

Lattice-Space Quantization of a Nonlinear Field Theory*

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A method for the approximate diagonalization of certain types of quantum field Hamiltonians is developed which is not limited to weakly nonlinear systems. It consists in omitting the gradient terms in zero order, and diagonalizing the resulting Hamiltonian by replacing the field defined in a continuum space by a field defined in a lattice space. This unperturbed system is equivalent to a countably infinite number of uncoupled nonlinear oscillators, which are then coupled together when the gradient terms are included as a perturbation. The method is applied to the quantization of the classical nonlinear meson theory that was introduced in an earlier paper to provide a qualitative explanation of the saturation of nuclear forces, according to which a positive ϕ^4 term is added to the field Hamiltonian. Although the quantized theory is manifestly noncovariant, it is found that a single-particle

solution exists that has an approximately relativistic relation between energy, momentum, and rest mass. It turns out to be essential that the lattice constant be kept finite, as all computed physical quantities become meaningless in the continuum limit (in which the lattice constant approaches zero). It is shown that these particles obey Einstein-Bose statistics, and that they scatter from each other. Nucleons are introduced as classical sources for the meson field, and calculations are made on the nucleon isobaric state, interaction of mesons with nucleons and heavy nuclei, and nucleon-nucleon interaction. Most of the results of the earlier classical theory have close counterparts in the present quantized theory. The possibility of extending the method to the quantization of both meson and nucleon fields when they are strongly coupled together is discussed briefly.

I. INTRODUCTION

THE quantization of non-interacting wave fields that satisfy linear equations can be carried through without difficulty, and energy eigenvalues can be obtained that correspond to particles of well-defined energy and momentum. No such simple and satisfactory treatment exists for wave fields that satisfy nonlinear equations, regardless of whether the nonlinearity arises from the interaction of otherwise linear fields or from an inherent property of the field itself. In most cases of physical interest, the difficulty does not appear in the formal process of quantization, which can be accomplished in the usual way, but rather in the diagonalization of the field Hamiltonian. Where the nonlinear term arises from an interaction and has a small coefficient, perturbation techniques can be used. The application of covariant mass and charge renormalization to quantum electrodynamics illustrates the high degree of success that can be achieved in this way. But where the nonlinear term is large, as in the meson-nucleon system, there is real doubt as to whether perturbation calculations provide even a qualitative guide to the actual situation. In such cases, calculations have sometimes been based on the Tamm-Dancoff method, the reliability of which is difficult to evaluate.

The present paper describes a different method, not limited to weakly nonlinear systems, for the approximate diagonalization of certain types of quantum field Hamiltonians. The method is based on the observation that for many field Hamiltonians, omission of the gradient terms leaves a system that can be regarded as a set of a continuously infinite number of uncoupled nonlinear oscillators, one at each point in space. A solution can be obtained if (1) the modified Hamiltonian

can be exactly diagonalized, (2) the gradient terms can justifiably be treated as a perturbation, and (3) a perturbation calculation can be carried through to sufficiently high order to yield physically interesting results. Step (1) requires that the field defined in a continuum space be replaced by a field defined in a lattice space, so that the continuously infinite number of oscillators is replaced by a countably infinite number, one at each lattice point. Step (2) imposes restrictions on the general structure of the Hamiltonian and on the lattice constant. Step (3) depends to a large extent on the quantity being calculated; we shall see that various orders of perturbation calculation are of interest in different situations.

The theory outlined above is manifestly noncovariant, because of the different treatment of space and time derivatives. It is not even Galilean-invariant, because of the introduction of the lattice. The second difficulty is not serious if we deal with solutions that do not vary appreciably from one lattice point to the next. The situation here is roughly analogous to the propagation of sound waves in a crystal; the lattice structure is not significant if the wavelength is large in comparison with the lattice constant. The first difficulty is of course a major defect of the present work. Nevertheless, it is possible to find solutions that have a kind of relativistic behavior.

