An Approximate Wave-Mechanical Description of Deuteron Stripping*

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An approximate wave function describing the deuteron stripping process is set up in a fashion similar to that introduced by S. T. Butler. This wave function is made to fit boundary conditions at the surface of the nucleus for the stripped-off particle. Thus the parameters characterizing the nucleus that enter this description are similar to those familiar from the theory of single-particle nuclear reactions. The cross section for the stripping process is calculated from the wave function by a method that gives the cross section for stripping by virtual levels as well as that for stripping by bound levels. Some discussion is devoted to the Coulomb effect.

I. INTRODUCTION

DEUTERON stripping refers to an interaction between a deuteron and a target nucleus involving the breaking up of the deuteron in which only one of the two particles making up the deuteron interacts directly with the target nucleus. The early work on this subject was concerned with estimating the total cross section for the process as a function of the energy of the incident deuterons.[1–4] The methods used were Born approximations employing crude approximate wave functions, and semiclassical models. Recent work, using more refined Born approximations[5,6] and approximate wave-mechanical methods,[7] has shown that there is a striking dependence of the angular distribution of the liberated particle on the orbital angular momentum of the stripped-off particle. The normalization of the cross section given by these methods does not lend itself to a direct interpretation in terms of the parameters usually used to characterize nuclei. Also, most of these methods are limited to the case in which the stripped-off particle is captured into a bound state, and all these methods neglect the Coulomb interaction between the proton and target nucleus.

In what follows, an approximate wave function describing the deuteron stripping process is set up in a fashion similar to that introduced by Butler.[7] This wave function is made to fit boundary conditions at the surface of the nucleus for the stripped-off particle. Thus the parameters characterizing the nucleus that enter this description are similar to those familiar from the theory of single-particle nuclear reactions. The cross section for the stripping process is calculated from the wave function in such a manner as not to limit the result to only stripping due to capture of the stripped-off particle into bound states. Some discussion is devoted to the Coulomb effect.

II. THE GENERAL THEORY

To start with, we make the following assumptions about the d-p stripping process:

1. The Coulomb interaction can be ignored.
2. The protons have no interaction with the target nucleus.
3. There is no interaction among the products of deuteron disintegrations or between these products and deuterons, so that deuteron fragments can be treated as free particles.

The neglect of the Coulomb interaction makes the theory symmetrical with respect to the possible roles of neutrons and protons.

The physical situation is represented by a six-dimensional wave function—three proton coordinates and three neutron coordinates. For neutrons outside the nucleus, the wave function must consist of a term representing incident and elastically scattered deuterons Ψ_D plus a term representing liberated proton-neutron pairs Ψ_F.

$$\psi(\mathbf{r}_p', \mathbf{r}_n') = \Psi_D + \Psi_F. \qquad (1)$$

The primed coordinates refer to the center of mass of the deuteron and the target nucleus. Because we wish to consider the interaction of the neutron with the target nucleus and to find the angular distribution of the liberated protons, it will be more convenient if we write $\mathbf{r}_n'$ and $\mathbf{r}_p'$ in terms of $\mathbf{r}_n$, the separation of the neutron and the target nucleus, and $\mathbf{r}_p''$, the separation of the proton from the center of mass of the target nucleus and the neutron. This is accomplished by the substitution

$$\mathbf{r}_p' = \frac{(M_n + M_I)}{(M_n + M_I + M_p)} \mathbf{r}_p'', \qquad (2)$$

$$\mathbf{r}_n' = \frac{M_I}{M_I + M_n} \mathbf{r}_n - \frac{M_p}{M_p + M_I + M_n} \mathbf{r}_p'', \qquad (3)$$

where M_n = the mass of the neutron, M_p = the mass of

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the proton, and M_I = the mass of the target nucleus. When $M_I \to \infty$, of course, the distinctions between r, r', and r'' disappear. So we write

$$\psi(\mathbf{r}_p', \mathbf{r}_n') = \Psi(\mathbf{r}_p'', \mathbf{r}_n) = \Psi_D + \Psi_F. \quad (4)$$

Now let us write these wave functions as Fourier integrals over the proton wave number, and let us also express the angular dependence of the neutron coordinate in terms of spherical harmonics:

$$\Psi_D = \sum_{l,m} \int d\mathbf{k} \Phi_{l,m}(\mathbf{K}, \mathbf{k}, r_n) Y_l^m(\theta_n, \varphi_n) e^{i\mathbf{k}\cdot\mathbf{r}_p''}, \quad (5)$$

$$\Psi_F = \sum_{l,m} \int d\mathbf{k} A_{l,m}(\mathbf{K}, \mathbf{k}) h_l^{(1)}(k_n r_n) \times Y_l^m(\theta_n, \varphi_n) e^{i\mathbf{k}\cdot\mathbf{r}_p''}, \quad r_n \geq R. \quad (6)$$

Y_l^m is the normalized spherical harmonic and $h_l^{(1)}$ is the spherical Hankel function of the first kind. $h_l^{(1)}$ is used to represent the free neutrons for $r_n \geq R$ since these particles are all created at the nuclear surface and consequently must be all outgoing in this region. $\mathbf{K}$ is the wave number of the plane wave of incident deuterons relative to the center of mass of the deuteron and the target nucleus, $\mathbf{k}$ is the proton wave number relative to the center of mass of the deuteron and the target nucleus, and $\mathbf{k}_n$ is the neutron wave number relative to the center of mass of the neutron and the target nucleus. The three are related by the conservation of energy for the deuteron fragments:

$$\frac{\hbar^2 k_n^2}{2M_{nI}} + \frac{\hbar^2 k^2}{2M_{pF}} = \frac{\hbar^2 K^2}{2M_{DI}} - \epsilon, \quad (7)$$

where $M_{AB} = M_A M_B/(M_A + M_B)$, $M_F = M_I + M_n$, and ϵ = magnitude of the deuteron binding energy. Equation (7) thus provides a definition of k_n for Eq. (6). Real values of k_n correspond to outgoing waves of neutrons while imaginary values of k_n cause $h_l^{(1)}$ to become a decreasing exponential representing the situation in which a neutron is captured into a bound state.

