Indeed it can be shown that, if the baryon core would satisfy Fierz-Pauli's theory of spin-$\frac{3}{2}$ particles,[18] one finds $k=\frac{2}{3}$ for such particles.[19]

In conclusion we may say that, until there is some further experimental evidence that spin-$\frac{3}{2}$ particles really exist, there may be little reason for accepting Sugawara's suggestion at all. If the need of consideration of (baryon+pion)-states would arise, contributions from such states to the nucleon magnetic moment should be calculated by the methods outlined above, and more likely with the value $k=\frac{2}{3}$ than with $k=2$.

[18] M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, 211 (1939).

[19] F. J. Belinfante, Phys. Rev. **92**, 994 (1953).

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Intrinsic Magnetic Moment of Elementary Particles of Spin $\frac{3}{2}$

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Fierz-Pauli's theory of spin-$\frac{3}{2}$ particles has been reformulated in a manner somewhat resembling the usual formulation of Dirac's equation for the electron. The discussion is simplified by complete reduction of the representation of the spatial rotation and reflection group by the field. The dependent variables can then be expressed in terms of the spin-$\frac{3}{2}$ field. The magnetic moment and the gyromagnetic ratio of "bare" spin-$\frac{3}{2}$ particles of charge q and mass m are found to be $(q\hbar/2mc)$ and $(q/3mc)$, respectively.

1. INTRODUCTION

IN a theory of the anomalous magnetic moments of nucleons, Sugawara[1] has recently tacitly assumed that the intrinsic magnetic moment of a spin-$\frac{3}{2}$, isobaric-spin-$\frac{3}{2}$ particle of slightly more than nucleon mass (a so-called *baryon*[2,3]) should be about six nuclear magnetons in its state of charge $2e$. Pauli and Dancoff's strong-coupling theory[4] of the excited states of the nucleoid[2] (=nucleore-pion system[5]) predicts a magnetic moment proportional to $(q-\frac{1}{2}e)/(j+1)$, which would make the total magnetic moment of a baryon of charge $2e$ (and with $j=\frac{3}{2}$) equal to 1.8× the magnetic moment of a proton ($j=\frac{1}{2}$). However, the Pauli-Dancoff theory of the magnetic moments of nucleoids is not only not trustworthy, as shown by its prediction that the neutron magnetic moment would be opposite and equal to the proton magnetic moment, but probably it is not even applicable in a theory like Sugawara's, in which the nucleon is assumed to be part of its time a nucleore, part of its time a nucleore with a pion cloud, and part of its time a baryon core and pion(s). Such an assumption becomes rather improbable, if one does not at the same time assume the baryon (core particle) to be an elementary particle itself,[3] like the proton-nucleore, neutron-nucleore, and pions figuring in Sugawara's theory. That is, we would have to assume that there is such a thing as a "bare elementary particle" of spin $\frac{3}{2}$, into which a nucleore could be transformed under emission or absorption of a pion.

The "bare baryon" would then have an intrinsic magnetic moment of its own—like the proton and neutron as "nucleores" are supposed to have magnetic moments of 1 and 0 nucleore magnetons respectively. The question then arises whether the magnetic moment to be expected for such bare baryon would have so large a value as assumed by Sugawara. We have reasoned that this is unlikely, and that it seems more plausible to guess that the gyromagnetic ratio of a spin-$\frac{3}{2}$ particle of charge q and mass m will be $q/3mc$, and its intrinsic magnetic moment $q\hbar/2mc$. (See reference 3.) It is the purpose of this paper to show[6] that this conjecture is correct, if for a "bare" particle of spin $\frac{3}{2}$ in interaction with an external electromagnetic field one assumes Fierz-Pauli's theory of such particles to be valid.[7]

2. FIELD COMPONENTS FOR ELEMENTARY PARTICLES OF SPIN $\frac{3}{2}$

For their Lorentz-covariant theory of particles of spin $\frac{3}{2}$ in interaction with a Maxwell field, Fierz and Pauli[7] formulated the field equations in a manifestly covariant form using spinor notation. The field has 16 complex components (not counting their conjugates). Between these 16 field components there are 8 relations ("subsidiary equations") not involving differentiation with respect to time, so that at some fixed initial time only eight field components can be chosen independently.

[1] M. Sugawara, Progr. Theoret. Phys. Japan **8**, 549 (1952). For a criticism of this theory see reference 3.

[2] F. J. Belinfante, Phys. Rev. **92**, 145 (1953).

[3] F. J. Belinfante, this issue Phys. Rev. **92**, 994 (1953).

[4] W. Pauli and S. M. Dancoff, Phys. Rev. **62**, 85 (1942).

[5] Nucleore=bare nucleon core; see R. G. Sachs, Phys. Rev. **87**, 1100 (1952).

[6] Without committing ourselves as to the value of Sugawara's suggestion. See also reference 3.

[7] M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, 211 (1939).

Once the relativistic covariance of a theory has been established, it often has certain advantages[8] to drop covariant notation.[9] One reason for this is the special part taken by the time in solving problems using an ordinary Schrödinger equation with a Hamilton operator. Also, in canonical quantization one likes to distinguish "derived variables" from "canonical variables,"[10] and this requires selection of a time axis (or of a time-like unit vector n^μ, in Schwinger's terminology[11]).

