Mu Meson as Nuclear Probe Particle[1]

JOHN A. WHEELER
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
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The gamma ray ("Chang radiation") given out in the transition of a meson to the $1s$ ground state from a $2p$ orbit around a heavy nucleus of nonzero quadrupole moment is concluded to consist of several components of appreciable separation. Measurements of their energies should give information on (1) nuclear radius, (2) nuclear quadrupole moments, (3) the magnetic moment of the meson, (4) the nonuniformity of the nuclear charge distribution, and (5) nuclear polarizability and compressibility.

IN experiments in this laboratory, Chang has obtained evidence[2] for the emission of the order of three gamma rays on the average, in the energy interval 1 Mev to 5 or more Mev, for each negative meson stopped in lead. On the basis of theory,[3,4] these radiations are assigned in roughly comparable number to *nuclear* rearrangement radiation *following* the charge exchange reaction of the meson with the nucleus ($\mu^- + p \rightarrow n + \nu$), and to *mesonic* jumps between Bohr orbits roughly 7×10^{-8} sec *before* this reaction.[5]

The purpose of this note is to stress the value of more refined measurements of the energy and structure of the principal mesonic line in the Chang gamma radiation for our knowledge of (1) nuclear radii, (2) nuclear quadrupole moments, (3) the magnetic moment of the meson, (4) the nonuniformity of the nuclear charge distribution, and (5) nuclear polarizability and compressibility.

For probing the nuclear electric field the μ^- meson is an ideal test particle. Its specific interaction with the nucleus is exceedingly weak. To it, the nucleus appears as a transparent cloud of electricity. The degree of transparency is remarkable, in view of the density of nuclear matter, 1 or 2×10^5 tons/(mm).[3] Thus a meson moving in the K orbit of lead[4] spends roughly half its time within the nucleus, and in this period of $\sim4\times10^{-8}$ sec traverses about 5 meters of nuclear matter, or $\sim10^{17}$ g/cm^2.[6] This circumstance means that the major features of the nuclear electric field uniquely determine the mesonic energy level diagram. Conversely, these features can be determined by the position of the mesonic states.

Specifically, consider the $2p\rightarrow1s$ jump. The expected energy of this transition varies from 0.14 Mev for $_8$O, through 0.53 Mev for $_{16}$S, 1.6 Mev for $_{26}$Fe, to a value in the neighborhood of 5 Mev for $_{82}$Pb—the precise value for this and other heavy nuclei depending appreciably upon the nuclear radius. Otherwise stated, the energy of the resonance radiation in heavy elements depends upon the *density of electricity* in the nuclear interior, approaching in the idealized limit of nuclei large in comparison with the extension of the mesonic orbits the value characteristic of a harmonic oscillator,

$$E_{\text{res}} = \hbar e/\mu^{\frac{1}{2}} r_p^{3/2},$$

where $(4\pi/3)r_p^3$ is the volume occupied by one proton (and the neutrons associated with it). This gamma ray is favored for study both by convenient energy and by high relative probability of emission. Thus, of the mesons which get caught in an atomic field and tumble down from one Bohr orbit to another and finally reach the K level, the majority make this jump and send out the corresponding gamma "resonance radiation." For a nucleus with no quadrupole moment, this line is split by spin-orbit coupling into two components, the one due to $2p_{\frac{1}{2}}$ being half as strong as that which originates from $2p_{\frac{3}{2}}$. This stronger line is further split when there is a quadrupole moment.

The meson is the more tightly bound in the $2p$ state, the more of its time it spends within the nucleus; i.e., the more nearly its plane of motion parallels the directions in which the nucleus has a more than average amount of electric charge. The angle between the two directions is determined by the angle between the

[1] The author owes to Professor Rainwater the opportunity to let this note appear simultaneously with the important experimental results reported by Rainwater and Fitch in an accompanying paper [Phys. Rev. **92**, 789 (1953)]. The present note was written in October 1949 while the author was John Simon Guggenheim Memorial Fellow on leave of absence from Princeton, and was circulated privately. In the present text some small additions are made and a numerical error in the quadrupole splitting connected with the definition of the quadrupole moment is corrected. Thanks are expressed to Professor Leprince-Ringuet for the hospitality of the Laboratoire de Physique de l'École Polytéchnique, Paris. The work reported here is associated with the program of cosmic-ray and elementary particle research at Princeton University, supported under the joint program of the U. S. Atomic Energy Commission and the U. S. Office of Naval Research.

