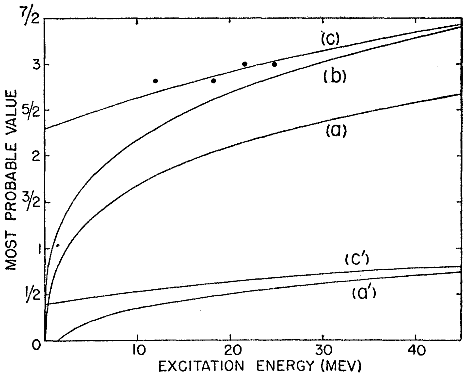

FIG. 3. The most probable angular momentum and spin or isotopic spin among the levels having an energy less than a given value. The dots indicate the values corresponding to Fig. 1. Curves (a) and (b) give the most probable angular momentum derived from the continuous theory, with the values of the parameters determined directly from the nucleon states and by means of the B.W.K. approximation, respectively. Curve (c) gives the values obtained from the method of Sec. 3. The curves (a′) and (c′) give the corresponding values of the most probable spin or isotopic spin. The curve (b′) would be very close to (c′), and, therefore, has been omitted.

sidered above. It should be noted that $J_m(0)$ is an important fraction of J_m in the interesting range of energy. The value of $J_m(0)$ depends entirely on the outer shell. It vanishes for a closed shell, and reaches a maximum when the outer shell is half-filled ($a=0$). For the $1d$ shell, the maximum value which occurs at $A=26$ is equal to $J_m=2.85$.

The coefficient B in (27), which gives the deviation of the distribution of the angular momenta from the Gaussian law, is easily computed by means of the expression given in the appendix. One finds that B has a very slow variation with energy, and is of the order of 10^{-5} at 25 Mev. Therefore, the predicted deviations from the Gaussian law are less than 2 percent for $J\leqslant 6$.

Finally, we shall return to the continuous theory for a more general discussion of the distribution of angular momenta among the nuclear levels. The discussion will be restricted here to nuclear models involving a spin-independent central potential $V(r)$. Then, the projections along the z axis m_l and m_s of the orbital momentum and of the spin of the nucleons are independent, and we have for the projection of the total angular momentum m the relation

$$\langle m^2\rangle_{\mathrm{Av}}=\langle m_l^2\rangle_{\mathrm{Av}}+\langle m_s^2\rangle_{\mathrm{Av}}=\langle m_l^2\rangle_{\mathrm{Av}}+\tfrac{1}{4}. \tag{39}$$

In the heavy nuclei, the contribution of the spin is small compared with the orbital term, since large orbital momenta occur near the top of the Fermi distribution. On the other hand, the orbital term is very simply related to the moment of inertia I of the nucleus[17] by

$$\langle m_l^2\rangle_{\mathrm{Av}}/\delta=\sum_{m_l} m_l^2\rho(\epsilon^0,m_l)=I/\hbar^2. \tag{40}$$

In this relation, I is the moment of inertia around an axis of the nucleus considered as a rigid body with a mass density at each point given by the usual expression for a Fermi gas in an external potential:

$$\mu(r)=4(4\pi/3)[2M(\epsilon^0-V)]^{\frac{3}{2}}Mh^{-3}.$$

The proof of the relation (40) starts from the B.W.K.formula,[18]

$$N_l(\epsilon^0)=(4/\pi)\int_{r_0}^{r_1} dr[(2M/\hbar^2)(\epsilon^0-V)-l^2/r^2]^{\frac{1}{2}},$$

giving the number of states of a nucleon with the angular momentum l and an energy less than ϵ^0. The density of states with a projection of the angular momentum on the z axis equal to m_l is given by

$$\rho(\epsilon^0,m_l)=\int_{|m_l|}^{l_{\max}} dl(d/d\epsilon^0)N_l(\epsilon^0).$$

An elementary integration yields, then, the identity

$$\int dm_l m_l^2\rho(\epsilon^0,m_l)=(2/3\hbar^2)\int_0^R d\tau r^2\mu(r);$$

where $d\tau$ is the volume element and R the nuclear radius defined as the distance at which V becomes equal to ϵ^0. This relation is equivalent to (40).

The moment of inertia of a nucleus of given mass and radius depends somewhat on the shape of the potential V. Potentials which attract the nucleons more strongly toward the center yield smaller moments of inertia. For example, the harmonic oscillator potential and the infinite square-well yield for I the values $\frac{1}{4}MR^2A$ and $\frac{2}{5}MR^2A$, respectively.