A further most important advantage of dropping manifest covariance is the following. The transformation of the spinor components of the field form a (two-valued) representation of the complete Lorentz group (including spatial reflections); therefore, also of its subgroup of spatial rotations and reflections. While the representation of the Lorentz group by the pair of symmetrical 3-index spinors used by Fierz and Pauli is irreducible,[12] the representation of the subgroup of spatial rotations and reflections is not. Therefore, it is natural to group the sixteen components of the field together in the following fashion: (1) An *undor*,[13] ($\gamma_5\chi$ in Eq. (1)), that is, a pair of a Kramers spinor and spin-conjugate spinor,[14] together transforming like a Dirac electron wave function in Kramers representation,[13–14] in particular transforming into each other under spatial reflection;[15] this undor corresponds to Fierz-Pauli's spinors c^α and $d_{\dot\alpha}$; (2) another such pair of Kramers spinor and spin-conjugate spinor, (the undor $\gamma_5\eta$ below); and (3) an 8-component quantity, consisting of a pair of what one might call a "four-spinor" and a "spin-conjugate four-spinor" (forming the representation $\mathfrak{D}^{(\frac{3}{2})}$ of the spatial rotation group;[12] compare $Z_1\cdots Z_8$ in Eq. (1) below). Such complete reduction of the representation of the spatial group by the field separates the dynamical field components Z forming the spin-$\frac{3}{2}$ field, from the dependent field components χ and η transforming as fields of spin-$\frac{1}{2}$ particles, and thus reduces the equations of motion and the subsidiary equations to a form in which they are most easily handled.

For the details of Fierz-Pauli's theory we refer to their original publication.[7] If under spatial reflection $x, y, z, ct \to -x, -y, -z, +ct$ (thence $\mathbf{p}\to-\mathbf{p}$, $p_0\to+p_0$ for the momentum in vector notation, or $p_{\dot\beta}{}^{\alpha}\to -p_{\dot\alpha}{}^{\beta}$ in spinor notation) we assume Fierz-Pauli's spinor $d_{\dot\alpha}$ to transform into ϵc^α (with $\epsilon^4=1$, see reference 15), the invariance of their Lagrangian requires transformation of their c^α into $\epsilon d_{\dot\alpha}$, of their $a_{\dot\alpha}{}^{\beta\nu}$ into $-\epsilon b^{\alpha}{}_{\dot\beta\dot\nu}$, and of their $b^{\alpha}{}_{\dot\beta\dot\nu}$ into $-\epsilon a_{\dot\alpha}{}^{\beta\nu}$. After reduction of the six-component representation of the spatial rotation group by $a_{\dot\alpha}{}^{\beta\nu}(\equiv a_{\dot\alpha}{}^{\nu\beta})$, into a representation by quartet and doublet states (by a "four-spinor" and an ordinary (two-)spinor respectively), and similarly for $b^{\alpha}{}_{\dot\beta\dot\nu}$, we shall find it convenient to introduce linear combinations of the two-spinor $d_{\dot\beta}$ with the two-spinor representing the doublet part in $a_{\dot\alpha}{}^{\beta\nu}$. [Compare η_1 and η_2 in Eq. (1).] Then, for ensuring complete reduction with respect to spatial reflections, we should similarly combine c^β with the corresponding two-spinor part in $-b^{\alpha}{}_{\dot\beta\dot\nu}$. [See $-\eta_3$ and $-\eta_4$ in Eq. (1).] Following Van der Waerden's interpretation of the dotted and undotted spinor indices,[16] and using Kramers' representation[17] of Dirac wave functions or undors,[13] we can identify $d_{\dot 1}$, $d_{\dot 2}$ with the first two components of such undor, and c^1, c^2 with the third and fourth component of such undor. To be consequent, we should then alter Fierz and Pauli's definition of $\sigma^4=iI$ into $\sigma^4=-iI$. (This also affects the definition of the momentum spinor in terms of the momentum four-vector.) With our conventions, Fierz-Pauli's "rest-mass" terms $-6\kappa c^{*\dot\alpha}d_{\dot\alpha}+\kappa(-b^*)^{\dot\alpha}{}_{\beta\nu}a_{\dot\alpha}{}^{\beta\nu}$ +conj. (with minus signs due to our lowering and raising of spinor indices in order to obtain a notation fitting our conventions) show a minus sign in front of the rest-mass term depending on the doublet wave function, but a plus sign in front of the terms $\kappa(-b)^*a$ depending on the quartet wave function. As the quartet terms are physically most important (containing the independent canonical variables), we want the latter to appear with a negative coefficient in the Lagrangian, as the mass term in the Dirac theory of electrons. Therefore, we now have to change the sign of the entire Lagrangian.

We shall now define the eight-component four-spinor pair Z and the two four-component undors (two-spinor pairs), by which we want to describe the field. The latter two undors we shall denote by $\gamma_5\chi$ and $\gamma_5\eta$,[18] where γ_5 is the Dirac matrix $i\gamma_1\gamma_2\gamma_3\gamma_0$, so that in the notation of footnote 17 in general $\boldsymbol{\alpha}=\gamma_5\boldsymbol{\sigma}$, and in

[8] Besides possible disadvantages.

[9] To some extent this has been done by Fierz and Pauli, reference 7, where they treat $s^{\dot\alpha}$ as equivalent to s_α in the discussion of the "subsidiary conditions."

[10] F. J. Belinfante, Physica **7**, 765 (1940).

[11] J. Schwinger, Phys. Rev. **74**, 1439 (1948).

[12] For a discussion of the theory of representations, see for instance E. Wigner, *Gruppentheorie* (F. Vieweg, Braunschweig, 1931). Watch, however, the completely different notation used by Wigner. About the representations of the Lorentz group, see B. L. Van der Waerden, *Die Gruppentheoretische Methode in der Quantenmechanik* (J. Springer, Berlin, 1932), in particular Sec. 20, pp. 78–86.