The nonlinear meson theory introduced earlier¹ provides a convenient example to which the present method can be applied. In this theory, a neutral scalar field ϕ interacts with itself through a positive term in the Hamiltonian proportional to $|\phi|^n$, where n is generally equal to 4, and may also interact with a classical source function which is related to the density of nucleons. Thus the present paper contains mainly a

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¹L. I. Schiff, *Phys. Rev.* **84**, 1 (1951), referred to here as I; W. E. Thirring, *Z. Naturforsch.* **7a**, 63, 379 (1952); B. J. Malenka, *Phys. Rev.* **86**, 68 (1952); E. M. Henley, *Phys. Rev.* **87**, 42 (1952).

particular kind of quantization of the classical theory developed previously.² The method can, however, also be applied to the quantization of certain coupled meson-nucleon fields, provided that a ϕ^4 term is introduced either *ad hoc* or because of renormalization.³ There need then be no limitation on the strength of the meson-coupling. This extension of the theory is discussed briefly in Sec. VI.

The lattice space and the quantization procedure are set up in Sec. II. The fact that solutions exist that correspond to particles is far from obvious in a theory in which the nonlinear term is large; it is established in Sec. III. The interaction of two such particles in the absence of sources (meson-meson scattering) is discussed in Sec. IV. Classical sources that correspond to nucleons are introduced in Sec. V, and calculations are made of the nucleon isobaric state, meson-nucleon scattering, nucleon-nucleon interaction, saturation of nuclear forces, and the interaction of mesons with nuclear matter.

II. LATTICE SPACE

The continuum field Hamiltonian with which we shall work is¹

$$H = \int \left[\frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} \mu^2 \phi^2 + \frac{1}{4} \alpha^2 \phi^4 \right] d\tau, \tag{1}$$

where ϕ is a real scalar field and π is its canonical momentum. μ is the field rest mass, α is the nonlinear parameter ($\alpha^2 > 0$), and we choose units such that $\hbar = c = 1$. The canonical commutation relation is

$$[\phi(\mathbf{r}, t), \pi(\mathbf{r}', t)] = i\delta(\mathbf{r} - \mathbf{r}'). \tag{2}$$

As is well known, limitation of the volume within which the field is defined has no essential effect on the quantization; if this volume is assumed to be a cube of edge length L , the limit $L \rightarrow \infty$ can later be taken without difficulty.

For simplicity, the lattice space is introduced in simple cubic form, with lattice constant l . The large cube of volume L^3 is still assumed, so that the space consists of $N = (L/l)^3$ points. As in the continuum case, the limit in which N and L become infinite while l is fixed can readily be taken later.

We now assert that the field is defined only at the lattice points, where it is characterized by quantum mechanical operators ϕ_s and π_s . Our first object is to find a lattice field Hamiltonian and commutation relations that become Eqs. (1) and (2), respectively, in the limit $l \rightarrow 0$. It is apparent that this cannot be done uniquely, since terms can always be included in the lattice Hamiltonian that vanish in this limit. We shall

² Quantum aspects have been treated from other points of view by D. R. Yennie, Phys. Rev. 88, 527 (1952); H. B. Rosenstock, Ph.D. thesis, University of North Carolina, 1951 (unpublished); D. Finkelstein, Ph.D. thesis, Massachusetts Institute of Technology, 1953 (unpublished).

³ P. T. Matthews, Phil. Mag. 42, 221 (1951).

adopt a form that is convenient and plausible, which was first introduced by Wentzel.⁴

