Magnetic Moments of Neutron and Proton

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After a discussion of the importance of results recently found by Sachs, some weaknesses of his theory are pointed out, and a contribution to the nucleon magnetic moment from the state with two-pion cloud in an S-state is calculated. A proposal of Sugawara to include admixtures of states with a one-pion cloud around a spin-$\frac{3}{2}$ ("baryon") core is criticized, and the contribution of such states to the nucleon magnetic moments is calculated. It is discussed what value might be taken for the magnetic moment of such "baryon" core.

1. INTRODUCTION

IN a recent paper, Sachs[1] has shown the importance of the two-pion admixture in the nucleon wave function for the explanation of the magnetic moments of the neutron and the proton. As a special example he considered a model consisting of 91.0 percent bare nucleon core (called "nucleore" by him) and 9.0 percent this nucleore accompanied by two pions each in a p state forming together a P state with "square" radial distribution.[2] In this case, the contributions to the neutron magnetic moment were[3] -0.020 from nucleore moment in the 9 percent two-pion state, -0.133 from pion orbits in the 9 percent two-pion state, and -1.792 from cross terms between two-pion and no-pion states; total -1.945 for the neutron.

Applying the mirror property to this model, we find for the corresponding proton state: $+0.910$ from nucleore moment in the 91 percent no-pion state, -0.010 from nucleore moment in the 9 percent two-pion state, $+0.133$ from pion orbits in the 9 percent two-pion state, and $+1.792$ from cross terms between two-pion and no-pion states; total $+2.825$ for the proton. These figures show the importance of the cross terms between two-pion and no-pion states arising from the pion-pair creation and annihilation terms in the expression for the magnetic moment of a pion field.[1]

Sachs' calculations neglect any separation of the nucleore from a fixed point chosen as the origin, and they neglect relativistic effects. This means that the large components of the Dirac wave-function of the nucleore are assumed to be s states. The p state small components are then neglected. Sachs makes it plausible that only pions in p states should be expected. His argument is clearly based on assumption of an interaction between nucleores and pions linear in the latter and coupling among each other nucleore states with s-state large components but not necessarily with the same spin. An example of such interaction is a single pseudo-vector coupling without further interactions. The interaction between pions and nucleores proposed by Foldy and collaborators[4] is of course of a different type: It can be formulated as a single linear coupling between nucleores and pions; but this pseudoscalar coupling connects states with s-state large components just as well (with comparable matrix element) with states with p-state large components, under emission of pions in s states.[5] On the other hand, after Foldy's transformation to pseudo-vector coupling there are all kinds of non-linear terms,[4,6] and there is no clear reason for restriction to p-state pions either. Therefore it may be that Sachs' nucleon model is a too simple one, even if the special assumptions leading him to the numerical values mentioned above are dropped. Nevertheless Sachs' results are interesting as they show that only a moderate admixture of a two-pion state (about 30 percent of the wave function) suffices for explanation of the magnetic moments of proton and neutron.

Sachs himself points out that the agreement[7] of his[3] value (-1.945) with the experimental magnetic moment (-1.910) of the neutron is fortuitous, as his assumptions (29)–(30) on the absence of pion clouds in S states, of one-pion states, and of two-pion P states antisymmetric in the two-pion radial function, as well as his assumption of a square form of the two-pion P-state symmetric radial function, are just one out of many ways in which the experimental facts could be explained in terms of the parameters of his theory.

In this connection it should be pointed out that in Sachs' general expression for the neutron magnetic moment (Eqs. (27)–(28)) a term

$$P_0^-(2)=P_0^-(2,\,\mathfrak{A})=\tfrac{2}{3}P_0(2,\,\mathfrak{A})$$

is missing. This is due to the fact that Sachs overlooked

[1] R. G. Sachs, Phys. Rev. **87**, 1100 (1952).

[2] By "square" radial distribution Sachs understands a sharp cutoff of the radial function at some fixed maximum value for the distance of the two pions from the nucleore frozen in the origin.