[13] F. J. Belinfante, Physica **6**, 849 (1939).

[14] H. A. Kramers, *Grundlagen der Quantentheorie, Quantentheorie des Elektrons und der Strahlung* (Akademische Verlagsgesellschaft, Leipzig, 1938), pp. 259–269; (compare also pp. 272–280).

[15] But for a factor ϵ such that $\epsilon^4=1$. See W. Pauli, Annales Institut H. Poincaré **6**, 130 (1936); E. Majorana, Nuovo cimento **14**, 171 (1937); G. Racah, Nuovo cimento **14**, 322 (1937). See also reference 13.

[16] B. L. Van der Waerden, Nachr. Akad. Wiss. Göttingen Math.-physik. Kl. IIa **1929**, 100 (1929); or O. Laporte and G. E. Uhlenbeck, Phys. Rev. **37**, 1380 (1931).

[17] H. A. Kramers, reference 14, pp. 280–294; or F. J. Belinfante, reference 13, pp. 850–851. In this representation of a Dirac wave function, the electron energy $mc^2\beta-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}$ is represented with $(\mathbf{v}/c=)\boldsymbol{\alpha}=\rho_3\boldsymbol{\sigma}$, and with $(mc^2/\mathcal{E}=)\beta=\rho_1$.

[18] The quantities $(\gamma_5\chi)$ and $(\gamma_5\eta)$ are meant to be undors under *spatial* rotations and reflections. Under Lorentz transformations, only $(\gamma_5\chi)$ transforms like an undor, but not $(\gamma_5\eta)$. The reason why we call these (spatial) undors here $(\gamma_5\chi)$ and $(\gamma_5\eta)$ instead of using a single letter, lies in the fact that most formulas contain γ_5 times these undors, and thus are simplified by our notation. The difference in transformation between $(\gamma_5\chi)$ and the "pseudo-undor" χ itself is only a minus sign in the spatial reflection.

Kramers representation[17] $\gamma_5=\rho_3$. Thus the components of $\gamma_5\chi$ are $\{\chi_1, \chi_2, -\chi_3, -\chi_4\}$ as listed in Eq. (1). In this Kramers representation, we define our field components in terms of the components of the Fierz-Pauli field by:

$$\left.\begin{aligned}
+\chi_1&=\sqrt{(8/3)}\cdot d_{\dot{1}},\\
+\chi_2&=\sqrt{(8/3)}\cdot d_{\dot{2}},\\
-\chi_3&=\sqrt{(8/3)}\cdot c^1,\\
-\chi_4&=\sqrt{(8/3)}\cdot c^2;\\
+\eta_1&=\sqrt{(2/27)}\cdot(a_{\dot{1}}{}^{12}-a_{\dot{2}}{}^{11})-(\sqrt{\tfrac{2}{3}})d_{\dot{1}},\\
+\eta_2&=\sqrt{(2/27)}\cdot(a_{\dot{1}}{}^{22}-a_{\dot{2}}{}^{12})-(\sqrt{\tfrac{2}{3}})d_{\dot{2}},\\
-\eta_3&=\sqrt{(2/27)}\cdot(b^2{}_{\dot{1}\dot{1}}-b^1{}_{\dot{1}\dot{2}})-(\sqrt{\tfrac{2}{3}})c^1,\\
-\eta_4&=\sqrt{(2/27)}\cdot(b^2{}_{\dot{1}\dot{2}}-b^1{}_{\dot{2}\dot{2}})-(\sqrt{\tfrac{2}{3}})c^2;\\
Z_1&=a_{\dot{1}}{}^{11},\\
Z_2&=(\sqrt{\tfrac{1}{3}})(2a_{\dot{1}}{}^{12}+a_{\dot{2}}{}^{11}),\\
Z_3&=(\sqrt{\tfrac{1}{3}})(a_{\dot{1}}{}^{22}+2a_{\dot{2}}{}^{12}),\\
Z_4&=a_{\dot{2}}{}^{22},\\
Z_5&=-b^1{}_{\dot{1}\dot{1}},\\
Z_6&=(-\sqrt{\tfrac{1}{3}})(2b^1{}_{\dot{1}\dot{2}}+b^2{}_{\dot{1}\dot{1}}),\\
Z_7&=(-\sqrt{\tfrac{1}{3}})(b^1{}_{\dot{2}\dot{2}}+2b^2{}_{\dot{1}\dot{2}}),\\
Z_8&=-b^2{}_{\dot{2}\dot{2}}.
\end{aligned}\right\}\quad(1)$$

Explanation: We may consider a^{11}, $a^{12}\sqrt{2}$, a^{22} as a "three-spinor" for triplet states (spin 1), and $a_{\dot{1}}$, $a_{\dot{2}}$ as a spinor for doublet states (spin $\frac{1}{2}$). The formulas for Z_1-Z_4 are then easily[19] seen to take the form of the linear combinations of wave functions of spin $\frac{3}{2}$ obtained by vector addition of spins 1 and $\frac{1}{2}$.—We defined η_1 and η_2 as a linear combination of the spinor $\{\chi_1, \chi_2\}$ and of the doublet states obtained by this same vector addition.[19] The reason for using this particular combination lies in the simple-to-solve form (20), (27) thus found for the subsidiary conditions. (Compare footnote 20.) The components $-\chi_3$, $-\chi_4$, $-\eta_3$, $-\eta_4$, Z_5, Z_6, Z_7, Z_8 are obtained from χ_1, χ_2, η_1, η_2, Z_1, Z_2, Z_3, Z_4 by spatial reflection and omission of the factors $\epsilon=(1)^{\frac{1}{4}}$ mentioned above. Normalization factors in χ, η, and Z have been chosen arbitrarily in such way as to give the coefficients in the Lagrangian (2) relatively simple values.