The coefficients $A_{l,m}$ appearing in Ψ_F can be expressed in terms of the boundary conditions for the neutron on the nuclear surface, $r_n = R$. Writing our wave function in the form

$$\Psi = \sum_{l,m} \int d\mathbf{k} (\Phi_{l,m} + A_{l,m} h_l^{(1)}) e^{i\mathbf{k}\cdot\mathbf{r}_p''} Y_l^m = \sum_{l,m} \Psi_{l,m} Y_l^m, \quad (8)$$

we require that

$$\left. \frac{\partial}{\partial r_n}(r_n \Psi_{l,m}) \right|_R = \mathfrak{f}_l [\Psi_{l,m}]_R. \quad (9)$$

The quantities $\mathfrak{f}_l$ are the parameters which characterize the nucleus in its neutron interactions. Because of the orthogonality of the plane waves $e^{i\mathbf{k}\cdot\mathbf{r}_p''}$, this can be written

$$\left. \frac{(\partial/\partial r_n)(\Phi_{l,m} + A_{l,m} h_l^{(1)})}{\Phi_{l,m} + A_{l,m} h_l^{(1)}} \right|_R = \frac{\mathfrak{f}_l - 1}{R}. \quad (10)$$

Solving for A,

$$A_{l,m} = -\left. \frac{R(\partial/\partial r_n)\Phi_{l,m} - (\mathfrak{f}_l - 1)\Phi_{l,m}}{R(\partial/\partial r_n) h_l^{(1)} - (\mathfrak{f}_l - 1) h_l^{(1)}} \right|_R. \quad (11)$$

To get an expression for the stripping cross section we will need the wave function describing the neutrons associated with protons having wave number $\mathbf{k}$. Our wave function $\Psi(\mathbf{r}_n, \mathbf{r}_p'')$ describes the neutrons associated with protons having the position $\mathbf{r}_p''$. We propose to change from a representation in $\mathbf{r}_n$, $\mathbf{r}_p''$ space to a representation in $\mathbf{r}_n$, $\mathbf{k}$ space. The wave function in $\mathbf{r}_n$, $\mathbf{k}$ space, i.e., the wave function for neutrons associated with protons of wave number $\mathbf{k}$, is gotten from $\Psi(\mathbf{r}_n, \mathbf{r}_p'')$ in the following manner:

$$\bar{\varphi}(\mathbf{k}, \mathbf{r}_n) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d\mathbf{r}_p'' \Psi(\mathbf{r}_n, \mathbf{r}_p'') e^{-i\mathbf{k}\cdot\mathbf{r}_p''}. \quad (12)$$

Performing this integration gives

$$\bar{\varphi}(\mathbf{k}, \mathbf{r}_n) = (2\pi)^{\frac{3}{2}} \sum_{l,m} \Phi_{l,m} Y_l^m + (2\pi)^{\frac{3}{2}} \sum_{l,m} A_{l,m} k_l^{(1)} Y_l^m = \bar{\varphi}_D + \bar{\varphi}_F = \bar{\varphi}_T, \quad r_n \geq R. \quad (13)$$

Using $\bar{\varphi}$, we calculate the radial neutron current density associated with protons of wave number $\mathbf{k}$:

$$J_\alpha = \frac{\hbar}{2M_{nI} i} \left(\bar{\varphi}_\alpha^* \frac{\partial}{\partial r_n} \bar{\varphi}_\alpha - \bar{\varphi}_\alpha \frac{\partial}{\partial r_n} \bar{\varphi}_\alpha^* \right), \quad (14)$$

where $\alpha = D, F, T$. Integrating $-J_T$ over the surface of a sphere concentric with the target nucleus and having a radius greater than or equal to R gives the net flux F_i into the nucleus of neutrons associated with protons having wave number $\mathbf{k}$. Choosing this sphere at the surface of the nucleus gives

$$F_i = \frac{i\hbar(2\pi)^3 R}{2M_{nI}} \sum_{l,m} |\Phi_{l,m} + A_{l,m} h_l^{(1)}|_R^2 (\mathfrak{f}_l - \mathfrak{f}_l^*)$$

$$= \frac{i\hbar(2\pi)^3 R^3}{2M_{nI}} \sum_{l,m} \frac{\left| \frac{\partial \Phi_{l,m}}{\partial r_n} - \frac{\Phi_{l,m}}{h_l^{(1)}} \frac{\partial h_l^{(1)}}{\partial r_n} \right|_R^2}{|\mathfrak{f}_l - \rho_l^{(1)}|^2} (\mathfrak{f}_l - \mathfrak{f}_l^*), \quad (15)$$

where

$$\rho_l^{(i)} = \Delta_l^{(i)} + i s_l^{(i)} = 1 + \left. \frac{R}{h_l^{(i)}(k_n r_n)} \frac{\partial h_l^{(i)}(k_n r_n)}{\partial r_n} \right|_R.$$

Integrating J_F over a sphere concentric with the target nucleus gives the net flux F_0 of the scattered free neutrons associated with protons of wave number **k**:

$$F_0=\frac{i\hbar(2\pi)^3R}{2M_{nI}}\sum_{l,m}|A_{l,m}h_l^{(1)}|_R{}^2(-2is_l^{(1)})$$

$$=\frac{i\hbar(2\pi)^3R^3}{2M_{nI}}\sum_{l,m}\left|\frac{\Phi_{lm,}}{R}-\frac{\dfrac{\partial\Phi_{l,m}}{\partial r_n}-\dfrac{\Phi_{l,m}}{h_l^{(1)}}\dfrac{\partial h_l^{(1)}}{\partial r_n}}{\mathfrak{f}_l-\rho_l^{(1)}}\right|_R^2 \times(-2is_l^{(1)}). \quad (16)$$