The same procedure gives for the parameter $1/\delta$ the value $1/\delta=(d/d\epsilon^0)N(\epsilon^0)$, where

$$N(\epsilon^0)=4(4\pi/3)h^{-3}\int_0^R d\tau[2M(\epsilon^0-V)]^{\frac{3}{2}}.$$

It is interesting to compare the values of the parameters of the continuous theory deduced from the B.W.K. approximation for $A=20$ with the values (38). One finds with the B.W.K. approximation $J_m=1.5U^{\frac{1}{4}}-\frac{1}{2}$ for the infinite square well, and $J_m=1.24U^{\frac{1}{4}}-\frac{1}{2}$ for the harmonic oscillator potential[19] (contribution of the spin included). The values corresponding to a finite well should lie between the later values. This shows that the B.W.K. approximation overestimates J_m, and, therefore, in the present case, it corrects to some extent the errors introduced by the continuous approximation. It follows that the values of J_m corresponding to an infinite well (curve b on Fig. 3) are more accurate than the values obtained with the parameters determined directly

TABLE I. Independent particles states for $A=20$.

State	$1s$	$1p$	$1d$	$2s$	$1f$
Energy ϵ_i (Mev)	0	10.3	23.1	28.5	37.6
$\sum_\alpha g_{i,\alpha}$	4	12	20	4	28
$\sum_\alpha m_\alpha^2 g_{i,\alpha}$	1	11	45	1	119

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[17] This relation is implicitly given by Bethe (see reference 1) for the special case of an infinite square-well potential.

[18] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), Sec. 28.

[19] The parameter of the potential is determined by the condition $V(R)=\epsilon^0$, where R is the nuclear radius.

from the nucleon states. This conclusion, however, does not hold for all values of A.

For the heavy nuclei, the contribution of the spin in (39) can be neglected in the first approximation. A simple physical interpretation of the relation (40) can then be obtained by introducing the quantity U^1 defined in Sec. 2. This quantity here is the excitation energy measured above the lowest nuclear level having an angular momentum M along the z axis. From (16) and (40) we obtain for the difference between U and U^1:

$$U^1 - U = -M^2\hbar^2/2I. \tag{41}$$

Thus U^1 is the remaining excitation energy when the kinetic energy due to the rotation of the nucleus as a rigid body around the z axis is subtracted from the total available excitation energy U.[20] It follows that although we have considered a pure independent-particle model, the distribution of angular momentum among the energy levels is the same as it would be for a rigid body. More specifically, this distribution could be obtained in the following manner: (a) we determine all the levels with $J=0$ (states of internal motion); (b) we deduce the levels with angular momentum J by adding the kinetic energy $J(J+1)\hbar^2/2I$ to each energy level with $J=0$.

This property, of course, does not imply that the Hamiltonian can be split into two nearly independent parts, one involving only the internal coordinates of the system, and one involving only the coordinates determining the position of the system in space. Such a separation is possible only in the case of an actual rigid body, and the detailed level scheme is then very different from that of an independent-particle model. This can easily be shown by considering a system consisting, for example, of four particles (α-particle model of O^{16}). For suitable kinds of interactions, the particles will form a tetrahedron which can rotate and vibrate. The behavior of the system depends, then, essentially on two parameters, its radius R, and the amplitude a of the zero-point vibrations. The spacing of the vibrational levels is of the order of $\hbar^2/Ma^2$, and that of the rotational levels of the order of $\hbar^2/MR^2$. In the case $a \ll R$, the system has a well-defined shape; it is a rigid body to a good approximation. The spacing of the vibrational levels is then very much larger than that of the rotational levels. Consequently, we shall observe well-separated groups of levels corresponding to the same vibrational state and different angular momenta. In the other extreme case where $a \approx R$, the particles are nearly independent. The spacing of the vibrational levels and of the rotational levels are then of the same order. Therefore, the groups of levels which existed in the case of the rigid body now overlap in such a way that it becomes actually impossible to distinguish between rotation and vibration. However, the statistical distribution of angular momentum among the levels is the same in both cases. It depends only on the moment of inertia and not on the ratio a/R.

V. THE EFFECT OF SYMMETRY

We shall now assume that, in addition to the central potential introduced previously, the nuclear potential energy contains symmetry-dependent terms. A simple approximate description of the dependence of the potential energy on the symmetry of the wave function is provided by Wigner's uniform model.[21]

Under the assumption that the nuclear forces do not depend on the spin and isotopic spin of the nucleons (pure Majorana and Wigner forces), the symmetry of the wave function of every nuclear level is characterized by three "partition quantum numbers"[22] $P \geqslant P' \geqslant |P''|$, and the symmetry-dependent part of the potential energy is given by

$$V_M = a_M[P(P+4) + P'(P'+2) + P''^2]. \tag{42}$$

From the discussion of the binding energies,[23] one finds for the value of the parameter a_M:

$$a_M = 20/A\,(\text{Mev}).$$

It is reasonable to assume that the same expression gives approximately the potential energy of the excited states.