The labels 1 and 3 on χ and η correspond to "spin up" ($m=+\frac{1}{2}$), and the labels 2 and 4 on χ and η to "spin down" ($m=-\frac{1}{2}$), as in the Dirac electron theory. For Z, the labels 1 and 5 mean $m=\frac{3}{2}$, 2 and 6 mean $m=\frac{1}{2}$, 3 and 7 mean $m=-\frac{1}{2}$, and 4 and 8 mean $m=-\frac{3}{2}$. We do not use the values of m itself as indices because fractions used as indices are hard on the eyes.

3. LAGRANGIAN FOR ELEMENTARY PARTICLES OF SPIN $\frac{3}{2}$ IN AN ELECTROMAGNETIC FIELD

With the notation just explained, and with the discussed alteration of the sign of σ_4 (and of derived quantities such as the momentum spinor) as well as of the sign of Fierz-Pauli's entire Lagrangian, we can find their (=minus our) Lagrangian by writing it out in components, substituting (1), and collecting the resulting terms. Thus we obtain

$$\left.\begin{aligned}
-L=Z^\dagger\{\kappa\beta+\pi_0-\tfrac{2}{3}\gamma_5\mathbf{S}\cdot\boldsymbol{\pi}\}Z+Z^\dagger\mathbf{K}\cdot\boldsymbol{\pi}\chi\\
+Z^\dagger\mathbf{K}\cdot\boldsymbol{\pi}\eta+\chi^\dagger\mathbf{K}^\dagger\cdot\boldsymbol{\pi}Z+\chi^\dagger\{3\boldsymbol{\alpha}\cdot\boldsymbol{\pi}-(9/2)\kappa\beta\}\eta\\
+\eta^\dagger\mathbf{K}^\dagger\cdot\boldsymbol{\pi}Z+\eta^\dagger\{3\boldsymbol{\alpha}\cdot\boldsymbol{\pi}-(9/2)\kappa\beta\}\chi\\
+\eta^\dagger\{(15/2)\boldsymbol{\alpha}\cdot\boldsymbol{\pi}-(9/2)\pi_0-9\kappa\beta\}\eta.
\end{aligned}\right\}(2)$$

Here, manifest covariance has been lost, but complete reduction with respect to the spatial rotation and reflection group has been gained. We have given the Lagrangian in units $\hbar c$, and $\kappa=mc/\hbar$. Further, $\beta=\rho_1$ and $\gamma_5=\rho_3$ in our representation (1), in which ρ_1 and ρ_3 transform the vertical (one-column) matrix

$$(\chi_1\chi_2,\ \chi_3\chi_4;\ \eta_1\eta_2,\ \eta_3\eta_4;\ Z_1Z_2Z_3Z_4,\ Z_5Z_6Z_7Z_8)$$

into the vertical (one-column) matrices

$$(\chi_3\chi_4,\ \chi_1\chi_2;\ \eta_3\eta_4,\ \eta_1\eta_2;\ Z_5Z_6Z_7Z_8,\ Z_1Z_2Z_3Z_4)$$

and

$$(\chi_1\chi_2,\ -\chi_3\quad -\chi_4;\ \eta_1\eta_2,\ -\eta_3\quad -\eta_4;\ Z_1Z_2Z_3Z_4,\ -Z_5\quad -Z_6\quad -Z_7\quad -Z_8),$$

respectively. The operators π_λ (kinetic momenta in units $\hbar$) are given by

$$\left.\begin{aligned}
&\pi_0=p_0-\mathfrak{e}A_0=-i\partial/c\partial t+\mathfrak{e}\Phi=-\mathcal{E}_{\text{kin}}/\hbar c,\\
&\boldsymbol{\pi}=\mathbf{p}-\mathfrak{e}\mathbf{A}=-i\boldsymbol{\nabla}-\mathfrak{e}\mathbf{A}=\text{kinetic momentum}/\hbar,\\
&\mathfrak{e}=q/\hbar c,
\end{aligned}\right\}(3)$$

where q is the charge of the particle. (For instance, $q=2e$, e, 0, $-e$ for a baryon.) The Pauli matrices $\boldsymbol{\sigma}$ operate on each of the four two-spinors $\chi_{1,2}$, $\chi_{3,4}$, $\eta_{1,2}$, and $\eta_{3,4}$. The Dirac matrices

$$\boldsymbol{\alpha}=\gamma_5\boldsymbol{\sigma}\ \text{and}\ \beta\ (\text{with}\ \gamma_5\beta+\beta\gamma_5=0,\ \beta^2=1,\ \text{etc.})\qquad(4)$$

operating on undors like $\gamma_5\chi$ and $\gamma_5\eta$ or on 4-component quantities like χ or η are then represented as described in footnote 17. The 4×2-matrices $\mathbf{K}$ and their 2×4 hermitian conjugates $\mathbf{K}^\dagger$ are defined by