Thus the total flux of neutrons either absorbed or scattered associated with protons of wave number **k** in the interval $d\mathbf{k}$ is

$$Fd\mathbf{k}=(F_i+F_0)d\mathbf{k}. \quad (17)$$

But there is a one-to-one correspondence between the liberated protons and neutrons. Hence $Fd\mathbf{k}$ is the total flux of liberated protons coming out with wave number **k** in $d\mathbf{k}$. Letting $\mathcal{J}$ represent the current density of the incident deuterons, the d-p cross section can now be written

$$d\sigma(\mathbf{k})=\frac{(F_i+F_0)}{\mathcal{J}}\frac{d\mathbf{k}}{d\Omega}=\frac{(F_i+F_0)}{\mathcal{J}}k^2dk$$

$$=\frac{(2\pi)^3\hbar R^3k^2dk}{\mathcal{J}M_{nI}}$$

$$\times\sum_{l,m}\left\{\frac{(\mathfrak{f}_l{}^*-\mathfrak{f}_l)}{2i}\frac{\left|\dfrac{\partial\Phi_{l,m}}{\partial r_n}-\dfrac{\Phi_{l,m}}{h_l^{(1)}}\dfrac{\partial h_l^{(1)}}{\partial r_n}\right|_R^2}{|\mathfrak{f}_l-\rho_l^{(1)}|^2}+s_l^{(1)}\left|\frac{\dfrac{\partial\Phi_{l,m}}{\partial r_n}-\dfrac{\Phi_{l,m}}{h_l^{(1)}}\dfrac{\partial h_l^{(1)}}{\partial r_n}}{\mathfrak{f}_l-\rho_l^{(1)}}-\frac{\Phi_{l,m}}{R}\right|^2\right\}. \quad (18)$$

If we were dealing with the interaction of free neutrons with a target nucleus, we would have a wave function of the form

$$\psi=\sum C_{lm}Y_l{}^m(h_l^{(2)}(k_nr)-\eta_lh_l^{(1)}(k_nr)). \quad (19)$$

Using this wave function, one finds

$$\mathfrak{f}_l=\left.\frac{h_l^{(2)}\rho_l^{(2)}-\eta_l\rho_l^{(1)}h_l^{(1)}}{h_l^{(2)}-\eta_lh_l^{(1)}}\right|_R, \quad (20)$$

$$\sigma_r^{(l)}=[(2l+1)\pi/k_n{}^2](1-\eta_l\eta_l{}^*) \quad (21)$$

= total reaction cross section for l neutrons,

$$\sigma_s^{(l)}=[(2l+1)\pi/k_n{}^2]|1-\eta_l|^2 \quad (22)$$

= total scattering cross section for l neutrons.

Now if we define an η_l by means of Eqs. (9) and (20), and then use Eq. (20) to eliminate $\mathfrak{f}_l$ from (18), we will get

$$d\sigma(\mathbf{k})=\frac{2\pi^2\hbar R^5k_n{}^2|k_n|^2k^2dk}{\mathcal{J}M_{nI}}\sum_{l,m}\frac{s_l^{(1)}|h_l^{(1)}h_l^{(2)}|_R{}^2}{(2l+1)}\times\{M^{(2)}\bar{\sigma}_s{}^{(l)}+M^{(1)}\bar{\sigma}_r{}^{(l)}\}, \quad (23)$$

where

$$M^{(i)}=\left|\frac{\partial\Phi_{l,m}}{\partial r_n}-\frac{\Phi_{l,m}}{h_l^{(i)}}\frac{\partial h_l^{(i)}}{\partial r_n}\right|_R^2,$$

$$\bar{\sigma}_r{}^{(l)}=\frac{(2l+1)\pi}{k_n{}^2}\begin{cases}(1-\eta_l\eta_l{}^*), & k_n{}^2>0,\\ [(\eta_l-\eta_l{}^*)/k_nRs_l^{(1)}(h_l^{(2)})^2] & k_n{}^2<0,\end{cases}$$

$$\bar{\sigma}_s{}^{(l)}=\frac{(2l+1)\pi}{k_n{}^2}|1-\bar{\gamma}\eta_l|^2,$$

$$\bar{\gamma}=\left[\frac{h_l^{(1)}\partial\Phi_{l,m}/\partial r_n-\Phi_{l,m}\partial h_l^{(1)}/\partial r_n}{h_l^{(2)}\partial\Phi_{l,m}/\partial r_n-\Phi_{l,m}\partial h_l^{(2)}/\partial r_n}\right]_R.$$

When $\Phi_{l,m}$ and $\partial\Phi_{l,m}/\partial r_n$ are both pure real or pure imaginary, as will be the case in the approximation introduced in Part IV, then

$$M^{(1)}=M^{(2)} \quad \text{and} \quad \bar{\gamma}=\exp(i\alpha) \quad \text{for } k_n{}^2>0.$$

In this case we can interpret Eq. (23) by saying each term of the stripping cross section consists of two factors: one factor represents the probability that a neutron with quantum numbers l and m arrives at the nuclear surface, and the other factor is the cross section for interaction of such neutrons. The appearance of the phase shift α in $\bar{\sigma}_s{}^{(l)}$ no doubt reflects the fact that the outgoing free neutrons do not interfere with free incident neutrons but with incident neutrons bound to protons.

The η_l appearing in Eq. (23) is not necessarily the same as the η_l appearing in Eq. (19). The η_l of Eq. (19) represents the interaction of free neutrons with a target nucleus. The η_l of Eq. (23) must describe the interaction of neutrons with a target nucleus in the presence of the accompanying proton. Nevertheless, it may not be unreasonable to assume that the two η_l's are similar. Such an assumption, while not necessary, is consistent with the previous neglect of proton interaction. It can be tested in the range of single-particle interactions if the reaction cross section is predominant. The scattering cross section should not be expected to be the same for single particle and deuteron reactions because of the phase shift α.

Up to this point, we have ignored the fact that the particles we have been discussing have intrinsic spins. To be exact, we have assumed that the interaction between the stripped-off particle (the neutron) and the target nucleus is independent of spin. It is our purpose now to generalize our result to allow for the possibility that the neutron and target nucleus interaction might depend on the total angular momentum J of the neutron and target nucleus system and on the difference in parity π between the final target nucleus and neutron system and the initial target nucleus. This generalization depends on the interpretation of our expression for the stripping cross section, say Eq. (18), as a sum of partial cross sections arising from the interaction with the target nucleus of neutrons of different l and m:

$$d\sigma=\sum_{l,m} d\sigma^{(l,m)}(\mathfrak{f}_l). \tag{24}$$

We effect our proposed generalization by representing each partial cross section $d\sigma^{(l,m)}$ as a sum of cross sections corresponding to all the different values of J and π attainable by neutrons of orbital angular momentum $l\hbar$, each of these terms being multiplied by the appropriate statistical weight:

$$d\sigma=\sum_{l,m}\sum_{\pi,J=||I-l|-\frac{1}{2}|}^{J=I+l+\frac{1}{2}}\frac{(2J+1)}{2(2I+1)(2l+1)}d\sigma^{(l,m)}(\mathfrak{f}_l(J,\pi)). \tag{25}$$