We introduce an averaging function

$$f(\mathbf{r} - \mathbf{r}_s) = L^{-3} \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot (\mathbf{r}_s - \mathbf{r})}, \tag{3}$$

where \mathbf{r}_s is a lattice point, and write

$$\phi_s \equiv a \int f(\mathbf{r} - \mathbf{r}_s) \phi(\mathbf{r}) d\tau, \tag{4}$$

$$\pi_s \equiv b \int f(\mathbf{r} - \mathbf{r}_s) \pi(\mathbf{r}) d\tau,$$

where the coefficients a and b are as yet undetermined. The summation variable \mathbf{k} in Eq. (3) is restricted to the points of a reciprocal lattice space which is also simple cubic; each component of \mathbf{k} ranges from $-\pi/l$ to π/l in steps of $2\pi/L$, so that there are N terms in the \mathbf{k} -sum. The averaging function f has unit integral over the large cube of volume L^3 , has its maximum value $1/l^3$ when its argument is zero, has linear dimensions of order l , and approaches a δ function when $l \rightarrow 0$. Equation (3) therefore states that the lattice field is an average of the continuum field over a distance of order l about the lattice point, multiplied by a coefficient a or b . In the following, we shall frequently use the relations:

$$\sum_{\mathbf{k}} e^{i\mathbf{k} \cdot (\mathbf{r}_s - \mathbf{r}_t)} = N\delta_{st}, \quad \sum_s e^{i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}_s} = N\delta_{\mathbf{k}\mathbf{k}'}. \tag{5}$$

The commutation relations associated with the lattice fields (4) are easily shown to be

$$[\phi_s, \pi_t] = iab l^{-3} \delta_{st}, \tag{6}$$

when use is made of Eqs. (2), (3), and (5). Thus for a canonical theory, we require that $ab = l^3$. The value chosen for a or for b is of no physical significance, since it appears only as an arbitrary scale factor in ϕ_s or π_s . It is convenient to choose $a = b = l^{\frac{3}{2}}$. When this is done, it is easily verified that

$$\sum_s \pi_s^2 \xrightarrow{l \rightarrow 0} \int \pi^2 d\tau, \quad \sum_s \phi_s^2 \xrightarrow{l \rightarrow 0} \int \phi^2 d\tau, \tag{7}$$

$$l^{-3} \sum_s \phi_s^4 \xrightarrow{l \rightarrow 0} \int \phi^4 d\tau.$$

There are several ways in which the gradient term in Eq. (1) can be represented in the lattice space. With the form (3) for the smoothing function, the most convenient choice consists in replacing

$$\int (\nabla \phi)^2 d\tau$$

by⁴

$$\sum_s \sum_t A_{st} \phi_s \phi_t, \quad A_{st} \equiv N^{-1} \sum_{\mathbf{k}} k^2 e^{i\mathbf{k} \cdot (\mathbf{r}_s - \mathbf{r}_t)}, \tag{8}$$

⁴ G. Wentzel, Helv. Phys. Acta 13, 269 (1940).

which has the proper limit as $l \rightarrow 0$. The quantity A_{st} could also be represented as a second difference, for example, but this is less convenient for the following development, and makes little difference so long as we deal with solutions that do not vary appreciably from one lattice point to the next.

The lattice Hamiltonian can now be written:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}',$$

$$\mathcal{H}_0 = \sum_s \left[\frac{1}{2} \pi_s^2 + \frac{1}{2} (P + \mu^2) \phi_s^2 + \frac{\alpha^2}{4l^3} \phi_s^4 \right], \quad (9)$$

$$\mathcal{H}' = \frac{1}{2} \sum_s \sum_t' A_{st} \phi_s \phi_t, \quad P \equiv A_{ss} = N^{-1} \sum_k k^2 \xrightarrow{N \rightarrow \infty} \frac{\pi^2}{l^2},$$

where the dash on the summation over t means that the term $t=s$ is omitted. Because of (7) and (8), \mathcal{H} approaches the H given by Eq. (1) as $l \rightarrow 0$. The unperturbed lattice Hamiltonian \mathcal{H}_0 is separable and describes a set of uncoupled nonlinear oscillators. The perturbation \mathcal{H}' then couples these oscillators together.