$$K_x=\begin{pmatrix}-\sqrt{\frac{3}{2}} & 0\\ 0 & -\sqrt{\frac{1}{2}}\\ +\sqrt{\frac{1}{2}} & 0\\ 0 & +\sqrt{\frac{3}{2}}\end{pmatrix},\quad K_y=\begin{pmatrix}i\sqrt{\frac{3}{2}} & 0\\ 0 & i\sqrt{\frac{1}{2}}\\ i\sqrt{\frac{1}{2}} & 0\\ 0 & i\sqrt{\frac{3}{2}}\end{pmatrix},$$

$$K_z=\begin{pmatrix}0 & 0\\ \sqrt{2} & 0\\ 0 & \sqrt{2}\\ 0 & 0\end{pmatrix};\qquad(5a)$$

thence

$$K_y{}^\dagger=\begin{pmatrix}-i\sqrt{\frac{3}{2}} & 0 & -i\sqrt{\frac{1}{2}} & 0\\ 0 & -i\sqrt{\frac{1}{2}} & 0 & -i\sqrt{\frac{3}{2}}\end{pmatrix},\ \text{etc.}\qquad(5b)$$

[19] Compare for instance the table $(1/2)\times(1)$ in the Appendix to reference 2.

The hermitian 4×4-matrices $\mathbf{S}$ are defined by

$$S_x=\begin{bmatrix} 0 & \frac{1}{2}\sqrt{3} & 0 & 0 \\ \frac{1}{2}\sqrt{3} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2}\sqrt{3} \\ 0 & 0 & \frac{1}{2}\sqrt{3} & 0 \end{bmatrix},$$

$$S_y=\begin{bmatrix} 0 & -\frac{1}{2}i\sqrt{3} & 0 & 0 \\ \frac{1}{2}i\sqrt{3} & 0 & -i & 0 \\ 0 & i & 0 & -\frac{1}{2}i\sqrt{3} \\ 0 & 0 & \frac{1}{2}i\sqrt{3} & 0 \end{bmatrix}, \qquad (6)$$

$$S_z=\begin{bmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{bmatrix};$$

they obviously are the components of the spin-$\frac{3}{2}$ angular momentum in units $\hbar$. The factor $\frac{2}{3}$ in the term $-\frac{2}{3}\gamma_5\mathbf{S}\cdot\boldsymbol{\pi}$ in the Lagrangian (2) for the Fierz-Pauli theory is comparable to the factor 2 of the Dirac electron theory in its term with $-\boldsymbol{\alpha}\cdot\boldsymbol{\pi}=-2\gamma_5\,\mathbf{s}\cdot\boldsymbol{\pi}$, where $\mathbf{s}=\frac{1}{2}\boldsymbol{\sigma}$ is the spin-$\frac{1}{2}$ angular momentum in units $\hbar$.

4. COMMUTATION RELATIONS FOR THE "SPIN" MATRICES

One easily verifies the following relations between the matrices $\mathbf{K}$, $\mathbf{K}^\dagger$, $\mathbf{S}$, and $\boldsymbol{\sigma}$, and further relations obtained from the ones listed explicitly by cyclic permutation of x, y, and z:

$$2S_xK_x=K_x\sigma_x, \qquad (7a)$$

$$2K_x{}^\dagger S_x=\sigma_xK_x{}^\dagger, \qquad (7a^*)$$

$$2S_xK_y-K_x\sigma_y=-(2S_yK_x-K_y\sigma_x)=3iK_z, \qquad (7b)$$

$$2K_x{}^\dagger S_y-\sigma_xK_y{}^\dagger=-(2K_y{}^\dagger S_x-\sigma_yK_x{}^\dagger)=3iK_z{}^\dagger, \qquad (7b^*)$$

$$K_x{}^\dagger K_x=2, \qquad (8a)$$

$$K_x{}^\dagger K_y=-K_y{}^\dagger K_x=-i\sigma_z, \qquad (8b)$$

$$K_xK_x{}^\dagger=(9/4)-S_x{}^2, \qquad (9a)$$

$$K_xK_y{}^\dagger+K_yK_x{}^\dagger=-(S_xS_y+S_yS_x), \qquad (9b)$$

$$K_xK_y{}^\dagger-K_yK_x{}^\dagger=2iS_z, \qquad (9c)$$

$$S_xS_y-S_yS_x=iS_z. \qquad (9d)$$

For vector operators $\mathbf{A}$ and $\mathbf{B}$ we find from the above relations and the properties of the Pauli matrices $\boldsymbol{\sigma}$:

$$\mathbf{A}\cdot(\mathbf{K}\boldsymbol{\sigma}-2\mathbf{SK})\cdot\mathbf{B}=3i\mathbf{A}\cdot\mathbf{K}\times\mathbf{B}, \qquad (10)$$

$$\mathbf{A}\cdot(\boldsymbol{\sigma}\mathbf{K}^\dagger-2\mathbf{K}^\dagger\mathbf{S})\cdot\mathbf{B}=3i\mathbf{A}\cdot\mathbf{K}^\dagger\times\mathbf{B}, \qquad (10^*)$$

$$\mathbf{A}\cdot\mathbf{K}^\dagger\mathbf{K}\cdot\mathbf{B}=2\mathbf{A}\cdot\mathbf{B}+i\mathbf{A}\cdot\boldsymbol{\sigma}\times\mathbf{B}, \qquad (11)$$

$$\mathbf{A}\cdot(\mathbf{KK}^\dagger+\mathbf{SS})\cdot\mathbf{B}=(9/4)\mathbf{A}\cdot\mathbf{B}-\tfrac{3}{2}i\mathbf{A}\cdot\mathbf{S}\times\mathbf{B}, \qquad (12)$$

$$\mathbf{A}\cdot\boldsymbol{\sigma\sigma}\cdot\mathbf{B}=\mathbf{A}\cdot\mathbf{B}-i\mathbf{A}\cdot\boldsymbol{\sigma}\times\mathbf{B}. \qquad (13)$$