In Eq. (25), I is the spin of the target nucleus. Consequently Eq. (18) becomes

$$d\sigma(\mathbf{k})=\frac{(2\pi)^3\hbar R^3k^2dk}{\mathcal{G}M_{nI}}\sum_{l,m,\pi,J=||I-l|-\frac{1}{2}|}^{J=I+l+\frac{1}{2}}\frac{(2J+1)}{2(2I+1)(2l+1)}$$
$$\times\left\{\frac{(\mathfrak{f}_l^*-\mathfrak{f}_l)\left|\frac{\partial\Phi_{l,m}}{\partial r_n}-\frac{\Phi_{l,m}}{h_l^{(1)}}\frac{\partial y_l^{(1)}}{\partial r_n}\right|_R^2}{2i\qquad |\mathfrak{f}_l-\rho_l^{(1)}|^2}\right.$$
$$\left.+s_l^{(1)}\left|\frac{\frac{\partial\Phi_{l,m}}{\partial r_n}-\frac{\Phi_{l,m}}{h_l^{(1)}}\frac{\partial h_l^{(1)}}{\partial r_n}}{\mathfrak{f}_l-\rho_l^{(1)}}-\frac{\Phi_{l,m}}{R}\right|_R^2\right\}, \tag{26}$$

where $\mathfrak{f}_l$ is now regarded as a function of J and π as well as l.

III. INTEGRATION OF THE CROSS SECTION OVER A RESONANCE

Next let us assume that there is a resonance in the neutron-nucleus interaction and integrate the cross section over this resonance. Such a resonance corresponds to a level in the neutron+nucleus system. Our result will be the cross section for stripping by this level.

In the vicinity (on the neutron energy scale) of a resonance due to a level characterized by

$$J=J_1,\quad \pi=\pi_1,$$
$$\text{reduced width}=\gamma_l=\gamma_l(J,\pi)=\gamma\delta_{J,J_1}\delta_{\pi,\pi_1},$$
$$\text{radiation width}=\Gamma_r=\Gamma_r^{(l)}(J_1,\pi_1),$$
$$\text{energy}=E_1'=E_1-\gamma\Delta_{l_1}^{(1)},$$

$\mathfrak{f}_l$ has the form[8]

$$\mathfrak{f}_l=-(1/\gamma_l)(E-E_1+\tfrac{1}{2}i\Gamma_r). \tag{27}$$

[8] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), Chap. VIII.

Substituting Eq. (27) into Eq. (26) gives

$$d\sigma(\mathbf{k})=\frac{(2\pi)^3R^3M_{pF}kdE}{\hbar\mathcal{G}M_{nI}}\sum_{l,m,\pi,J}\frac{(2J+1)\gamma_l}{2(2I+1)(2l+1)}$$
$$\times\left\{\frac{\Gamma_r}{2}\frac{\left|\frac{\partial\Phi_{l,m}}{\partial r_n}-\frac{\Phi_{l,m}}{h_l^{(1)}}\frac{\partial h_l^{(1)}}{\partial r_n}\right|_R^2}{\{(E-E_1')^2+\Gamma^2/4\}}\right.$$
$$\left.+\frac{\Gamma_\alpha}{2}\left|\frac{\Phi_{l,m}}{\gamma_lR}+\frac{\frac{\partial\Phi_{l,m}}{\partial r_n}-\frac{\Phi_{l,m}}{h_l^{(1)}}\frac{\partial h_l^{(1)}}{\partial r_n}}{E-E_1'+\frac{1}{2}i\Gamma}\right|_R^2\right\}, \tag{28}$$

where $\Gamma_\alpha=2s_l^{(1)}\gamma_l$, $\Gamma=\Gamma_\alpha+\Gamma_r$, and Eq. (7) has been used to give

$$kdk=-(M_{pF}/\hbar^2)dE.$$

We will assume that the total level width Γ is small compared to the energy interval required to produce an appreciable change in the functions appearing in Eq. (28) (excluding, of course, the resonance denominators). Then we can carry out the integration over the resonance by means of the following relations:

$$\int_{-\chi/2}^{\chi/2}\frac{dy}{y^2+\frac{1}{4}\Gamma^2}=\frac{4}{\Gamma}\tan^{-1}\left(\frac{\chi}{\Gamma}\right)=\frac{4}{\Gamma}\left(\frac{\pi}{2}-\frac{\Gamma}{\chi}+\cdots\right), \tag{29}$$

$$\int_{-\chi/2}^{\chi/2}\left|A+\frac{B}{y+\frac{1}{2}i\Gamma}\right|^2dy=AA^*\chi$$
$$+[BB^*+\tfrac{1}{2}i\Gamma(AB^*-A^*B)]\frac{4}{\Gamma}\tan^{-1}\left(\frac{\chi}{\Gamma}\right). \tag{30}$$

Taking $\Gamma/\chi\to 0$, we get

$$\sigma(\mathbf{k}_1,\pi_1,J_1)=\frac{(2\pi)^3R^3k_1M_{pF}}{\hbar\mathcal{G}M_{nI}}\left\{\sum_{l,m}s_l^{(1)}\chi\left|\frac{\Phi_{l,m}}{R}\right|^2\right.$$
$$+\sum_{l,m}{}'\frac{(2J_1+1)\pi\gamma}{2(2I+1)(2l+1)}\left[\left|\frac{\partial\Phi_{l,m}}{\partial r_n}-\frac{\Phi_{l,m}}{h_l^{(1)}}\frac{\partial h_l^{(1)}}{\partial r_n}\right|^2\right.$$
$$-2(s_l^{(1)})^2\left|\frac{\Phi_{l,m}}{R}\right|^2+\frac{is_l^{(1)}}{R}\left(\Phi_{l,m}\frac{\partial\Phi_{l,m}^*}{\partial r_n}\right.$$
$$\left.\left.\left.-\Phi_{l,m}^*\frac{\partial\Phi_{l,m}}{\partial r_n}\right)\right]\right\}, \tag{31}$$

where χ is the range of integration and the sum over l in the primed sum must be limited to the range

$$||I-J|-\tfrac{1}{2}|\leqslant l\leqslant I+J+\tfrac{1}{2},$$

all the l's in the primed sum being even or odd depending on the value of π_1.