[3] In Sachs' formulas we substitute $\mathfrak{M}/\mu=6.64_2$ and $I_1(\mathfrak{S})=1$. We obtain $\mathfrak{M}_n=-1.94_5$, as contrasted to the figure -1.93 given by Sachs himself. The 91 percent to 9 percent mixture was adjusted by Sachs to the value 0.880 of $\mathfrak{M}_n+\mathfrak{M}_p$. All magnetic moments were measured in nuclear magnetons assumed to be practically identical with nucleore magnetons.

[4] L. L. Foldy, Phys. Rev. **84**, 168 (1951); Berger, Foldy, and Osborn, Phys. Rev. **87**, 1061 (1952).

[5] In this case the $\psi^\dagger\beta\gamma_5\psi$-interaction is purely relativistic, and Sachs' nonrelativistic approximation cannot be maintained. This necessitates explicit introduction of the space function for the nucleore, and thus Sachs' selection rule (3) is broken; thence also his Eq. (11).

[6] G. Wentzel, Phys. Rev. **86**, 802 (1952).

[7] In view of the effects neglected, such as states with more than two pions, recoil of nucleore, difference in mass between charged and neutron pions in applying charge-independence, etc., accuracy cannot be expected, so that $1.945\approx1.910$.

the contribution $\langle N=2, L=0|\mathfrak{M}_n|N=2, L=0\rangle$ to the neutron magnetic moment arising from the magnetic moment of the proton-nucleore in states with pion charge $C=-1$ in an S-state two-pion cloud. In this case, the proton-nucleore spin has always the direction of the total neutron spin, so that it contributes $P_0^-(2)$ nucleore magnetons, where $P_0^-(2)$ according to Sachs' notation is the probability of the state with a two-pion single-negative S-state pion cloud. In his special numerical example discussed above, this forgotten term drops out anyhow on account of the assumption expressed by Sachs' Eq. (29).

2. POSSIBLE NECESSITY OF EXTENDING SACHS' THEORY

In a complete theory one cannot just postulate values for the probabilities of the various nucleore-pion cloud states building up the nucleon wave function, or for the various overlap integrals and other radial integrals entering the theory; but one should take values for such quantities derived from some kind of meson theory, and then it should be hoped that these, on substitution in Sachs' formulas,[8] will yield the experimental values for the nucleon magnetic moments. Therefore it is worthwhile to consider what other terms beside those discussed by Sachs might possibly have to be taken into account in order to obtain such agreement.

It was already mentioned previously that it may be necessary to consider s-state pions, which means taking into account the "recoil" (the coordinate function) of the nucleore. Even if this recoil is neglected, s-state pions may be emitted in pairs by the interaction terms appearing in the Hamiltonian after Foldy's transformation.[4] Also there is no reason why states with more than two pions should not be important.

Further, there may be some reason to suspect the possibility of admixture of states in which the pion cloud surrounds a nuclear particle of a mass slightly more than the nucleon mass (maybe between 2350 and 2400 m_e), of ordinary spin $\frac{3}{2}$, and of isobaric spin $\frac{3}{2}$. For the sake of brevity in discussion, we have called such particule a *baryon*.[9] The possibility of its existence has been suggested by the results of scattering experiments and explanations proposed for these results by several authors.[10] It was recently proposed by Sugawara[11] that such baryon-pion states be incorporated in the calculation of the magnetic moment of the nucleon.

While it is possible that this suggestion makes some sense, Sachs has shown the incorrectness of Sugawara's further statement that the nucleon magnetic moments observed could not be explained simply by second-order pion interaction without consideration of baryon states.[11] For the rest, we do not agree with Sugawara's way of calculating these magnetic moments, simply adding the magnetic moments of the constituent particles in each state as in the calculation of a Paschen-Back effect, without considering the fact that (like in the anomalous Zeeman effect) the interactions between these constituent particles (between the nucleore spin and pion orbital moments) are much stronger than the individual interactions of those particles with the external magnetic field. Also, Sugawara's method never takes into account the cross terms between zero and two pions, which Sachs has shown to be so important. We should therefore use Sachs' method to calculate the contribution of an admixture of a (baryon+one pion)-state to the magnetic moments of a neutron and of a proton. This is done in the next chapter.