Here we used diadic notation in the left-hand members. Substituting $\mathbf{A}=\mathbf{B}=\boldsymbol{\pi}$, and using the first of the relations

$$\boldsymbol{\pi}\times\boldsymbol{\pi}=ie\mathfrak{B}, \quad \boldsymbol{\pi}\pi_0-\pi_0\boldsymbol{\pi}=ie\mathfrak{E} \qquad (14a\text{-}b)$$

following from (3) in an electromagnetic field $\mathfrak{E}$, $\mathfrak{B}$, we find

$$\boldsymbol{\pi}\cdot(\mathbf{K}\boldsymbol{\sigma}-2\mathbf{SK})\cdot\boldsymbol{\pi}=3e\mathfrak{B}\cdot\mathbf{K}, \qquad (15)$$

$$\boldsymbol{\pi}\cdot(\boldsymbol{\sigma}\mathbf{K}^\dagger-2\mathbf{K}^\dagger\mathbf{S})\cdot\boldsymbol{\pi}=3e\mathfrak{B}\cdot\mathbf{K}^\dagger, \qquad (15^*)$$

$$\boldsymbol{\pi}\cdot\mathbf{K}^\dagger\mathbf{K}\cdot\boldsymbol{\pi}=2\boldsymbol{\pi}^2+e\mathfrak{B}\cdot\boldsymbol{\sigma}, \qquad (16)$$

$$\boldsymbol{\pi}\cdot(\mathbf{KK}^\dagger+\mathbf{SS})\cdot\boldsymbol{\pi}=(9/4)\boldsymbol{\pi}^2-\tfrac{3}{2}e\mathfrak{B}\cdot\mathbf{S}, \qquad (17)$$

$$\boldsymbol{\pi}\cdot\boldsymbol{\sigma\sigma}\cdot\boldsymbol{\pi}=\boldsymbol{\pi}^2-e\mathfrak{B}\cdot\boldsymbol{\sigma}. \qquad (18)$$

Obviously the matrices $\mathbf{S}$, $\mathbf{K}$, $\mathbf{K}^\dagger$, and $\boldsymbol{\sigma}$ all commute with β and with γ_5.

5. FIELD EQUATIONS, SUBSIDIARY EQUATIONS, AND SOLUTION FOR THE INDEPENDENT VARIABLES

Independent variation of $Z^\dagger$, $\chi^\dagger$, and $\eta^\dagger$ in $\delta\int dt\int d^3\mathbf{x}\ L=0$ yields

$$\{\kappa\beta+\pi_0-\tfrac{2}{3}\gamma_5\mathbf{S}\cdot\boldsymbol{\pi}\}Z+\mathbf{K}\cdot\boldsymbol{\pi}\chi+\mathbf{K}\cdot\boldsymbol{\pi}\eta=0, \qquad (19)$$

$$\mathbf{K}^\dagger\cdot\boldsymbol{\pi}Z+\{3\boldsymbol{\alpha}\cdot\boldsymbol{\pi}-(9/2)\kappa\beta\}\eta=0, \qquad (20)$$

$$\mathbf{K}^\dagger\cdot\boldsymbol{\pi}Z+\{3\boldsymbol{\alpha}\cdot\boldsymbol{\pi}-(9/2)\kappa\beta\}\chi+\{(15/2)\boldsymbol{\alpha}\cdot\boldsymbol{\pi}-(9/2)\pi_0-9\kappa\beta\}\eta=0. \qquad (21)$$

Equation (20) forms the first four subsidiary equations, corresponding to Fierz-Pauli's Eqs. (28)–(29). In principle, (20) (with suitable boundary conditions of vanishing of fields at infinity) enables us to express the four-component $\eta(x,y,z,t)$ in terms of the field $Z(x',y',z',t)$ and the electromagnetic field, all at the same time t. Therefore, the components of η may be called "dependent variables," as contrasted to independent canonical variables. Also the "derived variables" well known from other field theories of elementary particles[10] are dependent variables, although of a more special kind, as those can be expressed as polynomials in the canonical variables, and spatial derivatives of finite order of them, all taken at the same point. On the other hand, $\eta(x,y,z,t)$ depends on $Z(x',y',z',t)$ in a spatial neighborhood of the point $\{x,y,z\}$.

Solution of Eq. (20) for η by successive approximations in powers of the charge q becomes easy after multiplication of (20) by $\{\frac{1}{3}\boldsymbol{\alpha}\cdot\boldsymbol{\pi}-\frac{1}{2}\kappa\beta\}$, which gives

$$\{\tfrac{1}{3}\boldsymbol{\pi}\cdot\boldsymbol{\alpha}-\tfrac{1}{2}\kappa\beta\}\mathbf{K}^\dagger\cdot\boldsymbol{\pi}Z+\{\boldsymbol{\pi}\cdot\boldsymbol{\sigma\sigma}\cdot\boldsymbol{\pi}+(9/4)\kappa^2\}\eta=0$$

or

$$\{\boldsymbol{\pi}^2-e\mathfrak{B}\cdot\boldsymbol{\sigma}+(9/4)\kappa^2\}\eta=4\pi f^{(0)}[Z], \qquad (22)$$

where

$$f^{(0)}[Z(x,y,z,t)]=\{\tfrac{3}{2}\kappa\beta-\boldsymbol{\pi}\cdot\boldsymbol{\alpha}\}\mathbf{K}^\dagger\cdot\boldsymbol{\pi}Z(x,y,z,t)/12\pi. \qquad (23)$$