[2] W. Y. Chang, Revs. Modern Phys. **21**, 166 (1949); Phys. Rev. **75**, 1315 (1949).

[3] J. Tiomno and J. A. Wheeler, Revs. Modern Phys. **21**, 153 (1949).

[4] J. A. Wheeler, Revs. Modern Phys. **21**, 133 (1949).

[5] See Keuffel, Harrison, Godfrey, and Reynolds, Phys. Rev. **87**, 942 (1952) for measurements of this mean time for medium and heavy elements and references to such time measurements for lighter substances.

[6] The μ meson is not unique in this respect. The electron shows a comparable behavior. In fact, the nucleus is even somewhat more transparent to electrons than to μ mesons, because the energy release is smaller in the reaction, $e^- + p \rightarrow n + \nu$ (when it is energetically possible!) than in the corresponding mesonic charge exchange reaction, $\mu^- + p \rightarrow n + \nu$. But the *coupling constant* for the mesonic reaction is of the same order as that for the electronic charge exchange reaction. Of course, the K electron is much less useful than the K meson as a probe particle because the fraction of its time spent within the nucleus is roughly $[(\mu/m)^3 = 10^7]$-fold smaller.

angular momenta I of the nucleus and $j=\frac{3}{2}$ of the meson. When I is extremely large, the angle in question is the same in absolute value when $F=|\mathbf{F}|=|\mathbf{I}+\mathbf{j}|$ is equal to $I+\frac{1}{2}$ or $I-\frac{1}{2}$; and when $F=I+\frac{3}{2}$ or $I-\frac{3}{2}$. In this limit the quadrupole interaction splits $2p_{\frac{3}{2}}$ into two components of equal strength; but the values of I characteristic of actual nuclei give rise to three ($I=1$ or $\frac{3}{2}$) or four ($I>\frac{3}{2}$) components in the $2p_{\frac{3}{2}}$ level. With an electric quadrupole moment there is, of course, always associated a nuclear magnetic dipole moment and consequently a further displacement of the mesonic levels. However, this additional shift has already been shown[4] to be negligible (10 ev for Al^{27}) and is therefore disregarded here.

The energy of the two, four, or five components of the resonance radiation is determined by four quantities in the following order of importance (Fig. 1):

(1) A change in the *nuclear radius* of 10 percent alters the energy of all of the components of the lead line by about 0.4 Mev in 4.6 Mev (4.6 Mev is an approximate value calculated from a meson mass of $210m$ and the conventional nuclear radius, $R=1.4\times10^{-13}$ cm $A^{\frac{1}{3}}$; see reference 4; the experimental value is closer to 6 Mev, according to the accompanying report by Fitch and Rainwater). Consequently, a measurement of the center of gravity of the structure to an accuracy not out of reason, say 0.05 Mev, should determine the *effective* radius of the nucleus for *electric* interactions to 1 or 2 percent, a precision apparently better than that obtainable by the use of other probe particles.

(2) The *quadrupole moments* of many nuclei and the corresponding deviations of the nuclear electric fields from spherical symmetry produce percentagewise enormously greater effects in mesonic spectra than in electronic levels, owing to the greater fraction of the time spent by the meson within the nucleus.[6] For choice of elements for study of the splitting, one has to be guided by the known periodicities in the quadrupole moment as a function of mass and charge number:[7] the largest known moments are near Lu^{176} and Ta^{181}; there is possibly another maximum, with still larger moments, among the very heaviest nuclei.