3. BARYON-PION CONTRIBUTION TO NUCLEON MAGNETIC MOMENTS

We shall follows Sachs in assuming the pion to be in a p state. We use a notation explained earlier,[9,12] according to which states of "spin" (orbital angular momentum, spin, or isobaric spin) J with z-component J_z are indicated by $[J, J_z]$. Subscript 1 will indicate the nucleore or baryon core; subscript 2 will here indicate the pion. In our case, $[\frac{1}{2}, \pm\frac{1}{2}]_1$, $[\frac{3}{2}, \pm\frac{1}{2}]_1$, and $[\frac{3}{2}, \pm\frac{3}{2}]_1$ may be considered as ordinary spin functions, and $[1, 0]_2$ and $[1, \pm 1]_2$ to be normalized spherical harmonics $\Upsilon_1^m(\theta, \varphi)$ for the pion orbit, with the sign convention used by Rojansky.[13] The isobaric spin functions we shall here distinguish from these space and spin functions by enclosing them between boldface braces $\{\ \}$.

For a baryon-pion state of total angular momentum $\frac{1}{2}$ along the positive z axis, and with isobaric spin $\frac{1}{2}$ in the $-\zeta$ direction as for a neutron, the wave function is found from the tables **(1)×(3/2)** in the Appendix of reference 9:

$$\begin{aligned}\Psi_n^{+\frac{1}{2}} = {} & f_{12}[\tfrac{1}{2}, \tfrac{1}{2}]_{12}\{[\tfrac{1}{2}, -\tfrac{1}{2}]_{12}\} = \\ & f_{12}\big[[\tfrac{3}{2}, -\tfrac{1}{2}]_1[1, 1]_2\sqrt{\tfrac{1}{6}} - [\tfrac{3}{2}, \tfrac{1}{2}]_1[1, 0]_2\sqrt{\tfrac{1}{3}} + \\ & [\tfrac{3}{2}, \tfrac{3}{2}]_1[1, -1]_2\sqrt{\tfrac{1}{2}}\big] \times \{[\tfrac{3}{2}, -\tfrac{3}{2}]_1[1, 1]_2\sqrt{\tfrac{1}{2}} - \\ & [\tfrac{3}{2}, -\tfrac{1}{2}]_1[1, 0]_2\sqrt{\tfrac{1}{3}} + [\tfrac{3}{2}, \tfrac{1}{2}]_1[1, -1]_2\sqrt{\tfrac{1}{6}}\}. \quad (1)\end{aligned}$$

Here, f_{12} is a radial wave function of the baryon-pion system.

Since we don't consider two-pion states with the baryon,[14] we need not consider here the pion-pair terms in the pion magnetic moment. We shall further assume that the baryon magnetic moment is proportional to $\mathbf{S}$=its spin angular momentum in units $\hbar$,

[8] Corrected by the addition of $P_0^-(2)=\frac{2}{3}P_0(2, \mathfrak{A})$. See the introduction.

[9] F. J. Belinfante, Phys. Rev. **92**, 145 (1953).

[10] K. A. Brueckner, Phys. Rev. **86**, 106 (1952); G. S. Janes and W. L. Kraushaar, Phys. Rev. **90**, 341 (1953); B. T. Feld, Phys. Rev. **90**, 342 (1953); G. Wentzel, Phys. Rev. **86**, 437 (1952).

[11] M. Sugawara, Progr. Theoret. Phys. (Japan) **8**, 549 (1952).

[12] While in reference 9 this notation was used for isobaric spin functions only, we shall use it here for angular momenta as well.

[13] V. Rojansky, *Introductory Quantum Mechanics* (Prentice-Hall, Inc., New York, 1938), p. 414. Compare Eq. (A.1) of the Appendix of reference 9.