Equation (22) with (3) can be rewritten as

$$\{\boldsymbol{\nabla}^2-(3\kappa/2)^2\}\eta=-4\pi f \qquad (24)$$

with

$$f=f^{(0)}[Z]+\{e\mathfrak{B}\cdot\boldsymbol{\sigma}+\mathbf{p}^2-\boldsymbol{\pi}^2\}\eta/4\pi. \qquad (25)$$

The solution of Eq. (24) satisfies

$$\eta(x,y,z,t)=\int d^3\mathbf{x}'\exp(-\tfrac{3}{2}\kappa r)f(x',y',z',t)/r, \quad (26)$$

with $r=|\mathbf{x}-\mathbf{x}'|$.

By (25)–(26) we solve (24) for η in successive approximations. By $e\rightarrow 0$ in Eq. (25), we first find $f^{(0)}$ as zero-order approximation to f. Inserting $f^{(0)}$ into (26) then yields the first approximation $\eta^{(1)}$ to η. Similarly, (26) expresses $\eta^{(n+1)}$ in terms of $f^{(n)}$, which in turn is given by (25) in terms of $f^{(0)}$ and $\eta^{(n)}$. By (26), $\eta(x,y,z,t)$ —or at least the first few terms in its expansion—is thus found to depend mainly on the fields in a small region with a radius of the order of magnitude $(1/\kappa)$ around the point (x,y,z) in three-dimensional space. From Eq. (20) it is seen that η vanishes for a particle of zero kinetic momentum ($\boldsymbol{\pi}Z=0$), and therefore η will be much smaller than Z for slow particles. (Compare Eqs. (26) and (23).)

In order to find the other four subsidiary equations, we multiply Eqs. (19), (20), and (21) by $\{-\boldsymbol{\pi}\cdot\mathbf{K}^\dagger\}$, $\{2\kappa\beta-\boldsymbol{\pi}\cdot\boldsymbol{\alpha}+\pi_0\}$, and $\{\tfrac{2}{3}\boldsymbol{\pi}\cdot\boldsymbol{\alpha}-\kappa\beta\}$, respectively, and we add the results. Using Eqs. (4), (14b), (15*), (16), (18) we thus find

$$-\mathfrak{e}\mathfrak{B}\cdot\mathbf{K}^\dagger\gamma_5 Z-i\mathfrak{e}\mathfrak{E}\cdot\mathbf{K}^\dagger Z-3\mathfrak{e}\mathfrak{B}\cdot\boldsymbol{\sigma}\chi +(9/2)\kappa^2\chi-3\mathfrak{e}\mathfrak{B}\cdot\boldsymbol{\sigma}\eta-3i\mathfrak{e}\mathfrak{E}\cdot\boldsymbol{\alpha}\eta=0,$$

or, with $\lambda=2\mathfrak{e}/3\kappa^2=2q\hbar/3m^2c^3$,

$$\{1-\lambda\mathfrak{B}\cdot\boldsymbol{\sigma}\}\chi=\lambda(\gamma_5\mathfrak{B}+i\mathfrak{E})\cdot(\boldsymbol{\alpha}\eta+\tfrac{1}{3}\mathbf{K}^\dagger Z). \quad (27)$$

We solve this equation for χ by multiplying it from the left by [20]

$$\{1-\lambda\mathfrak{B}\cdot\boldsymbol{\sigma}\}^{-1}\equiv\{1-\lambda^2\mathfrak{B}^2\}^{-1}\{1+\lambda\mathfrak{B}\cdot\boldsymbol{\sigma}\}. \quad (28)$$

We thus have solved for all dependent variables χ and η in terms of the independent canonical variables Z and the electromagnetic field. The equations of motion for the field Z are then given by Eq. (19) with the solutions for χ and η substituted.

6. THE MAGNETIC MOMENT OF SPIN-$\frac{3}{2}$ PARTICLES

Multiplying Eqs. (19), (20), and (21) by $\{\kappa\beta-\pi_0-\tfrac{2}{3}\gamma_5\boldsymbol{\pi}\cdot\mathbf{S}\}$, by $\{\tfrac{2}{3}\boldsymbol{\pi}\cdot\mathbf{K}\}$, and by $\{-(2/9)\boldsymbol{\pi}\cdot\mathbf{K}\}$, respectively, and adding the three products to $\pi_0{}^2Z$, we find by Eqs. (4), (14b), (15), and (17):

$$\pi_0{}^2Z=\{\kappa^2+\boldsymbol{\pi}\cdot\boldsymbol{\pi}-\tfrac{2}{3}\mathfrak{e}\mathfrak{B}\cdot\mathbf{S}-\tfrac{2}{3}i\mathfrak{e}\mathfrak{E}\cdot\mathbf{S}\gamma_5\}Z +\mathfrak{e}(\gamma_5\mathfrak{B}+i\mathfrak{E})\cdot\mathbf{K}\eta+\{\mathbf{K}\cdot\boldsymbol{\pi}(2\kappa\beta-\pi_0-\boldsymbol{\alpha}\cdot\boldsymbol{\pi}) +\mathfrak{e}(\gamma_5\mathfrak{B}+i\mathfrak{E})\cdot\mathbf{K}\}\chi. \quad (29)$$