A particularly appropriate case for discussion (Fig. 1) is that of $_{73}Ta^{181}$: heavy enough to pull the meson into close interaction with the nucleus; not heavy enough to have passed out of the region of large quadrupole moments; available in suitable physical form; quadrupole moment already measured via splitting of electronic energy levels. Some of the transuranic elements ought to show even greater quadrupole moments, and still larger splittings. Here no quantitative predictions can be made because of the unfortunate absence of quadrupole data for these most interesting of nuclei. Possibly the Chang radiation itself will turn out to give the most convenient means to determine nuclear spins and quadrupole moments in the transuranic region.

Especially instructive is a comparison of the quadrupole-induced line separation in the electronic and mesonic cases. If the nucleus were as small relative to the scale of meson orbits as it is compared to atomic dimensions, only the potential energy outside the nucleus,

$$V(r)=-(Ze^2/r)-[1.5\cos^2(\mu,I)-0.5](q/2)(e^2/r^3), \quad (1)$$

would count in either case; and one could get no more information about the quadrupole phenomenon from one measurement than from the other. Actually a meson in the K orbit of Ta spends a significant fraction of its time inside the nucleus. There the quadrupole field is not only much smaller than what would be predicted from Eq. (1) but also depends in an important way on the distribution of the charge which is responsible for the quadrupole moment.

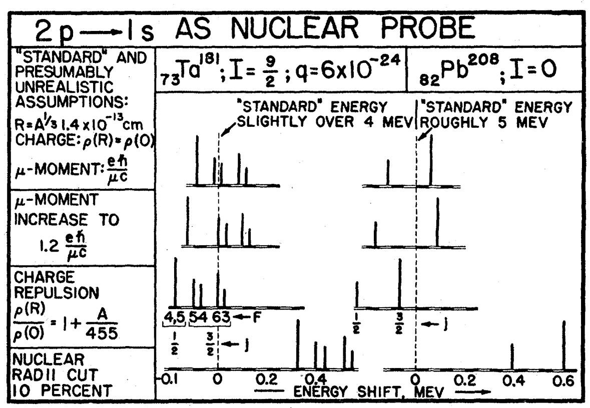

FIG. 1. Fine structure of $2p\to1s$ lines of Ta and Pb as indicator of nuclear and mesonic properties. The second line shows the effect of a 20 percent anomaly in the magnetic moment of the meson; the third line shows the influence of nonuniformity of the charge density of the nucleus (40 percent greater at the center than at the surface for Ta, and 45 percent greater for Pb); the fourth line shows the effect of a 10 percent decrease in nuclear radius. All the numbers are approximate and illustrative only; they are estimates based on lowest relevant order of perturbation theory; details are given in the text. *Note*: Later information indicates that the spin of Ta^{181} is 7/2 rather than 9/2. The resulting change in the quadrupole splitting pattern is relatively small, as seen by inspection of the quadrupole splitting coefficients in Table I.

The level splittings plotted in Fig. 1 were calculated from the following picture[8] of the origin of quadrupole moments. One or more nucleons are considered—from the point of view of the individual particle model in its zeroth approximation—to be outside of closed shells and to be described by individual particle wave functions, not all of whose nodal surfaces are spherical. Each such particle exerts on the nuclear surface a non-

[7] The empirical data are summarized by W. Gordy, Phys. Rev. **76**, 139 (1949) and by Townes, Foley, and Low, Phys. Rev. **76**, 1415 (1949); theoretical discussions are given by D. Hill and J. A. Wheeler, Phys. Rev. **89**, 1132 (1953); and K. Ford, Phys. Rev. **90**, 29 (1953).

[8] J. Rainwater, Phys. Rev. **79**, 432 (1950); D. Hill and J. A. Wheeler, Phys. Rev. **89**, 1102 (1953).

uniform pressure, the magnitude of which is greatest in the directions normal to the largest number of nodal surfaces. As a consequence of such unbalanced pressures acting against the surface tension, the nucleus takes on an ellipsoidal form. The seat of the quadrupole moment is idealized as an ellipsoid with a uniform volume distribution of electric charge.