[14] This approximation made by Sugawara is feasible insofar as cross terms to no-pion states are here impossible, since conservation of isobaric spin forbids (baryon+no pion)-states.

as well as to $(t_\zeta+\frac{1}{2})$ = its charge (q) in units e:

$$\mathfrak{M}_b(\text{in nuclear magnetons}) = k(t_\zeta+\tfrac{1}{2})\mathbf{S}. \quad (2)$$

The contribution to the z component of the magnetic moment of the neutron by the admixture $\Psi_n^{+\frac{1}{2}}$ in the neutron wave function is then given by the matrix element of the operator,

$$k(t_\zeta+\tfrac{1}{2})S_z+(\mathfrak{M}/\mu)L_zC, \quad (3)$$

between $\Psi_n^{+\frac{1}{2}}$ and $\Psi_n^{+\frac{1}{2}}$, where $\mathfrak{M}/\mu$ is the ratio of proton mass to pion mass, where $\mathbf{L}$ is the orbital angular momentum of the pion, and C is its charge in units e. By $S_z[s, m_s]_1 = m_s[s, m_s]_1$, $L_z[l, m]_2 = m[l, m]_2$, $t_\zeta\{[t, m_t]_1\} = m_t\{[t, m_t]_1\}$, $C\{[T, m_T]_2\} = m_T\{[T, m_T]_2\}$, this matrix element is easily found to be

$$\delta\mathfrak{M}_n = -P(b, 1)[(5/2)k+\mathfrak{M}/\mu]/9, \quad (4)$$

where

$$P(b, 1) = \int |f_{12}|^2 r^2 dr \quad (5)$$

is the probability of the (baryon+one pion)-state. While (4) is to be added to Sachs' expression (27) or (28) increased by $\frac{2}{3}P_0(2, \mathfrak{a})$, (see footnote 8), at the same time $P(b, 1)$ is to be included in P_1 in the expression $(1-P_0-P_1)$ occurring in Sachs' Eqs. (27)–(28).

Similarly, the corresponding admixture,

$$\begin{aligned}\Psi_p^{+\frac{1}{2}} = {} & f_{12}[\tfrac{1}{2}, \tfrac{1}{2}]_{12}\{[\tfrac{1}{2}, \tfrac{1}{2}]_{12}\} = \\ & f_{12}[[\tfrac{3}{2}, -\tfrac{1}{2}]_1[1, 1]_2\sqrt{\tfrac{1}{6}} - \\ & [\tfrac{3}{2}, \tfrac{1}{2}]_1[1, 0]_2\sqrt{\tfrac{1}{3}} + \\ & [\tfrac{3}{2}, \tfrac{3}{2}]_1[1, -1]_2\sqrt{\tfrac{1}{2}}] \times \\ & \{[\tfrac{3}{2}, -\tfrac{1}{2}]_1[1, 1]_2\sqrt{\tfrac{1}{6}} - \\ & [\tfrac{3}{2}, \tfrac{1}{2}]_1[1, 0]_2\sqrt{\tfrac{1}{3}} + \\ & [\tfrac{3}{2}, \tfrac{3}{2}]_1[1, -1]_2\sqrt{\tfrac{1}{2}}\}, \end{aligned} \quad (6)$$

to the proton state with spin in positive z direction contributes to the z component of the magnetic moment of the proton an amount given by the matrix element of the same operator (3) between $\Psi_p^{+\frac{1}{2}}$ and $\Psi_p^{+\frac{1}{2}}$. The amount of this contribution is

$$\delta\mathfrak{M}_p = +P(b, 1)[10k+\mathfrak{M}/\mu]/9. \quad (7)$$

The effect of all this on Sachs' Eq. (22) for $\mathfrak{M}_n+\mathfrak{M}_p$ is the following. In the derivation of (22), we must include $P(b, 1)$ in P_1(=probability of P-state pion cloud) where we write $(1-P_0-P_1)$ for the probability of the no-pion state just above Sachs' Eq. (22); but we must *not* include $P(b, 1)$ in the contribution $-\frac{1}{3}P_1$ from the (nucleore+P-state pion cloud)-state. Instead, our two expressions (4) and (7) contribute to $\mathfrak{M}_n+\mathfrak{M}_p$. Thus, Sachs' Eq. (22) is replaced by

$$\mathfrak{M}_n+\mathfrak{M}_p = 1-(4/3)\{P_1(1)+P_1(2)\} - [1-(5k/6)]P(b, 1), \quad (8)$$

where $P_1(N)$ = probability for a P-state N-pion cloud around a regular (spin-$\frac{1}{2}$) nucleore, and $P(b, 1)$ = probability for a P-state one-pion cloud around a baryon (other pion clouds around a baryon not considered).