Spin and isotopic spin-dependent forces can also be considered. Their contribution to the potential energy of the nucleus depends on the total spin or isotopic spin of the levels according to[21]

$$V_S = a_S S(S+1), \quad V_T = a_T T(T+1). \tag{43}$$

The first effect to be considered, is the displacement of the ground state. Since the excitation energy is measured from the ground state, one must in all formulas replace U by

$$U' = U + V^0, \tag{44}$$

where V^0 is the part of the potential energy of the ground state which is not included in the central potential. With the uniform model, for example,

$$V^0 = a_M[P^0(P^0+4) + P'^0(P'^0+2) + (P''^0)^2].$$

A consequence of this effect is the fact that among isobar nuclei, the level density increases with T_ζ for small T_ζ.[24] In Secs. 2 and 3, we have found that the level density was decreasing with T_ζ according to a Gaussian law. However, U' increases with T_ζ, and this results in an increase of the level density which for small T_ζ overbalances the Gaussian factor.

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[20] J. H. Jensen (private communication) has proposed to take this simple property as a starting point for the calculation of the distribution of angular momenta in statistical theory.

[21] E. Feenberg and E. P. Wigner, Phys. Rev. **51**, 95 (1937).

[22] E. P. Wigner, Phys. Rev. **51**, 106 (1937).

[23] E. P. Wigner, *University of Pennsylvania Bicentennial Conference* (University of Pennsylvania Press, Philadelphia, 1941).

[24] See, for example, Table VI in Critchfield and Oleksa (reference 4).

The second, more complex effect is the displacement of the excited states by variable amounts depending on their symmetry. In order to illustrate the method followed here, we shall first consider a potential energy of the type V_S, for example. The more realistic expression V_M is mathematically more complex to handle, and it will be discussed later.

Let $\rho(U')$ be the total level density when the spin-dependent potential energy is neglected. The density of states with given S_z can be expressed by a formula of the type

$$\rho(U',S_z)=(2\pi)^{-\frac{1}{2}}\sigma^{-1}\rho(U')\exp(-S_z^2/2\sigma^2), \tag{45}$$

where σ is an appropriate dispersion coefficient to be determined by the methods of Secs. 2 or 3. The density of states of spin S is then

$$\rho_S(U')=\rho(U',S)-\rho(U',S+1), \tag{46}$$

$$=(2\pi)^{-\frac{1}{2}}\sigma^{-3}(S+\tfrac{1}{2})\rho(U')\exp[-(S+\tfrac{1}{2})^2/\sigma^2]. \tag{47}$$

Under the influence of the spin-dependent forces, the energy of the states of spin S is increased by V_S. Consequently, the density of states of spin S at the excitation energy U becomes

$$\rho_S'(U')=\rho_S(U'-V_S). \tag{48}$$

If we want now the density of states of given S_z, we have to compute the sum

$$\rho'(U',S_z)=\sum_{S=|S_z|}^{\infty}\rho_S(U'-V_S), \tag{49}$$

and for the new total level density we get

$$\rho'(U')=\sum_{S=0}^{\infty}(2S+1)\rho_S(U'-V_S). \tag{50}$$

These summations can be carried out very simply in the first approximation. We shall assume that $\log\rho(U')$ is a sufficiently smooth function over the interval V_S so that we can write[25]

$$\log\rho(U'-V_s)=\log\rho(U')-V_s d\log\rho(U')/dU'. \tag{51}$$

Actually, V_S becomes arbitrarily large with increasing S. However, if a_S is sufficiently small, the approximation (51) fails only for values of S so large that the density of the corresponding levels is practically zero. Thus, the condition for the validity of (51) is that the third term in the Taylor expansion should be very small for $S+\frac{1}{2}\approx\sigma$. This reads

$$a_S^2\sigma^4|(d/dU')^2\log\rho(U')|\ll 1.$$

Substitution of (51) into (48) yields

$$\begin{aligned}\rho_S'(U')=&(2\pi)^{-\frac{1}{2}}\sigma^{-3}(S+\tfrac{1}{2})\rho(U')\\&\times\exp[-(S+\tfrac{1}{2})^2/2\sigma^2\\&-a_S S(S+1)d\log\rho(U')/dU']\\=&(2\pi)^{-\frac{1}{2}}\sigma^{-3}(S+\tfrac{1}{2})\rho(U')C\exp[-(S+\tfrac{1}{2})^2/2\sigma'^2],\end{aligned} \tag{52}$$