It is convenient to adopt a representation in which $\pi_s = -i(\partial/\partial\phi_s)$, which is consistent with the commutation relations (6). Then the unperturbed Schrödinger equation may be separated as follows:

$$\mathcal{H}_0 \Psi = E \Psi, \quad \Psi = \prod_s u_{n_s}(\phi_s), \quad E = \sum_s \epsilon_{n_s},$$

$$\left[-\frac{1}{2} \frac{d^2}{d\phi^2} + \frac{1}{2} (P + \mu^2) \phi^2 + \frac{\alpha^2}{4l^3} \phi^4 \right] u_n(\phi) = \epsilon_n u_n(\phi). \quad (10)$$

The energies ϵ_n are the eigenvalues for the one-dimensional motion of a particle in a potential that becomes positively infinite for large displacements from the equilibrium point ($\phi=0$); they therefore form a discrete set extending from a smallest positive value to $+\infty$. The eigenfunctions u_n form a complete orthonormal set with parities $(-1)^n$.

We can put this equation in dimensionless form by substituting

$$x = (\alpha^3/2^{1/6}l^{3/2})\phi, \quad \lambda_n = (2^{4/3}l/\alpha^3)\epsilon_n,$$

whence

$$\frac{d^2 u_n}{dx^2} + \left[\lambda_n - \frac{2^{3/2}}{\alpha^{4/3}} (\pi^2 + \mu^2 l^2) x^2 - x^4 \right] u_n = 0. \quad (11)$$

This form is well-adapted to a study of the situation in which $\alpha \gg 1$, since then the dominant character of the x^4 term is evident (this assumes that μl is not large in comparison with unity). For very large α , the x^2 term may be neglected, and the λ 's are a well-defined set of dimensionless numbers. Then the ϵ 's are of order α^3/l , and all matrix elements of ϕ are of order $l^{1/3}/\alpha^{1/3}$. Thus if we treat \mathcal{H}' as a perturbation on \mathcal{H}_0 , a suitable parameter for gauging the validity of the perturbation calculation is the ratio of a typical matrix element of \mathcal{H}' to the spacing of the eigenvalues of \mathcal{H}_0 ; this ratio is a

rough measure of the extent to which other states mix in with a particular unperturbed state. In cases of physical interest, only a small number of terms of the series in \mathcal{H}' will contribute, so a typical matrix element is of order $(1/l^2)(l^{1/3}/\alpha^{1/3})^2 = (1/\alpha^3)$, and the above ratio is of order $1/\alpha^{4/3}$. Thus the perturbation treatment envisaged here is useful if α is large, regardless of the magnitude of l .

It is of interest to see how this perturbation treatment of \mathcal{H}' is modified if the nonlinearity is chosen to have the form $\alpha^2 |\phi|^n$, where n is not necessarily equal to 4. The ratio of perturbation to eigenvalue spacing is then of order $l^{4(n-4)/(n+2)}/\alpha^{8/(n+2)}$. Thus if $n > 4$ the validity of the perturbation approach is improved if l is made small, and if $n < 4$ it is improved if l is made large; only for $n=4$ is it independent of l . This explains why Wentzel⁴ was forced to assume large l for the linear field theory (which is equivalent to $n=2$).

It is also interesting to note that the unperturbed energy eigenvalues are of order $\alpha^{4/(n+2)}/l^{3(n-2)/(n+2)}$ in the general case. Thus only in the case $n=2$ (linear field theory) do they remain finite as $l \rightarrow 0$. In this case, the continuum field Hamiltonian has positively infinite energy eigenvalues, but only for the trivial reason that there is an infinite number of degrees of freedom ($N \rightarrow \infty$) each of which has a finite zero-point energy. In the case $n=4$, all the eigenvalues for each of the N degrees of freedom become positively infinite as $l \rightarrow 0$. Thus so long as the perturbation theory is valid (large α), we can use this result as a demonstration that the continuum field Hamiltonian (1) has energy eigenvalues that are positively infinite in a nontrivial way.⁵ This means that sensible results can be obtained only by some sort of renormalization or cut-off procedure. We have attempted various types of renormalization, without success; an obvious difficulty is that covariance cannot be used as a guide. A cutoff is provided automatically by the lattice constant l ; so long as l is finite, all results of the theory are finite and, as we shall see in the next section, there exists a maximum momentum of order $1/l$. We therefore reconcile ourselves to the idea that the limit $l \rightarrow 0$ cannot actually be taken, and regard l as a parameter of the theory.