4. THE MAGNETIC MOMENT OF THE BARYON CORE

In Eqs. (2), (4), and (7), the value of k would have to be taken from some theory of the properties of a baryon. Here, two different points of view are possible.

One is to consider the baryon itself to be a nucleoid,[9] that is, a compound particle consisting of an ordinary nucleore and a pion cloud, like the excited states of the nucleore-pion system discussed by Pauli and Dancoff.[15] (Since Pauli and Dancoff did not succeed in explaining by their model the observed magnetic moments of nucleons, the question arises whether a value of k for a baryon calculated on the basis of such model could be trusted.)

Use of such compound baryon as the core in a certain admixture to a nucleon state makes sense only if the pion cloud inside the compound baryon is bound to the nucleore much stronger than the pion we assumed to surround the baryon. Thus our admixture would be a nucleoid state with "pions in layers" (some pions close to the core and one pion loosely bound at the outskirts of the nucleoid), and the internal pion shells in such state might be said to be "incomplete."[16] It does not seem probable that the ground state of a nucleoid would contain important admixtures of such kind.

Therefore Sugawara's suggestion of considering (baryon+pion)-admixtures to the nucleon ground state would seem rather meaningless, unless one decides to take a second point of view, according to which the baryon core is to be treated as a new elementary particle of spin $\frac{3}{2}$, different from a nucleore, even though reactions baryon+pion⇄nucleon and baryon⇄nucleon +pion may be allowed. This point of view seems to have been taken by Sugawara himself in so far as he postulated an *a priori* value for k in Eq. (2). Neglecting the difference in mass between nucleon and baryon, he chose $k=2$, by an unjustifiable analogy to the Thomas factor of spin-$\frac{1}{2}$ particles.

This value 2 postulated by Sugawara seems unreasonably high. For vector mesons of mass $\mathfrak{m}$ and charge q one finds a magnetic moment[17] $(q\hbar/2\mathfrak{m}c)\mathbf{S}$, where $\mathbf{S}$ is the meson spin. Apart from the mass ratio, this corresponds to $k=1$ for vector mesons. As in the same sense we have $k=2$ for electrons, this seems to suggest $k=1/S$ in general for particles of spin $S\neq 0$.

[15] W. Pauli and S. M. Dancoff, Phys. Rev. **62**, 85 (1942).

[16] The word "incomplete" is used here loosely. Of course, as pions satisfy Einstein-Bose statistics, there is no such thing as a "complete" pion shell. We mean to say only that the internal shells have a total orbital angular momentum different from zero, which combines with the nucleore spin to a total angular momentum $\frac{3}{2}$, different from the nucleore spin. The same is true for the isobaric spin.

[17] Yukawa, Sakata, and Taketani, Proc. Phys.-Math. Soc. Japan **30**, 319 (1938); F. J. Belinfante, Physica **6**, 870 (1939).

Indeed it can be shown that, if the baryon core would satisfy Fierz-Pauli's theory of spin-$\frac{3}{2}$ particles,[18] one finds $k=\frac{2}{3}$ for such particles.[19]

In conclusion we may say that, until there is some further experimental evidence that spin-$\frac{3}{2}$ particles really exist, there may be little reason for accepting Sugawara's suggestion at all. If the need of consideration of (baryon+pion)-states would arise, contributions from such states to the nucleon magnetic moment should be calculated by the methods outlined above, and more likely with the value $k=\frac{2}{3}$ than with $k=2$.

[18] M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, 211 (1939).

[19] F. J. Belinfante, Phys. Rev. **92**, 994 (1953).