where the new dispersion coefficient σ' is given by

$$\sigma'=\sigma(1+2a_S\sigma^2 d\log\rho(U')/dU')^{-\frac{1}{2}}, \tag{53}$$

and where C is the correction factor

$$C=\exp[(a_S/4)d\log\rho(U')/dU']. \tag{54}$$

This factor which originates in the replacement of $S(S+1)$ by $(S+\frac{1}{2})^2$ is usually very near unity. Since the expression (52) for ρ_S' has the same dependence on S as ρ_S, we can immediately go back to the total level density and to the density of states of given S_z. One finds immediately from (45) and (47) that

$$\begin{aligned}\rho'(U')&=C(1+2a_S\sigma^2 d\log\rho/dU')^{-3/2}\rho(U'),\\\rho'(U',S_z)&=(2\pi)^{-1/2}\sigma'^{-1}\rho'(U')\exp(-S_z^2/2\sigma'^2).\end{aligned} \tag{55}$$

It should be noted that σ is a function of U', and that strictly speaking U' should be replaced by $U'-V_S$ also in σ. However, σ is a very slowly varying function of the energy and this would only introduce a very small correction. The above expressions simplify if we use the results of the continuous theory. We have then, roughly:

$$\sigma^2=(1/4\pi)(6U'/\delta)^{\frac{1}{2}},$$

$$d\log\rho(U')/dU'\cong\pi(6U'\delta)^{-\frac{1}{2}}.$$

Hence,

$$1+2a_S\sigma^2 d\log\rho/dU'=1+a_S/2\delta, \tag{56}$$

which is independent of the energy. Thus, it is seen from (53) and (55) that in the present approximation the dispersion coefficient and the total level density are simply multiplied by constant factors, except for the small additional correction of the latter quantity due to C.

In the case of the Majorana forces we have to consider the constants of motion S_z, T_ζ, Y_ζ. The density of levels with given values of these quantities can be written

$$\begin{aligned}\rho(U',S_z,T_\zeta,Y_\zeta)=&(2\pi)^{-\frac{3}{2}}\sigma^{-3}\rho(U')\\&\times\exp[-(S_z^2+T_\zeta^2+Y_\zeta^2)/2\sigma^2].\end{aligned} \tag{57}$$

The problem of deducing from (57) the density of levels with given symmetry character has been treated by Bardeen and Feenberg.[26] The result is more conveniently expressed in terms of the partition numbers Λ_1, Λ_2, Λ_3, Λ_4. These numbers are related to the partition quantum numbers by

$$\begin{aligned}2P&=\Lambda_1+\Lambda_2-\Lambda_3-\Lambda_4,\\2P'&=\Lambda_1-\Lambda_2+\Lambda_3-\Lambda_4,\\2P''&=\Lambda_1-\Lambda_2-\Lambda_3+\Lambda_4,\\A&=\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4.\end{aligned} \tag{58}$$

Similarly, we replace S_z, T_ζ, Y_ζ by λ_1, λ_2, λ_3, λ_4 such

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[25] The validity of this approximation is clearly shown by the fact that the curves on Fig. 4 are almost straight lines over large intervals of energy.

[26] J. Bardeen and E. Feenberg, Phys. Rev. 54, 809 (1938).

that

$$\begin{aligned} 2S_z &= \lambda_1+\lambda_2-\lambda_3-\lambda_4, \\ 2T_\zeta &= \lambda_1-\lambda_2+\lambda_3-\lambda_4, \\ 2Y_\zeta &= \lambda_1-\lambda_2-\lambda_3+\lambda_4, \\ A &= \lambda_1+\lambda_2+\lambda_3+\lambda_4. \end{aligned}$$

With the new notation, the relation (57) reads

$$\rho(U',\lambda_1,\lambda_2,\lambda_3,\lambda_4) = (2\pi)^{-\frac{3}{2}}\sigma^{-3}\rho(U') \exp[(\tfrac{1}{4}A^2-\sum_1^4 \lambda_i^2)/2\sigma^2]. \quad (59)$$

Let E_i ($i=1,2,3,4$) be a set of operators which increase the variables λ_i by one unit according to

$$E_1^n f(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = f(\lambda_1+n,\lambda_2,\lambda_3,\lambda_4), \text{ etc.}$$

The density of levels with the symmetry defined by the partition numbers $(\Lambda)=(\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4)$ is then given by[27]

$$\rho_{(\Lambda)}(U') = \begin{vmatrix} 1 & E_2^{-1} & E_3^{-2} & E_4^{-3} \\ E_1 & 1 & E_3^{-1} & E_4^{-2} \\ E_1^2 & E_2 & 1 & E_4^{-1} \\ E_1^3 & E_2^2 & E_3 & 1 \end{vmatrix} \times\rho(U',\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4). \quad (60)$$