The potential energy of a meson in the field of such an ellipsoidal charge distribution, of radius R and quadrupole moment[9] q, is given by Eq. (1) outside the nucleus and inside by

$$V(r) = -(3Ze^2/2R) + (Ze^2r^2/2R^3) - [1.5\cos^2(\mu, I) - 0.5](q/2)(e^2r^2/R^5). \quad (2)$$

The rapid falloff of the quadrupole field inside the nucleus cuts down the level splitting—relative to its value for a negligibly small nucleus of the same moment—by a form factor,

$$f_q \doteqdot 1/[1+0.1(RZ\mu e^2/\hbar^2)^2]^2 (\sim 1/2.4 \text{ for Ta}), \quad (3)$$

so that the width of the pattern produced by the quadrupole interaction has the order of magnitude,

$$\Delta E_{\text{quad}} = 1 \text{ Mev}(q/10^{-24}\text{ cm}^2)(Z/237)^3 f_q. \quad (4)$$

One can look forward to an experimental determination of the form factor f_q in Eq. (4). For this purpose it is necessary to measure the quadrupole-induced splitting of the $2p_{\frac{3}{2}}$ level of a nucleus for which the quantity q is already known from the hyperfine structure of electronic levels. Then one will be in a position to check the theoretical formula (3) for f_q—or still better, to do a precise numerical integration of the Dirac equation and to compare the more accurate resulting value for f_q with experiment. In this way an observational test can be made of the view that the quadrupole moments of heavy nuclei arise in first approximation from an ellipsoidal deformation of a uniformly charged nuclear substance.

3. The *magnetic moment* of the meson has been assumed in the discussion so far to have the value, $2(e/2\mu c)\times(\hbar/2)$, appropriate for a particle whose probability amplitude function satisfies the Dirac equation. A value of the mesonic moment which is greater by the factor $(1+\epsilon)$ will produce a splitting between the $2p_{\frac{1}{2}}$ level and the center of gravity of the $2p_{\frac{3}{2}}$ pattern roughly equal to

$$\Delta E_{\text{spin}} = 1 \text{ Mev} \times (1+2\epsilon)(Z/101)^4 f_s, \quad (5)$$

where the form factor f_s for a uniformly charged sphere of electricity has approximately the value

$$f_s \doteqdot 1/[1+0.2(RZ\mu e^2/\hbar^2)^2] (\sim 1/2.1 \text{ for Ta}). \quad (6)$$

Thus f_s is a quantity known within certain limits from existing determinations of nuclear radii and susceptible of more precise evaluation as in Item 1. Consequently a measurement of the spin splitting of the resonance line should give direct information about the magnetic moment of the μ meson.

4. The *distribution of charge* over the nucleus has been assumed uniform in the foregoing estimates. However, the charge density near the surface of a nucleus of mass number A is greater than the density at the center by a factor

$$(\rho_{\max}/\rho_{\min}) = 1+\delta \doteqdot 1+0.0022A \; (=1.45 \text{ for Pb}), \quad (7)$$

according to early considerations[9] of Feenberg and Wigner. This effect does not significantly alter the quadrupole form factor f_q, and slightly decreases the spin-orbit coupling form factor, f_s. The principal effect of the nonuniformity in charge, however, is to raise the $1s$ level ($\sim$0.14 Mev for Pb). By overlooking this effect, one would deduce from the $2p-1s$ energy difference (Item 1) a nuclear radius about 3 percent too great. The appropriate correction can be calculated easily enough, of course, by adopting Feenberg and Wigner's theoretical expressions for the nonuniformity of charge. For an *experimental* means to determine independently the magnitude of the nonuniformity and the radius of the nucleus, it would be sufficient to know the positions—relative to the $2p$ triplet—not only of the $1s$ level, as already discussed, but also of the $2s$ state: the two mesonic states of zero angular momentum respond in different measure to the parameters R and $(\rho_{\max}/\rho_{\min})$. No calculations on this point are reported here as the transitions $2s\rightarrow 2p$ (energy for case of Pb estimated[4] to be of order of 1 Mev) are one state further removed from observation. Thus only a fraction of the captured mesons ever arrive at the $2s$ level, while the great majority pass through the $2p$ state on their way to the ground level.