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Intrinsic Magnetic Moment of Elementary Particles of Spin $\frac{3}{2}$

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Fierz-Pauli's theory of spin-$\frac{3}{2}$ particles has been reformulated in a manner somewhat resembling the usual formulation of Dirac's equation for the electron. The discussion is simplified by complete reduction of the representation of the spatial rotation and reflection group by the field. The dependent variables can then be expressed in terms of the spin-$\frac{3}{2}$ field. The magnetic moment and the gyromagnetic ratio of "bare" spin-$\frac{3}{2}$ particles of charge q and mass m are found to be $(q\hbar/2mc)$ and $(q/3mc)$, respectively.

1. INTRODUCTION

In a theory of the anomalous magnetic moments of nucleons, Sugawara[1] has recently tacitly assumed that the intrinsic magnetic moment of a spin-$\frac{3}{2}$, isobaric-spin-$\frac{3}{2}$ particle of slightly more than nucleon mass (a so-called *baryon*[2,3]) should be about six nuclear magnetons in its state of charge $2e$. Pauli and Dancoff's strong-coupling theory[4] of the excited states of the nucleoid[2] (=nucleore-pion system[5]) predicts a magnetic moment proportional to $(q-\frac{1}{2}e)/(j+1)$, which would make the total magnetic moment of a baryon of charge $2e$ (and with $j=\frac{3}{2}$) equal to 1.8× the magnetic moment of a proton ($j=\frac{1}{2}$). However, the Pauli-Dancoff theory of the magnetic moments of nucleoids is not only not trustworthy, as shown by its prediction that the neutron magnetic moment would be opposite and equal to the proton magnetic moment, but probably it is not even applicable in a theory like Sugawara's, in which the nucleon is assumed to be part of its time a nucleore, part of its time a nucleore with a pion cloud, and part of its time a baryon core and pion(s). Such an assumption becomes rather improbable, if one does not at the same time assume the baryon (core particle) to be an elementary particle itself,[3] like the proton-nucleore, neutron-nucleore, and pions figuring in Sugawara's theory. That is, we would have to assume that there is such a thing as a "bare elementary particle" of spin $\frac{3}{2}$, into which a nucleore could be transformed under emission or absorption of a pion.

The "bare baryon" would then have an intrinsic magnetic moment of its own—like the proton and neutron as "nucleores" are supposed to have magnetic moments of 1 and 0 nucleore magnetons respectively. The question then arises whether the magnetic moment to be expected for such bare baryon would have so large a value as assumed by Sugawara. We have reasoned that this is unlikely, and that it seems more plausible to guess that the gyromagnetic ratio of a spin-$\frac{3}{2}$ particle of charge q and mass m will be $q/3mc$, and its intrinsic magnetic moment $q\hbar/2mc$. (See reference 3.) It is the purpose of this paper to show[6] that this conjecture is correct, if for a "bare" particle of spin $\frac{3}{2}$ in interaction with an external electromagnetic field one assumes Fierz-Pauli's theory of such particles to be valid.[7]

2. FIELD COMPONENTS FOR ELEMENTARY PARTICLES OF SPIN $\frac{3}{2}$

For their Lorentz-covariant theory of particles of spin $\frac{3}{2}$ in interaction with a Maxwell field, Fierz and Pauli[7] formulated the field equations in a manifestly covariant form using spinor notation. The field has 16 complex components (not counting their conjugates). Between these 16 field components there are 8 relations ("subsidiary equations") not involving differentiation with respect to time, so that at some fixed initial time only eight field components can be chosen independently.

[1] M. Sugawara, Progr. Theoret. Phys. Japan **8**, 549 (1952). For a criticism of this theory see reference 3.

[2] F. J. Belinfante, Phys. Rev. **92**, 145 (1953).

[3] F. J. Belinfante, this issue Phys. Rev. **92**, 994 (1953).

[4] W. Pauli and S. M. Dancoff, Phys. Rev. **62**, 85 (1942).

[5] Nucleore=bare nucleon core; see R. G. Sachs, Phys. Rev. **87**, 1100 (1952).

[6] Without committing ourselves as to the value of Sugawara's suggestion. See also reference 3.

[7] M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, 211 (1939).