This relation is the generalization of (46). It involves a sum of 24 terms, and hence it is not very easy to handle in general. To a good approximation, however, if the function in the right-hand side of (60) is sufficiently smooth, we can replace the operators E_i by differential operators according to

$$E_i^n = 1+n(\partial/\partial\lambda_i)+(n^2/2\,!)(\partial/\partial\lambda_i)^2+\cdots. \quad (61)$$

For greater accuracy, it is desirable to transform the determinant (60) in such a way as to make the arguments n in the Taylor series as small as possible. Therefore, we rewrite the determinant as follows:

$$E_1^{\frac{3}{2}}E_2^{\frac{1}{2}}E_3^{-\frac{1}{2}}E_4^{-\frac{3}{2}} \begin{vmatrix} E_1^{-\frac{3}{2}} & E_2^{-\frac{3}{2}} & E_3^{-\frac{3}{2}} & E_4^{-\frac{3}{2}} \\ E_1^{-\frac{1}{2}} & E_2^{-\frac{1}{2}} & E_3^{-\frac{1}{2}} & E_4^{-\frac{1}{2}} \\ E_1^{\frac{1}{2}} & E_2^{\frac{1}{2}} & E_3^{\frac{1}{2}} & E_4^{\frac{1}{2}} \\ E_1^{\frac{3}{2}} & E_2^{\frac{3}{2}} & E_3^{\frac{3}{2}} & E_4^{\frac{3}{2}} \end{vmatrix}.$$

We now replace in the new determinant the E_i by the expansions (61). After a few algebraic manipulations, we obtain as the first nonvanishing term in the density of states of given symmetry character:

$$\rho_{(\Lambda)}(U') = E_1^{\frac{3}{2}}E_2^{\frac{1}{2}}E_3^{-\frac{1}{2}}E_4^{-\frac{3}{2}} \begin{vmatrix} 1\frac{1\,0}{1\,1} & 1\frac{1\,0}{1\,1} & 1\frac{1\,0}{1\,1} & 1\frac{1\,0}{1\,1} \\ \partial_1^1 & \partial_2^1 & \partial_3^1 & \partial_4^1 \\ \partial_1^2 & \partial_2^2 & \partial_3^2 & \partial_4^2 \\ \partial_1^3 & \partial_2^3 & \partial_3^3 & \partial_4^3 \end{vmatrix} \times\rho(U',\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4),$$

where ∂_i^m stands for $(\partial/\partial\lambda_i)^m$. If we use for $\rho(U',\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ the expression (59), we get[28]

$$\begin{aligned} \rho_{(\Lambda)}(U') = {} & (2\pi)^{-\frac{3}{2}}\sigma^{-15}E_1^{\frac{3}{2}}E_2^{\frac{1}{2}}E_3^{-\frac{1}{2}}E_4^{-\frac{3}{2}}(\Lambda_1-\Lambda_2) \\ & \times(\Lambda_1-\Lambda_3)(\Lambda_1-\Lambda_4)(\Lambda_2-\Lambda_3)(\Lambda_2-\Lambda_4)(\Lambda_3-\Lambda_4) \\ & \times\rho(U') \exp[(\tfrac{1}{4}A^2-\sum_1^4 \Lambda_i^2)/2\sigma^2]. \end{aligned}$$

By transforming this expression back into the notation of the partition quantum numbers, we finally obtain

$$\begin{aligned} \rho_{PP'P''}(U') = {} & (2\pi)^{-\frac{3}{2}}\sigma^{-15}[(P+4)^2-(P'+1)^2] \\ & \times[(P+2)^2-P''^2][(P'+1)^2-P''^2]\rho(U') \\ & \times\exp\{-[(P+2)^2+(P'+1)^2+P''^2]/2\sigma^2\}. \end{aligned}$$

This expression is the required generalization of (46). It gives the distribution of symmetry types among the levels.