III. ENERGY AND MOMENTUM OF A FREE PARTICLE

Lattice Energy

The lowest state of the unperturbed system is that in which each lattice-point oscillator has its smallest energy eigenvalue. This state is nondegenerate, and has energy $N\epsilon_0$. The first excited state of the unperturbed system is that in which all but one lattice-point oscillator has its smallest energy eigenvalue, and that one has its second eigenvalue. This state is N -fold degenerate, and has energy $(N-1)\epsilon_0 + \epsilon_1$, or an excitation energy of $(\epsilon_1 - \epsilon_0)$ above the lowest state. If we apply

⁵ This supplements the similar result obtained by D. R. Yennie (reference 2) for the case in which the nonlinear term can be treated as a perturbation (small α).

first-order perturbation theory, and remember that the oscillator eigenfunctions $u_n(\phi)$ of Eq. (10) have parities $(-1)^n$, it is apparent that the lowest state is unaffected. This state is

$$\Psi_0 = \prod_t u_0(\phi_t). \tag{12}$$

The N -degenerate first-excited states of the unperturbed system are

$$U_1(s) = u_1(\phi_s) \prod_{t \neq s} u_0(\phi_t), \quad s = 1, \dots, N, \tag{13}$$

where the dash on the product means that the term $t=s$ is omitted. Diagonalization of the matrix for \mathcal{H}' computed from the functions (13) leads to the following first-order normalized eigenfunctions and perturbation matrix:

$$\Psi_1(\mathbf{K}) = N^{-1/2} \sum_s e^{i\mathbf{K} \cdot \mathbf{r}_s} U_1(s), \tag{14}$$

$$\mathcal{H}'(\mathbf{K}, \mathbf{K}') = (K^2 - P)\phi_{01}^2 \delta(\mathbf{K}, \mathbf{K}').$$

Here \mathbf{K} is any one of the set of vectors \mathbf{k} of the reciprocal lattice space, and

$$\phi_{01} \equiv \int_{-\infty}^{\infty} \bar{u}_0(\phi) \phi u_1(\phi) d\phi$$

is a real matrix element since the u 's are taken to be real. Thus the excitation energy through first order is

$$\epsilon_1 - \epsilon_0 = P\phi_{01}^2 + K^2\phi_{01}^2. \tag{15}$$

The wave function $\Psi_1(\mathbf{K})$ given by Eq. (14) is a kind of wave of excitation running through the lattice; it is analogous to the excitation waves in the theory of solids or to the spin waves in the theory of ferromagnetism.

Lattice Momentum

We now consider the momentum contained in the lattice. The continuum field momentum is

$$\mathbf{G} = -\frac{1}{2} \int [\pi(\nabla\phi) + (\nabla\phi)\pi] d\tau, \tag{16}$$

and it is natural to write for the lattice momentum

$$\mathcal{G} = -\sum_s \sum_t \mathbf{B}_{st} \pi_s \phi_t, \quad \mathbf{B}_{st} \equiv N^{-1} \sum_{\mathbf{k}} i\mathbf{k} e^{i\mathbf{k} \cdot (\mathbf{r}_s - \mathbf{r}_t)}, \tag{17}$$

where $\mathbf{B}_{st} = -\mathbf{B}_{ts}$ and $\mathbf{B}_{ss} = 0$; moreover, \mathcal{G} is Hermitian and approaches \mathbf{G} as $l \rightarrow 0$. It turns out, however, that \mathcal{G} defined by Eq. (17) does not commute with \mathcal{H} , so that it is not a constant of the motion. In the absence of external forces, we expect the momentum to be a constant, so that we cannot reasonably regard \mathcal{G} as the momentum of the lattice.