The fine structure of the $2s\rightarrow 2p$ and $2p\rightarrow 1s$ lines will be expected to show a characteristic intensity pattern, assisting in identification of the components: (1) intensity *summed* over the components of $2p_{\frac{3}{2}}\rightarrow 1s$ very nearly twice the intensity of $2p_{\frac{1}{2}}\rightarrow 1s$ because the probability of *arrival* at any sublevel of $2p$ is approximately fixed by the statistical weight of that level; (2) intensity distribution *within* the components of $2p_{\frac{3}{2}}$ proportional to $2F+1$ for each component (details in Table I); (3) the small number of photons in each component of $2s\rightarrow 2p$, divided by the cube of the frequency of that component, proportional to the much larger number of photons in the corresponding com-

[9] Here the following symbols for the quadrupole moment are used: $q(\text{cm}^2)$ times $e = Q(\text{cm}^2\cdot\text{franklin}) = \sum e_k(3z_k^2 - r_k^2) = 2\sum e_k r_k^2 \times[1.5\cos^2(k, I) - 0.5] = (2Ze/5)(c^2-a^2) = 6ZeR_0\alpha/5$, where the last two expressions refer to the case of a uniform charge distribution bounded by the surface $R = R_0[1+\alpha P_2(\cos\theta)]$, with equatorial semi-axis $a = R_0(1-\alpha/2)$ and semi-axis of symmetry $c = R_0(1+\alpha)$. E. Wigner, Bicentennial Symposium, University of Pennsylvania, 1940; E. Feenberg, Phys. Rev. **59**, 593 (1941). Smaller values for the coefficient of A in the redistribution formula of Eq. (7)—perhaps only half as great—are given by employing the later treatment of E. Feenberg, Rev. Mod. Phys. **19**, 239 (1947), as shown in detail in Table II of reference 11, as kindly pointed out by D. L. Hill.

ponent of $2p \rightarrow 1s$. This characteristic mesonic radiation is, of course, separate in time from the other half of the Chang radiation due to de-excitation of the residual nucleus *following* the charge exchange reaction. However, if the experimental time discrimination is not complete, the characteristic intensity pattern should distinguish the desired radiation from the de-excitation radiation, which will (a) normally in a heavy nucleus be divided among many lines and (b) not vary in energy monotonically with nuclear charge number.

5. The *polarizability* and *compressibility* of nuclear matter[10] can be deduced if one can determine the charge repulsion factor δ of Eq. (7), Item 4, and the swelling of a nucleus due to addition of a proton. This swelling effect has recently been called on by Wilets, Hill, and Ford[11] to explain some of the anomalies in the isotope shift of atomic spectral lines. The effect should be very much greater percentagewise in the case of the mesonic radiation. The experimentally relevant quantity is the shift in the center of gravity of the $2p \rightarrow 1s$ line as a result of changes in the number of neutrons and protons in the nucleus. Knowing the charge, one can, by calculation, determine what effective radius is required to fit the observations, thus determining the dependence of radius upon Z and A, and thereby permitting an estimate of the compressibility. However, in any such analysis one must allow for an important effect in the interpretation of the isotope effect first brought out by Wilets, Hill, and Ford. The presence of a nuclear quadrupole moment, described for instance by a deformation $R=R_0[1+\alpha P_2(\cos\theta)]$, increases the apparent extension of the nucleus—as regards its effect on the $1s$-level—by a fractional amount proportional to the *square* of α. This effect, moreover, is due to the intrinsic deformation of the nucleus—the deformation as seen by a nucleon which responds to the changing orientation of the nuclear boundary—and will therefore occur even in cases ($I=0$ or $I=\frac{1}{2}$) when the quadrupole moment as seen by an atomic electron averages out to zero. This deformation and the associated correction of the nuclear radius can be calculated or estimated from other information as described by Wilets, Hill, and Ford. It is the thus corrected radius which should serve as the starting point for an analysis of nuclear compressibility.

In conclusion, the spectroscopy of the characteristic mesonic gamma radiation, initiated by Chang and made into a science by Fitch and Rainwater, promises to teach interesting lessons about both the meson and the nucleus.