The unprimed sum in 31 represents the contribution of potential scattering, the first term after the primed summation sign results from resonance scattering and absorption, and the second and third terms following the primed summation sign arise from interference between potential scattering and resonance scattering. Clearly, the difference between the $k_{n1}^2<0$ ($s_l^{(1)}=0$) case and the $k_{n1}^2>0$ case is all due to the appearance of potential scattering when $k_{n1}^2>0$.

We see that the only nuclear parameters entering our final expression are the radius R and the reduced width γ. For virtual states, $k_{n1}^2>0$, we can interpret γ in terms of the partial width for neutron emission, Γ_α:

$$\gamma=\Gamma_\alpha/2s_l^{(1)}. \tag{32}$$

For bound states, $k_{n1}^2<0$, we may use a suggestion of Feshbach *et al.*[9] They suggest that the quantity

$$D^*=\pi K_0 R\gamma, \quad K_0\approx 1.0\times 10^{13}\ \mathrm{cm}^{-1}, \tag{33}$$

may correspond to the approximate average energy level separation at the excitation energy E_1'.

Since the reduced width γ of a level enters the expression for the cross section essentially as a normalizing factor, the use of stripping experiments to determine γ will not be so sensitive to errors introduced by poor resolution as are the usual methods for measuring level widths.

IV. AN APPROXIMATION FOR THE DEUTERON WAVE FUNCTION

To apply our results, we must know the deuteron wave function Ψ_D. If the cross sections are sufficiently small, the deuteron wave function can be well approximated by the incident plane wave. Suppose the internal wave function of the deuteron is

$$\left[\frac{\alpha\beta(\alpha+\beta)}{2\pi(\alpha-\beta)^2}\right]^{\frac{1}{2}}\frac{(e^{-\alpha r}-e^{-\beta r})}{r}=\frac{N(e^{-\alpha r}-e^{-\beta r})}{r}, \tag{34}$$

where $\mathbf{r}=\mathbf{r}_p-\mathbf{r}_n$ ($\mathbf{r}_p$ = separation of proton from target nucleus). This is equivalent to the assumption that the proton-neutron interaction is given by the Hulthèn potential. Then, on the basis of the above assumptions,

$$\Psi_D=e^{i\mathbf{K}\cdot\mathbf{R}}N(e^{-\alpha r}-e^{-\beta r})/r, \tag{35}$$

where

$$\mathbf{R}=(M_n\mathbf{r}_n+M_p\mathbf{r}_p)/(M_n+M_p).$$

This approximation for Ψ_D is also used by Butler.[7]

The deuteron current density becomes

$$\mathcal{J}=K\hbar/M_D, \tag{36}$$

and

$$\Phi_{l,m}=\frac{1}{(2\pi)^3}\int d\mathbf{r}_p''d\Omega_n e^{-i\mathbf{k}\cdot\mathbf{r}_p''}Y_l^{-m}(\theta_n,\varphi_n)\Psi_D$$

$$=\frac{2N}{\pi}\left\{\frac{1}{\alpha^2+k_d^2}-\frac{1}{\beta^2+k_d^2}\right\}i^l j_l(\kappa r_n)Y_l^{-m}(\theta_\kappa,\varphi_\kappa), \tag{37}$$

where

$$\boldsymbol{\kappa}=\mathbf{K}-\frac{M_I}{M_I+M_n}\mathbf{k},$$

$$\mathbf{k}_d=\mathbf{k}-\frac{M_{np}}{M_n}\mathbf{K}.$$

$Y_l^m(\theta_\kappa,\varphi_\kappa)$ is defined relative to the same axis as the $Y_l^m(\theta_n,\varphi_n)$ which appears in Eqs. (5) and (6). We can choose this axis parallel to any function of $\mathbf{K}$ and $\mathbf{k}$ we like. We choose the axis to be parallel to $\mathbf{K}-\mathbf{k}$. Then

$$\Phi_{l,m}=\frac{N}{\pi^{\frac{3}{2}}}(2l+1)^{\frac{1}{2}}\delta_{m,0}i^l\left\{\frac{1}{\alpha^2+k_d^2}-\frac{1}{\beta^2+k_d^2}\right\}j_l(\kappa r_n). \tag{38}$$

Substitution into Eq. (31) gives

$$\sigma(\mathbf{k}_1,\pi_1,J_1)=\frac{M_{DI}M_{pF}8N^2R^3k_1}{K\hbar^2M_{nI}}\left\{\frac{1}{\alpha^2+k_{d1}^2}-\frac{1}{\beta^2+k_{d1}^2}\right\}^2$$
$$\times\left\{\sum_l(2l+1)s_l^{(1)}\chi\left|\frac{j_l(\kappa_1R)}{R}\right|^2+\sum_l{}'\frac{(2J_1+1)\gamma\pi}{2(2I+1)}\right.$$
$$\times\left[\left|\frac{\partial j_l(\kappa_1r_n)}{\partial r_n}-\frac{j_l(\kappa_1r_n)}{h_l^{(1)}(k_{n1}r_n)}\frac{\partial h_l^{(1)}(k_{n1}r_n)}{\partial r_n}\right|_R^2\right.$$
$$\left.\left.-2(s_l^{(1)})^2\left|\frac{j_l(\kappa_1R)}{R}\right|^2\right]\right\}. \tag{39}$$

The quantity

$$\sum_l{}'\left\{\frac{1}{\alpha^2+k_{d1}^2}-\frac{1}{\beta^2+k_{d1}^2}\right\}^2$$
$$\times\left|\frac{\partial j_l(\kappa_1r_n)}{\partial r_n}-\frac{j_l(\kappa_1r_n)}{h_l^{(1)}(k_{n1}r_n)}\frac{\partial h_l^{(1)}(k_{n1}r_n)}{\partial r_n}\right|_R^2$$

is just the angular distribution derived by Butler. Thus our theory is identical to Butler's in so far as the relationship between the orbital angular momentum of the captured particle and the angular distribution of the liberated particle for stripping by bound levels is concerned. The agreement of this relationship with experiment is surprisingly good in view of the fact that the Coulomb effect has been neglected.

V. THE COULOMB EFFECT

To include the effect of the electrostatic interaction between the proton and the target nucleus in our theory we must replace the free space proton functions used in our analysis by their Coulomb analogs, and Ψ_D must be replaced by a wave function $\Psi_D{}^c$, which describes the motion of the incident and scattered deuterons without neglecting the Coulomb field. These functions[10] will be

[9] Feshbach, Peaslee, and Weisskopf, Phys. Rev. 71, 145 (1947).