The Majorana forces increase the energy of levels with the partition quantum numbers P, P', P'' by the amount V_M given by (42). Hence, the density of these levels becomes

$$\rho'_{PP'P''}(U') = \rho_{PP'P''}(U'-V_M). \quad (63)$$

The density of the levels with given S_z, T_ζ, Y_ζ is now

$$\rho'(U',S_z,T_\zeta,Y_\zeta) = \sum_{P,P',P''} N_{PP'P''}(S_z,T_\zeta,Y_\zeta) \times\rho_{PP'P''}(U'-V_M), \quad (64)$$

where $N_{PP'P''}(S_z,T_\zeta,Y_\zeta)$ is the number of states with given values of S_z, T_ζ, Y_ζ in the supermultiplet P, P', P''. The total level density is similarly

$$\rho'(U') = \sum_{P,P',P''} N_{PP'P''}\rho_{PP'P''}(U'-V_M), \quad (65)$$

where $N_{PP'P''}$ is the total number of states in the supermultiplet P, P', P''. For a very accurate calculation, the sums (64) or (65) should be carried out numerically. However, if $\rho(U')$ and a_M are such that an expression similar to (51) can be used, the dependence of $\rho'_{PP'P''}$ on $P,P'P''$ is the same as that of $\rho_{PP'P''}$. Hence, we can immediately deduce $\rho'(U',S_z,T_\zeta,Y_\zeta)$ and $\rho'(U')$. The new coefficient of dispersion is given by

$$\sigma' = \sigma[1+2a_M\sigma^2 d\log\rho(U')/dU']^{-\frac{1}{2}}, \quad (66)$$

and the total level density becomes

$$\rho'(U') = C[1+2a_M\sigma^2 d\log\rho(U')/dU']^{-15/2}\rho(U'), \quad (67)$$

where C is now

$$C = \exp[(5a_M/2)d\log\rho(U')/dU']. \quad (68)$$

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[27] This relation is to be compared with Eq. (13) of Bardeen and Feenberg (reference 26) where the expansion of the determinant given here is written explicitly.

[28] Each term of the determinant is the product of a first-order derivative, a second-order derivative, and a third-order derivative with respect to the λ_i. These derivatives introduce the factors σ^{-2}, σ^{-4}, σ^{-6}, respectively. A factor σ^{-3} is present in the expression (59) for $\rho(U',\lambda_1,\lambda_2,\lambda_3,\lambda_4)$. Thus, altogether one gets the factor σ^{-15}.

Finally, the density of levels with given S_z, T_ζ, Y_ζ is

$$\rho'(U', S_z, T_\zeta, Y_\zeta) = (2\pi)^{-\frac{3}{2}}\sigma'^{-3}\rho'(U') \times\exp[-(S_z^2+T_\zeta^2+Y_\zeta^2)/2\sigma'^2]. \quad (69)$$

If the continuous approximation is used, the above expressions become

$$\sigma' = \sigma(1+a_M/2\delta)^{-\frac{1}{2}},$$
$$\rho'(U') = C(1+a_M/2\delta)^{-15/2}\rho(U'), \quad (70)$$
$$C = \exp[(5\pi a_M/2)(6U'\delta)^{-\frac{1}{2}}],$$

and the relation (69) remains unchanged.

It should be mentioned that the three kinds of forces considered above can be introduced simultaneously. The procedure consists then in taking into account first the Majorana forces, and in introducing the effect of the spin and isotopic spin-dependent forces into the formulas (67) or (70). As a result, the level density is multiplied by three factors corresponding to the three kinds of forces. The dispersion occuring in the factors (55) caused by the spin and isotopic spin-dependent forces has to be replaced by the expression (66).

As a first application, we can estimate the effect of Majorana forces, for example, on the distribution of angular momentum. It is clear that, instead of computing directly as in Sec. 4 the density of the levels having a total angular momentum M along the z axis, one can proceed in two steps. First, one determines the density of the levels having an orbital momentum M_L and a spin M_S along the z axis. This yields a two-variables Gaussian distribution with two dispersions σ_L and σ_S. The second step consists in deducing from the two-variables distribution law the density of the levels for which M_L+M_S has a given value M. The result is a Gaussian law with a dispersion σ_J given by

$$\sigma_J^2 = \sigma_L^2+\sigma_S^2. \quad (71)$$

This procedure gives, of course, the same result as the method of Sec. 4. If Majorana forces are introduced, the Gaussian distribution law of M_L and M_S remains valid, with the only change that σ_S is replaced by σ_S' according to (66) or (70). However, the spin contribution σ_S to the dispersion of the total angular momentum is usually much smaller than the orbital term σ_L. Therefore, the substitution of σ_S' into (71) introduces only a small correction of σ_J. For the nuclei of mass $A=20$, the continuous theory yields

$$\sigma_L^2/\sigma_S^2 = \langle m_L^2\rangle_{\mathrm{Av}}/\langle m_S^2\rangle_{\mathrm{Av}} = 6.7,$$

when the values (38) of the parameters are used. With $a_M=1$, we get from (70):

$$\sigma_S^2/\sigma_S'^2 = 1.7.$$

One finds then:

$$(\sigma_J-\sigma_J')/\sigma_J = 0.03. \quad (72)$$

The level density computed by means of the method of Sec. 3, increases more slowly with energy than that obtained from the continuous theory (see, for example, Fig. 4). It follows from (66) that the correction to σ_J due to the Majorana forces, is somewhat smaller in the more exact treatment than expressed by (72). Actually, it is of the order of 2 percent at 25 Mev.