DETAILS

All the estimates reported in Fig. 1 and in the text have been made in the lowest relevant order of perturbation theory, using hydrogenic functions for the $2p$ state, employing for the $1s$ state of Pb a wave function found by numerical integration,[4] and estimating shifts of the $1s$ level of Ta, where necessary, by applying crude correction factors to the corresponding figures for lead. For any accurate analysis of experimental results, it is necessary to redo the calculations with wave functions determined by numerical integration of the wave equation in the appropriate field of force. Not to attempt this extensive task, but only to indicate how the present estimates were made, the following details are subjoined:

The shift in the $2p$ levels due to spin and quadrupole effects is given by

$$
\begin{aligned}
E_{sq} &= [j(j+1)-l(l+1)-s(s+1)](\hbar^2/4\mu^2c^2)(dV/rdr)_{\mathrm{Av}} \\
&\quad -(qe^2/2)[1.5\cos^2(\mu,j)-0.5]_{\mathrm{Av}} \\
&\quad \times[1.5\cos^2(j,I)-0.5]_{\mathrm{Av}}\left\{\begin{matrix} r^2/R^5 \text{ for } r<R \\ 1/r^3 \text{ for } r>R \end{matrix}\right\}_{\mathrm{Av}} \\
&= \left\{\begin{matrix} 1 \text{ for } j=\frac{3}{2} \\ -2 \text{ for } j=\frac{1}{2} \end{matrix}\right\}(\mu c^2/96)(Z/137)^4 f_s(x) \\
&\quad +\left\{\begin{matrix} 1 \text{ for } j=\frac{3}{2} \\ 0 \text{ for } j=\frac{1}{2} \end{matrix}\right\}[1.5\cos^2(j,I)-0.5]_{\mathrm{Av}} \\
&\quad \times(\mu c^2/240)[q/(e^2/\mu c^2)^2][Z/(137)^2]^3 f_q(x) \\
&= \left\{\begin{matrix} \frac{1}{3} \text{ for } j=\frac{3}{2} \\ -\frac{2}{3} \text{ for } j=\frac{1}{2} \end{matrix}\right\} 1\text{ Mev}(Z/101)^4 f_s \\
&\quad +\left\{\begin{matrix} 1 \text{ for } j=\frac{3}{2} \\ 0 \text{ for } j=\frac{1}{2} \end{matrix}\right\}[1.5\cos^2(j,I)-0.5]_{\mathrm{Av}} 0.5\text{ Mev} \\
&\quad \times(q/10^{-24}\text{ cm}^2)(Z/237)^3 f_q.
\end{aligned}
$$

The evaluation is made with the hydrogenic wave function for the $2p$ state,

$$F_p = cr^2\exp(-Z\mu e^2 r/2\hbar^2),$$

normalized so that $\int F_p{}^2 dr=1$. Form factors dependent on the finite size of the nucleus are expressed in terms of the dimensionless variable, $x=RZ\mu e^2/\hbar^2$:

$$
\begin{aligned}
f_s(x) &= \left\{x^{-3}\int_0^x x^4 + \int_x^\infty x\right\}\exp(-x)dx \\
&= 24x^{-3}-(24x^{-3}+24x^{-2}+12x^{-1}+3)\exp(-x) \\
&= 1,\ 0.824,\ 0.564,\ 0.290 \text{ for } x=0,\ 1,\ 2,\ 3.5 \\
&\doteqdot 1/(1+0.2x^2) \\
&= 1,\ 0.833,\ 0.556,\ 0.290 \text{ for } x=0,\ 1,\ 2,\ 3.5;
\end{aligned}
$$

[10] Reference 9 and W. J. Swiatecki, Proc. Phys. Soc. (London) **A63**, 1208 (1950).
[11] Wilets, Hill, and Ford, Phys. Rev. **91**, 1488 (1953).