There is nevertheless a close correspondence between the translational properties of \mathbf{G} and \mathcal{G} .⁶ As is well

⁶ The writer is indebted to F. Bloch for pointing out the properties of the displacement operator.

known, \mathbf{G} is the generator of displacements of the continuum field variables in that the unitary transformation $S_{\mathbf{a}} \equiv \exp(-i\mathbf{a} \cdot \mathbf{G})$ displaces $\phi(\mathbf{r})$ and $\pi(\mathbf{r})$ by an arbitrary vector \mathbf{a} . In the same way, the unitary transformation $S_{\mathbf{a}} \equiv \exp(-i\mathbf{a} \cdot \mathcal{G})$ displaces ϕ_s and π_s by the vector \mathbf{a} , where, however, \mathbf{a} must now be one of the lattice vectors. In both the continuum and the lattice cases, the fact that the field Hamiltonian is invariant with respect to displacements means that it commutes with the displacement operator; for a space of finite extent, it is of course always necessary to assume that there are periodic boundary conditions at the edges. This commutativity can be used, in the continuum case, to show that \mathbf{G} itself commutes with H , by allowing the displacement vector \mathbf{a} to become infinitesimal. In the lattice case, however, there is no infinitesimal lattice vector, so that we cannot prove in this way that \mathcal{G} and \mathcal{H} commute, and in fact they do not.

We wish to find a momentum operator that is Hermitian, reduces to \mathbf{G} in the limit $l \rightarrow 0$, and is a constant of the motion. A simple choice for the x component that meets all these requirements is

$$p_x = [\exp(il\mathcal{G}_x) - \exp(-il\mathcal{G}_x)]/2il = (\sin l\mathcal{G}_x)/l, \tag{18}$$

with corresponding expressions for the other two components. Any energy eigenfunction, if nondegenerate, must also be an eigenfunction of Eq. (18), and of $S_{lx} \equiv \exp(-il\mathcal{G}_x)$, since both of these operators commute with \mathcal{H} . If an energy eigenfunction is not an eigenfunction of p_x , it must be degenerate, and linear combinations of these degenerate energy eigenfunctions can then be chosen that diagonalize p_x , i.e., that are also eigenfunctions of p_x . The corresponding eigenvalues can be found as follows.

Consider a wave function that is an eigenfunction of both \mathcal{H} and \mathbf{p} . It is then also an eigenfunction of S_{lx} , S_{ly} , and S_{lz} . If we operate on it L/l times with S_{lx} , we displace it by the distance L along the positive x direction, and hence bring it back into itself (periodic boundary conditions). Thus the eigenvalue of S_{lx} is one of the (L/l) th roots of unity: $\exp(-2\pi i\nu_x l/L)$, where ν_x is an integer; the eigenvalues for S_{ly} and S_{lz} have the same set of possible values. Then if we define a vector κ by putting $\kappa_x = 2\pi\nu_x/L$, etc., it is one of the vectors \mathbf{k} of the reciprocal lattice; the eigenvalue of S_{lx} is $\exp(-i\kappa_x)$, and similarly for S_{ly} and S_{lz} . The possible eigenvalues of Eq. (18) are then $(\sin l\kappa_x)/l$, with analogous eigenvalues for p_y and p_z . So long as κ is small in comparison with $1/l$, the eigenvalue of \mathbf{p} is very nearly equal to κ , and we shall usually assume that κ is small. It should be noted, however, that the eigenvalue of any component of \mathbf{p} cannot exceed $1/l$, and that the maximum value occurs when the corresponding component of κ is equal to $\pi/2l$.