[10] *Tables of Coulomb Wave Functions*, Volume 1, National Bureau of Standards: Applied Mathematics, Series 17.

defined in terms of the two solutions, $F_l(\rho)$ and $G_l(\rho)$, of the equation

$$\left\{\frac{d^2}{d\rho^2}+\left[1-\frac{2\bar{\eta}}{\rho}-\frac{l(l+1)}{\rho^2}\right]\right\}y=0. \quad (40)$$

F_l is the regular solution, G_l is the irregular solution, and they are normalized so that they have the asymptotic forms

$$F_l(\rho)\underset{\rho\to\infty}{\longrightarrow}\sin[\rho-\bar{\eta}\ln 2\rho-\tfrac{1}{2}l\pi+\sigma_l],$$
$$G_l(\rho)\underset{\rho\to\infty}{\longrightarrow}\cos[\rho-\bar{\eta}\ln 2\rho-\tfrac{1}{2}l\pi+\sigma_l], \quad (41)$$

where

$$\sigma_l=\arg\{\text{gamma function }(l+1+i\bar{\eta})\}. \quad (42)$$

Now we list the free space functions we have used in our analysis together with their Coulomb analogs:

$$\bar{\eta}\to 0,$$
$$f_l(kr)=\frac{F_l(kr)e^{-i\sigma_l}}{kr}\longrightarrow j_l(kr),$$
$$g_l(kr)=-\frac{G_l(kr)e^{-i\sigma_l}}{kr}\longrightarrow n_l(kr),$$
$$y_l^{(1)}(kr)=f_l+ig_l\to h_l^{(1)}(kr), \quad (43)$$
$$y_l^{(2)}(kr)=f_l^*-ig_l^*\to h_l^{(2)}(kr),$$
$$\mathcal{C}(\mathbf{kr})=4\pi\sum_{l,m}i^lY_l^m(\theta_r,\varphi_r)Y_l^{-m}(\theta_k,\varphi_k)\times f_l^*(kr)\to e^{i\mathbf{k}\cdot\mathbf{r}},$$
$$\mathfrak{C}(\mathbf{kr})=4\pi\sum_{l,m}i^lY_l^m(\theta_r,\varphi_r)Y_l^{-m}(\theta_k,\varphi_k)\times f_l(kr)\to e^{i\mathbf{k}\cdot\mathbf{r}}.$$

For our problem

$$\bar{\eta}=Ze^2M_{pI}/\hbar^2k, \quad (44)$$

where e is the charge of the electron. $\mathfrak{C}$ is constructed so that its *outgoing* part is asymptotically a pencil of particles of wave number $\mathbf{k}$. $\mathcal{C}$ is constructed so that its *incoming* part is asymptotically a pencil of particles of wave number $\mathbf{k}$.

Now if we re-examine our derivation in part II, we will find that the only properties of the free space functions that played a role in the argument were:

(a) the Wronskian property of the h's [Eq. (11)],
(b) the orthonormality of the plane waves,
(c) the interpretation of the outgoing part of a plane wave as a pencil of particles.

It can be easily verified that the y's have the same Wronskian property as the h's and the $\mathfrak{C}$'s have the same orthonormality property as plane waves. Asymptotically the outgoing part of $\mathfrak{C}$ has the same interpretation as the outgoing part of a plane wave. Thus we can take the Coulomb effect into account with a single stroke—we merely take our previous results and substitute $\Psi_D{}^c$ for Ψ_D and substitute y_l for h_l and $\mathfrak{C}(\mathbf{kr})$ for $e^{i\mathbf{k}\cdot\mathbf{r}}$ when the argument contains the proton coordinate. Thus, if we write the deuteron wave function in the form

$$\Psi_D{}^c=\sum_{l,m}\int d\mathbf{k}\Phi_{l,m}{}^{cp}(\mathbf{K},\mathbf{k},r_n)Y_l^m(\theta_n,\varphi_n)\mathfrak{C}(\mathbf{k},r_p'')$$
$$=\sum_{l,m}\int d\mathbf{k}\Phi_{l,m}{}^{cn}(\mathbf{K},\mathbf{k},r_p)Y_l^m(\theta_p,\varphi_p)e^{i\mathbf{k}\cdot\mathbf{r}_n''}, \quad (45)$$

the d-p cross section is identical with Eq. (31) except that $\Phi_{l,m}$ must be replaced by $\Phi_{l,m}{}^{cp}$, and the d-n cross section results from Eq. (31) when $\Phi_{l,m}$ is replaced by $\Phi_{l,m}{}^{cn}$ and $s_l^{(1)}$ is replaced by

$s_l^{(1)c}$= imaginary part of

$$\left[1+\frac{R}{y_l^{(1)}(k_p,r_p)}\frac{\partial y_l^{(1)}(k_p,r_p)}{\partial r_p}\right]_R. \quad (46)$$

Actually this procedure introduces an additional approximation for the d-p case. The calculation for the d-p case involves using $\mathfrak{C}(\mathbf{k}_p,\mathbf{r}_p'')$ to represent liberated protons. Such a representation would be correct if the electric charge of the target nucleus resided at $r_p''=0$ rather than at $r_p=0$. Our approximation is justified when the neutron is captured into a bound state or when the target nucleus is heavy.

Application of these Coulomb expressions to the interpretation of experiments must wait upon the discovery of suitable approximations to the $\Phi_{l,m}{}^{cp}$ and $\Phi_{l,m}{}^{cn}$ defined in Eqs. (45).

VI. COMRARISON WITH EXPERIMENT

According to the results of Sec. IV, the angular dependence of the cross section for stripping due to a certain level in the residual stripped-off particle and target nucleus system can be used to identify the orbital angular momentum of the stripped-off particle in that level, while the normalization of the cross section determines the reduced width of the level. We will consider two experiments—one involving stripping by bound levels and the other involving stripping by virtual levels. We will evaluate l, γ, and D^* for these levels.