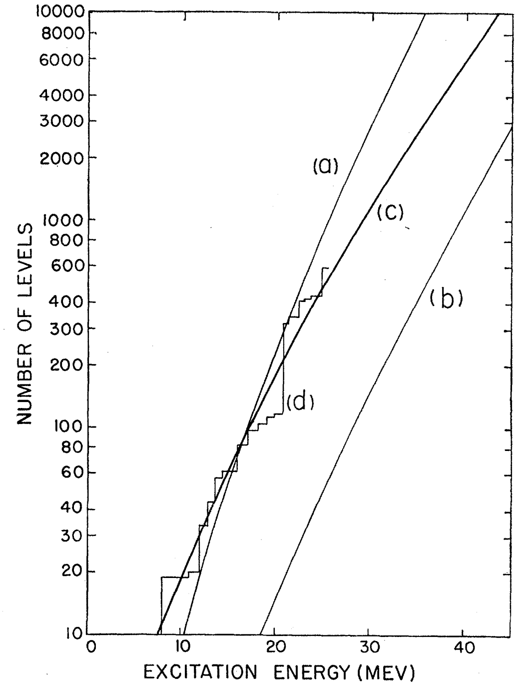

FIG. 4. The number of levels having an energy less than a given value and the most probable angular momentum. Curves (a), (b), and (c) correspond to the continuous theory neglecting the Majorana forces (Bethe), to the continuous theory with Majorana forces included, and to the theory of Sec. 3 with Majorana forces included. The results of Critchfield and Oleksa are represented by (d).

Finally, we have to discuss the absolute values of the level density. Therefore, the numbers of levels of energy less than a given value and *having the most probable angular momentum* corresponding to each energy have been plotted on Fig. 4. These numbers refer to levels of Ne^{20} only, i.e., with $T_\zeta=0$. The corresponding theoretical expression is

$$(2\pi)^{-1}\sigma_J^{-2}\sigma_T^{-1}N(U)e^{-\frac{1}{2}}, \quad (73)$$

where $N(U)$ is the total level density computed according to the procedure indicated at the end of Sec. 3, σ_J and σ_T are the dispersion coefficients of angular momentum and isotopic spin, respectively. For the present case $U'=U$. The results obtained from the continuous theory, without taking into account the Majorana forces (curve *a*), are in good agreement with the results of the explicit counting as was already mentioned by Critchfield and Oleksa.[4] The Majorana forces, however, according to (70) and (73) reduce the number of levels by a factor of the order of $(1.7)^7\approx 40$, and the resulting level density is then completely wrong (curve *b*). The method of Sec. 3, however, gives very much larger level densities than the continuous theory. This is essentially due to the term S_0 in the entropy. This term, which does not exist in the continuous theory, is of the order of 10 in the present case. Therefore, the level density is much larger at low excitation energy in this method than in the continuous approximation. The explanation of this larger density is that the nucleons in the incomplete 1*d* shell can be rearranged in many different ways with no excitation energy. On the other hand, of course, the level density obtained from the method of Sec. 3 increases more slowly with energy, since the two theories should agree at very high energy. However, the energy at which the agreement occurs is very much outside the range of energy considered here. When the Majorana forces are introduced into the theory of Sec. 3, the results agree very well with those of Critchfield and Oleksa (curve *c*).

Thus, it is seen that the agreement with the exact counting of the continuous theory without Majorana is the result of compensation of two very large errors. This does not occur for other nuclei, or even for a different strength of the Majorana forces. Moreover, even in Ne^{20}, the simplified theory does not yield exact results above 25 Mev.