Similarly we derive, by adding $(2/9)(\pi_0+\boldsymbol{\pi}\cdot\boldsymbol{\alpha}-\kappa\beta)$ times the difference {Eq. (21)—Eq. (20)} to $\pi_0{}^2\eta$, and by using Eqs. (4), (14b), and (18):

$$\pi_0{}^2\eta=\{\kappa^2+\boldsymbol{\pi}\cdot\boldsymbol{\pi}-\mathfrak{e}\mathfrak{B}\cdot\boldsymbol{\sigma}\}\eta+\{\kappa^2-\kappa\beta\pi_0 +\tfrac{2}{3}\pi_0\boldsymbol{\pi}\cdot\boldsymbol{\alpha}+\tfrac{1}{3}\kappa\beta\boldsymbol{\alpha}\cdot\boldsymbol{\pi}+\tfrac{2}{3}\boldsymbol{\pi}\cdot\boldsymbol{\pi}-\tfrac{2}{3}\mathfrak{e}\mathfrak{B}\cdot\boldsymbol{\sigma}\}\chi. \quad (30)$$

According to Eq. (20), for slow particles η is smaller than Z by a factor v/c, while by Eq. (27) χ vanishes in absence of electromagnetic fields. If we neglect terms quadratic in the electromagnetic field as well as terms bilinear in the electromagnetic field and in (v/c), Eqs. (29)–(30) reduce to

$$\pi_0{}^2Z\approx\{\kappa^2+\boldsymbol{\pi}^2-\tfrac{2}{3}\mathfrak{e}(\mathfrak{B}+i\gamma_5\mathfrak{E})\cdot\mathbf{S}\}Z, \quad (31)$$

$$\pi_0{}^2\eta\approx\{\kappa^2+\mathbf{p}^2\}\eta. \quad (32)$$

In the following we shall further neglect the imaginary term with $\gamma_5\mathfrak{E}$ in Eq. (31).

Since $\mathcal{E}=-\hbar c\pi_0$ was the kinetic energy, $m=\kappa\hbar/c$ is apparently the rest mass of our spin-$\frac{3}{2}$ particles. We solve for the nonrelativistic energy $\mathcal{W}$ by squaring the expression

$$-\pi_0=\kappa+(\mathcal{W}-q\Phi)/\hbar c, \quad (33)$$

which defines $\mathcal{W}$, and by equating the result to $\pi_0{}^2\approx\kappa^2+\boldsymbol{\pi}^2-\tfrac{2}{3}\mathfrak{e}\mathfrak{B}\cdot\mathbf{S}$ from Eq. (31). Neglecting the square of the nonrelativistic energy $\mathcal{W}$, and putting $\mathfrak{p}=\hbar\boldsymbol{\pi}$ for the kinetic momentum in cgs-units, we thus find

$$\{\mathcal{W}-q\Phi\}Z\approx(\hbar c/2\kappa)\{\boldsymbol{\pi}^2-\tfrac{2}{3}\mathfrak{e}\mathfrak{B}\cdot\mathbf{S}\}Z \approx[\mathfrak{p}^2/2m-(q\hbar/3mc)\mathfrak{B}\cdot\mathbf{S}]Z, \quad (34)$$

so that our spin-$\frac{3}{2}$ particles apparently have a magnetic moment

$$\boldsymbol{\mu}=(q\hbar/3mc)\mathbf{S}. \quad (35)$$

Remembering that $\hbar\mathbf{S}$ was the spin angular momentum of the spin-$\frac{3}{2}$ part of the field, with z-component $\frac{3}{2}\hbar$, $\frac{1}{2}\hbar$, $-\frac{1}{2}\hbar$, or $-\frac{3}{2}\hbar$ according to (6), we find the gyromagnetic ratio for our particles to be $(q/3mc)$ and we find $(q\hbar/2mc)$ for the maximum value of μ_z.

The η components of the field, which become important for fast particles only, behave like a spin-$\frac{1}{2}$ field with spin angular momentum $\frac{1}{2}\hbar\boldsymbol{\sigma}$ and magnetic moment $q\hbar\boldsymbol{\sigma}/2mc$ as seen by treating (30) similar to (29) by Eq. (33) going one approximation beyond Eq. (32). The gyromagnetic ratio of this part of the field is q/mc; but the maximum possible value for the corresponding μ_z value is again $q\hbar/2mc$.

Our results, together with those found for Dirac electrons and for vector mesons,[21] suggest a spin magnetic moment $q\hbar/2mc$ for *all* elementary particles of charge q, rest mass m, and low kinetic energy, for any spin different from zero. Thus a larger spin would always be compensated by a correspondingly smaller gyromagnetic ratio.

[20] Our χ corresponds to Fierz-Pauli's spinor field $d_{\dot\alpha}$, $-c^{\alpha}$; see Eq. (1). Their Eqs. (23)–(24), translated into our notation, express χ in terms of the components Z and $(\chi+2\eta)$ of the $a_{\dot\alpha}{}^{\beta\nu}$, $b^{\alpha}{}_{\dot\beta\dot\nu}$ field. We prefer to express χ in terms of Z and η as in (27) with (28), although this destroys the relativistic appearance of the formula. The reason for our preference is, of course, the fact that the other identities (20), by (23)–(26), express η and not $(\chi+2\eta)$ in terms of Z.

[21] Yukawa, Sakata, and Taketani, Proc. Phys.-Math. Soc. Japan **20**, 319 (1938); F. J. Belinfante, Physica **6**, 870 (1939).