As our first example we will consider an experiment involving capture into bound states. We use the data of Fulbright *et al.*,[11] who measured the angular distributions for the $Be^9(d,p)Be^{10}$ and $Be^9(d,p)Be^{10*}$ reactions. The differential cross sections for these processes were measured with an accuracy of only 20 percent. The angular distributions can both be fitted by choosing $l=1$, $R=4.5\times10^{-13}$ cm. Using these parameters and their data, we get the following values for the reduced

[11] Fulbright, Bruner, Bromley, and Goldman, Phys. Rev. 88, 700 (1952).

widths of these levels:

$\gamma = 36/(2J+1)$ kev for the ground state,

$\gamma = 41/(2J+1)$ kev for the excited state.

With these values of γ the "approximate average level spacing" D^* is

$D^* = 620/(2J+1)$ kev for the ground state,

$D^* = 690/(2J+1)$ kev for the excited state.

In these expressions J is the spin of the residual nucleus. Since the spin of Be^9 is $\frac{3}{2}$ and since the neutron is captured with $l=1$, J must have a value between 3 and 0.

So far as we know, the only investigation of stripping angular distributions due to virtual levels has been done by Goldberg.[12] Using 4-Mev deuterons to initiate the $Mg^{24}(d,n)Al^{25}$ reaction, he has measured the angular distribution for three virtual levels in Al^{25}.

To analyze Goldberg's data we will assume that the stripped-off particle can be captured into a given level with only one value of l which is characteristic of that level. Then the cross section for the d-n reaction becomes

$$\sigma_l = \frac{(4\pi^4)R^3 k_1 M_{nF}(2J+1)\gamma}{\hbar \mathfrak{g} M_{pI}(2l+1)(2I+1)} \times \sum_m \left\{ \left| \frac{\partial \Phi_{l,m}{}^{cn}}{\partial r_p} - \frac{\Phi_{l,m}{}^{cn}}{y_l{}^{(1)}} \frac{\partial y_l{}^{(1)}}{\partial r_p} \right|_R^2 + \frac{s_l{}^{(1)c} i}{R} \left(\Phi_{l,m}{}^{cn} \frac{\partial \Phi_{l,m}{}^{cn*}}{\partial r_p} - \Phi_{l,m}{}^{cn*} \frac{\partial \Phi_{l,m}{}^{cn}}{\partial r_p} \right) - 2(s_l{}^{(1)c})^2 \left| \frac{\Phi_{l,m}{}^{cn}}{R} \right|^2 + \frac{2(2l+1)(2I+1)}{\gamma\pi} \times s_l{}^{(1)c} \chi \left| \frac{\Phi_{l,m}{}^{cn}}{R} \right|^2 \right\}. \quad (47)$$

The Q's for the three virtual levels measured are

$Q_1 = -2.44$ Mev, $Q_2 = -2.67$ Mev, $Q_3 = -3.04$ Mev,

and the incident deuterons had a mean energy of 3.97 Mev in the laboratory. We see that the protons are captured with very small energies:

$$E = -Q - \epsilon = -Q - 2.226 \text{ Mev},$$

$$E_1 = 0.21 \text{ Mev}, \quad E_2 = 0.44 \text{ Mev}, \quad E_3 = 0.81 \text{ Mev}.$$

As a consequence, the penetration factor,

$$s_l{}^{(1)c} = \frac{k_{p1}R}{F_l{}^2(k_{p1}R) + G_l{}^2(k_{p1}R)},$$

will be quite small. For instance, choosing $R = 5.8 \times 10^{-13}$ cm, $l=1$, and using E_3, one finds that

$$s_l{}^{(1)c} = 0.5 \times 10^{-2}.$$

[12] E. Goldberg, Phys. Rev. **89**, 760 (1953).

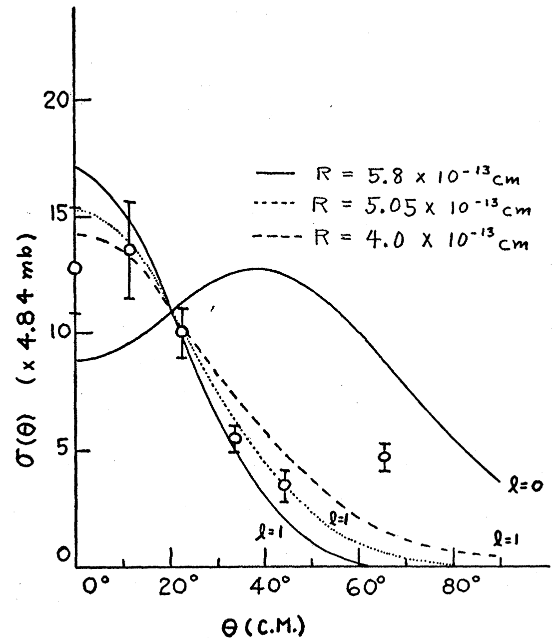

FIG. 1. Angular distribution (with respect to the center-of-mass system) of neutrons from the $Mg^{24}(d,n)Al^{25}$ reaction. Deuteron energy in the laboratory system was 3.973 Mev. The Q was -2.44 Mev.

Thus we can take $s_l{}^{(1)c} = 0$ and incur an error of less than 1 percent. In this approximation we have

$$\sigma_l = \frac{4\pi^4 R^3 M_{nF} k_1 (2J+1)\gamma_l}{\hbar \mathfrak{g} M_{pI}(2l+1)(2I+1)} \times \sum_m \left| \frac{\partial}{\partial r_p} \Phi_{l,m}{}^{cn} - \frac{(\Delta_l{}^{(1)c} - 1)}{R} \Phi_{l,m}{}^{cn} \right|^2. \quad (48)$$

Having used the Coulomb effect to eliminate the penetration factor we will now, somewhat inconsistently, neglect Coulomb effects. Neglecting the Coulomb effect allows us to use

$$\Phi_{l,m}{}^{cn} = \frac{N}{\pi^{\frac{1}{2}}} \delta_{m0} i^l (2l+1)^{\frac{1}{2}} \left\{ \frac{1}{\alpha^2 + k_d{}^2} - \frac{1}{\beta^2 + k_d{}^2} \right\} j_l(\kappa r_p), \quad (49)$$

with

$$\mathbf{k}_d = \mathbf{k} - \frac{M_{np}}{M_p} \mathbf{K},$$

$$\boldsymbol{\kappa} = \mathbf{K} - \frac{M_I}{M_F} \mathbf{k} = \mathbf{K} - \frac{M_I}{M_I + M_p} \mathbf{k},$$

giving us

$$\sigma_l = \frac{4\pi N^2 R^3 M_{nF} M_{DI} (2J+1)}{\hbar M_{pI}(2I+1)K} k_1 \gamma_l \left\{ \frac{1}{\alpha^2 + k_{d1}{}^2} - \frac{1}{\beta^2 + k_{d1}{}^2} \right\}^2 \times \left[\frac{\partial j_l(\kappa r_p)}{\partial r_p} - \frac{\Delta_l{}^{(1)c} - 1}{R} j_l(\kappa r_p) \right]_R^2. \quad (50)$$