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APPENDIX

For accurate calculations, additional terms in the expansions of Sec. 3 are necessary. These are easily obtained from (22) by the derivation outlined in the latter section. Thus one gets for S_1, U, and D:

$$\begin{aligned}S_1=Ut+\bar{F}-(\bar{F}'^2/2F'')-(\bar{F}'^3/6F''^3)\\ -(\bar{F}'^4F^{(3)2}/8F''^5)+(\bar{F}'^4F^{(4)}/24F''^4)+\cdots,\end{aligned}\tag{29'}$$

$$\begin{aligned}U=G'-(\bar{F}'G''/F'')+(\bar{F}'^2/2F''^3)(1+\bar{F}'F^{(3)}/F''^2)\\ \times(F''G^{(3)}-F^{(3)}G'')-(\bar{F}'^3/6F''^4)\\ \times(F''G^{(4)}-F^{(4)}G'')+\cdots,\end{aligned}\tag{30'}$$

$$\begin{aligned}D=[1-(G''-\bar{F}'G^{(3)}/F''^2)^2/(F''-\bar{F}'F^{(3)}/F'')\\ \times(H''-\bar{F}'H^{(3)}/F'')][H''-(\bar{F}'/F'')\\ \times(1+\bar{F}'F^{(3)}/2F''^2)H^{(3)}+(\bar{F}'/F'')^2(H^{(4)}/2)]\\ \times\prod_{k=0}^{K}\sigma_k^2.\end{aligned}\tag{32'}$$

The expansion of σ_k^2 includes terms of different kinds. The terms of the first kind are the coefficients of the terms in M_k^2 appearing in the entropy when this quantity is computed as a function of t and M_k. The corresponding expression for σ_k is

$$\begin{aligned}\sigma_k=F_k''-(\bar{F}'F_k^{(3)}/F'')(1+\bar{F}'F^{(3)}/2F''^2)\\ +\bar{F}'^2F_k^{(4)}/2F''^2+\cdots.\end{aligned}\tag{31'}$$

However, the entropy should be computed as a function of U and not of t. Since t is a function of U and M_k, it is seen that the entropy depends also implicitly on M_k through t. This yields the following correction which should be added to (31′):

$$\delta\sigma_k=(\bar{F}'G''/F''^2H'')(F''G_k^{(3)}-G''F_k^{(3)})+\cdots.$$

In an accurate calculation one finds that the determinant D depends also on the M_k. We can then write the determinant as

$$D(1-\sum_1^K\alpha_kM_k^2+\cdots),$$

where the α_k are small. The final formula for the level density is multiplied by the power $-\frac{1}{2}$ of the determinant. This introduces the factor

$$(1-\sum_1^K\alpha_kM_k^2)^{-\frac{1}{2}}\cong\exp(\sum_1^K\alpha_kM_k^2/2),$$

which can be considered as producing an additional correction $\delta^{(1)}\sigma_k$ of σ_k. This correction is given by

$$(\sigma_k+\delta\sigma_k+\delta^{(1)}\sigma_k)^{-2}=(\sigma_k+\delta\sigma_k)^{-2}-\alpha_k,$$

or

$$\delta^{(1)}\sigma_k=(\alpha_k/2)(\sigma_k+\delta\sigma_k)^3.$$

The α_k are given by

$$\begin{aligned}F_k''^2\alpha_k=F_k^{(3)}H^{(3)}/2(H''F''-\bar{F}'H^{(3)})\\ +\sum_{l=1}^{K}(F_k^{(3)}F_l^{(3)}/F''-F_{kl}^{(4)})/2(F_l''-\bar{F}'F_l^{(3)}/F'')\\ +[F_k^{(3)}(1+\bar{F}'F_k^{(3)}/F''F_k'')-\bar{F}'F_k^{(4)}/F'']^2/\\ \times(F_k''-\bar{F}'F_k^{(3)}/F'')(F''-\bar{F}'F^{(3)}/F'')\\ +[G_k^{(3)}(1+\bar{F}'F_k^{(3)}/F''F_k'')-\bar{F}'G_k^{(4)}/F'']^2/\\ \times(F_k''-\bar{F}'F_k^{(3)}/F'')(H''-\bar{F}'H^{(3)}/F'')\\ -2(G''-\bar{F}'G^{(3)}/F'')[(F_k^{(3)}G^{(3)}/F'')-(G_k^{(4)}/2)]/\\ \times(F''-\bar{F}'F^{(3)}/F'')(H''-\bar{F}'H^{(3)}/F'').\end{aligned}$$

Here, the two first lines are the contributions from the diagonal terms in the determinant, the other lines are the contributions of the terms $\partial^2\Phi/\partial x_0\partial x_k$, $\partial^2\Phi/\partial t\partial x_k$, $\partial^2\Phi/\partial x_0\partial t$, respectively. It should be mentioned, finally, that a very small correction is produced by the dependence of D on M_k implicitly through t.

The coefficients B_{kl} in (27) are given in the first approximation by

$$F_k''^2F_l''^2B_{kl}=-(1/8F'')F_k^{(3)}F_l^{(3)}+\begin{cases}F_{kl}^{(4)}/8 & \text{if } k\neq l,\\ F_{kk}^{(4)}/24 & \text{if } k=l.\end{cases}$$