and

$$f_q(x) = \left\{ x^{-5}\int_0^x x^6 + \int_x^\infty x \right\} \exp(-x)dx$$

$$= 720x^{-5} - (720x^{-5} + 720x^{-4} + 360x^{-3} + 120x^{-2} + 30x^{-1} + 5)\exp(-x)$$

$$= 1,\ 0.796,\ 0.508,\ 0.341 \text{ for } x = 0,\ 1,\ 2,\ 2.76$$

$$\doteqdot 1/(1+0.1x^2)^2$$

$$= 1,\ 0.826,\ 0.510,\ 0.322 \text{ for } x = 0,\ 1,\ 2,\ 2.76.$$

The factor $[1.5\cos^2(j, I) - 0.5]_{\mathrm{Av}}$ takes on very simple values in the case of a nuclear spin I, so large that it can be treated as infinite—a limit never attained in practice. Then the factor in question takes on the limiting value $+1$ for the pair of states characterized by the values, $F = I + \frac{3}{2}$ and $F = I - \frac{3}{2}$, of the quantum number F for sum of angular momentum of meson and nucleus. A similar degeneracy occurs for the only other states possible when $j = \frac{3}{2}$. Then $F = I + \frac{1}{2}$ or $I - \frac{1}{2}$ and the angular factor in question has the value -1. In this limit of large j, the state $j = \frac{3}{2}$ is therefore split up by interaction with the nuclear quadrupole moment symmetrically into two components. Their statistical weights are equal both to each other and to the statistical weight of the $j = \frac{1}{2}$ level, which is unaffected by the quadrupole interaction. In the contrary limit where the nuclear spin is 0 or $\frac{1}{2}$, there is, of course, no quadrupole splitting. For $I = 1$ and $I = \frac{3}{2}$ there are three components. For every larger but still finite value of I there are four distinct components, $F = I + \frac{3}{2}$, $I + \frac{1}{2}$, $I - \frac{1}{2}$, $I - \frac{3}{2}$, with statistical weights $2F + 1$ (compared to the weight $4I + 2$ of the component $j = \frac{1}{2}$ of the Chang radiation), and with energies calculated by use of the formula of Casimir,

TABLE I. Value of the factor $[1.5\cos^2(I, j) - 0.5]_{\mathrm{Av}}$ which appears in the formula for the quadrupole splitting of the $2p_{\frac{3}{2}}$ mesonic level. I = nuclear quantum number; F = resultant of I and of the angular momentum, $j = \frac{3}{2}$, of the meson.

I	$F = I + \frac{3}{2}$	$I + \frac{1}{2}$	$I - \frac{1}{2}$	$I - \frac{3}{2}$
0 or 1/2	no quadrupole splitting			
1	1	−4	5	no level
3/2	1	−3	1	5
2	1	−5/2	0	7/2
5/2	1	−11/5	−2/5	14/5
3	1	−10/5	−3/5	12/5
7/2	1	−13/7	−5/7	15/7
4	1	−49/28	−22/28	55/28
9/2	1	−10/6	−5/6	11/6
5	1	−24/15	−13/15	26/15
11/2	1	−85/55	−49/55	91/55
6	1	−33/22	−20/22	35/22
13/2	1	−19/13	−12/13	20/13
∞	1	−1	−1	1

$$[1.5\cos^2(I, j) - 0.5]_{\mathrm{Av}} = \frac{1.5K(K+1) - 2I(1+1)j(j+1)}{I(2I-1)j(2j-1)},$$

where $K = F(F+1) - I(I+1) - j(j+1)$.

The *shift* of levels due to a change δR in the radius R of a nucleus idealized as having a uniform charge density is

$$E_{\delta R} = (3Ze^2\delta R/2R^2)(1 - r^2/R^2)_{\mathrm{Av}}.$$

With a nonuniform charge density,

$$\rho = (3Ze/4\pi R^3)(1 + R^{-2}r^2\delta)/(1+0.6\delta),$$

greater at the surface than at the center by the factor $(1+\delta)$, is associated an energy *increase* over the eigenvalues[4] for a uniform charge distribution, equal to

$$E_{\text{nonuniform}} = (Ze^2/R)[(1 - r^2/R^2)^2]_{\mathrm{Av}}(3\delta/20)/(1+0.6\delta).$$