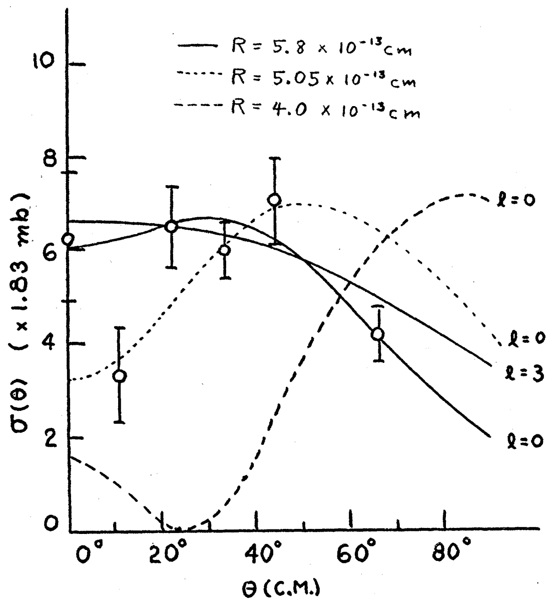

FIG. 2. Angular distribution (with respect to the center-of-mass system) of neutrons from the $Mg^{24}(d,n)Al^{25}$ reaction. Deuteron energy in the laboratory system was 3.973 Mev. The Q was -2.67 Mev.

We have plotted this angular distribution for various choices of R and l and compared it to Goldberg's data in Figs. 1, 2, and 3. The values of the parameters used in the calculations are

$$\alpha=2.32\times10^{12}\ \text{cm}^{-1},\qquad I=0,$$
$$\beta=16.24\times10^{12}\ \text{cm}^{-1},\qquad M_n=1.01\ \text{amu},$$
$$\epsilon=2.226\ \text{Mev},\qquad M_p=1.01\ \text{amu},$$
$$M_I=24.00\ \text{amu},\qquad M_D=2.01\ \text{amu},$$
$$M_F=25.00\ \text{amu},\qquad 1\ \text{amu}=1.66\times10^{-27}\ \text{kg}.$$

It is seen that the best fit to the data is secured by choosing

$$l=1,\quad R=5.05\times10^{-13}\ \text{cm},\quad \text{for } Q_1;$$
$$l=0,\quad R=5.8\times10^{-13}\ \text{cm},\quad \text{for } Q_2;$$
$$l=1,\quad R=4\times10^{-13}\ \text{cm},\quad \text{for } Q_3.$$

With these choices one gets the following values for the reduced widths γ_l and for D^*:

$$\gamma_1=327/(2J+1)\ \text{kev},\quad D^*=5180/(2J+1)\ \text{kev, for } Q_1;$$
$$\gamma_0=108\ \text{kev},\quad D^*=1965\ \text{kev, for } Q_2;$$
$$\gamma_1=207/(2J+1)\ \text{kev},\quad D^*=2600/(2J+1)\ \text{kev, for } Q_3.$$

Level 3 has been studied by Koester[13] by the elastic scattering of protons. Koester classifies this level as a $P_{\frac{3}{2}}$ state having a reduced width $\gamma=436$ kev. Using this assignment of J we get $\gamma=52$ kev. The values we get for D^* may be compared with the level spacing observed[14] in Mg^{25}. These turn out to be about 200 kev at the same excitation as the levels in Al^{25} discussed above.

The fact that we cannot fit the three curves using the same value of R is disappointing.

We have already noted that the boundary condition for the quantity $\Phi+Ah$ are not necessarily identical to that for a "free" neutron with the corresponding energy. Consequently, the reduced width that enters our expressions is not necessarily equal to that measured in single particle reactions. Nevertheless, it is to be expected that the two reduced widths are of the same order of magnitude.

The fact that we neglect the Coulomb interaction in our approximation for Φ leads us to underestimate the

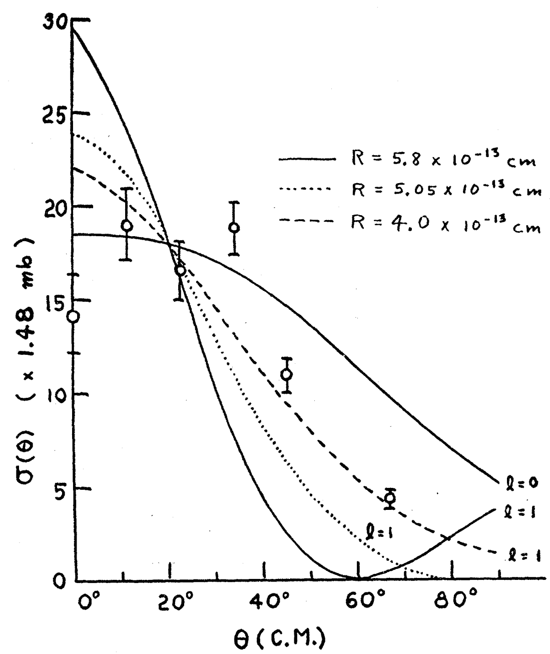

FIG. 3. Angular distribution (with respect to the center-of-mass system) of neutrons from the $Mg^{24}(d,n)Al^{25}$ reaction. Deuteron energy of the laboratory system was 3.973 Mev. The Q was -3.04 Mev.

reduced widths. This is because we expect the true expression for the stripping cross section to be smaller than ours by a factor giving the diminution of the square of the wave function of the incident deuterons resulting from the partial penetration of the deuterons into the Coulomb barrier.

VII. ACKNOWLEDGMENTS

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[13] L. J. Koester, Jr., Phys. Rev. **85**, 643 (1952).

[14] D. E. Alburger and E. M. Hafner, Revs. Modern Phys. **22** 373 (1950).