The wave function (14) provides an instructive example of the effect of operating with S_{lx} . We denote by s' the lattice point whose x coordinate is larger by

l than the x coordinate of s . We then have

$$\begin{aligned} S_{lx}\Psi_1(\mathbf{K}) &= N^{-\frac{1}{2}} \sum_s e^{i\mathbf{K}\cdot\mathbf{r}_s} U_1(s') \\ &= N^{-\frac{1}{2}} e^{-ilKx} \sum_s e^{i(\mathbf{K}\cdot\mathbf{r}+lKx)} U_1(s') \\ &= N^{-\frac{1}{2}} e^{-ilKx} \sum_{s'} e^{i\mathbf{K}\cdot\mathbf{r}_s'} U_1(s') \\ &= e^{-ilKx} \Psi_1(\mathbf{K}), \end{aligned}$$

so that $\Psi_1(\mathbf{K})$ is in fact an eigenfunction of S_{lx} , and hence also of \mathbf{p} . This shows also that the momentum associated with the state (14) is very nearly equal to the vector \mathbf{K} that appears in $\Psi_1(\mathbf{K})$, so long as \mathbf{K} is not comparable with $1/l$.

Interpretation as a Particle

We are now in a position to relate the expression (15) for the first-order energy of the first excited state of the lattice, to the lattice momentum \mathbf{K} in the same state. We first assume that the nonlinear parameter α is large compared to unity. The term $(\epsilon_1 - \epsilon_0)$ in Eq. (15) is of order α^3/l , and the term $P\phi_{01}^2$ is of order $1/(l\alpha^3)$, so that the latter can be neglected for the present. We interpret $(\epsilon_1 - \epsilon_0)$ as the rest mass of a particle, $K^2\phi_{01}^2$ as the kinetic energy of that particle, and \mathbf{K} as its momentum. In other words, we say that the first excited state of the lattice, which possesses the above energy and momentum with respect to the lowest state of the lattice, is to be interpreted as a particle which has made its appearance in a vacuum. This interpretation is limited by the nonrelativistic form of Eq. (15), but this is not surprising since the kinetic energy term in (15) is the result of a first-order calculation, and it might be expected that higher-order terms in \mathcal{H}' would yield higher-order relativistic corrections to the simple formula (15). Equation (15) is at least consistent with the present view insofar as the kinetic energy term is small (of order $\alpha^{-4/3}$) in comparison with the rest mass term.

The crucial test of Eq. (15) as a description of a particle is the requirement that the kinetic mass $1/(2\phi_{01}^2)$, which is the ratio of the square of the momentum to twice the kinetic energy, be equal to the rest mass $(\epsilon_1 - \epsilon_0)$. This means that the quantity $2\phi_{01}^2(\epsilon_1 - \epsilon_0)$ should be equal to unity. If we use the substitutions associated with Eq. (11) to put this in dimensionless form, it becomes $x_{01}^2(\lambda_1 - \lambda_0)$, where $x_{01} \equiv \int_{-\infty}^{\infty} \bar{u}_0(x) x u_1(x) dx$ is a real matrix element since the u 's which satisfy Eq. (11) are taken to be real. The following sum rule is easily established:

$$\sum_n x_{mn}^2 (\lambda_n - \lambda_m) = 1. \quad (19)$$

If we take $m=0$, each term in Eq. (19) is positive, and we can conclude that $x_{01}^2(\lambda_1 - \lambda_0) \leq 1$. The equality holds only if all higher matrix elements x_{03}, x_{05} , etc. are zero. This is true in the linear case, in which the x^4 term does not appear in Eq. (11), since then the u 's are harmonic oscillator wave functions; thus the present

methods can deal with the linear field (although they are of course unnecessarily complicated in this case).

The actual value of $x_{01}^2(\lambda_1 - \lambda_0)$ in the nonlinear case must be found by numerical integration of Eq. (11). For large α , where the x^2 term in Eq. (11) can be neglected, several numerical values are collected in the next subsection. The above quantity is equal to 0.9884, so that the kinetic and rest masses of our particle differ by only about one percent.⁷ This agreement is not only gratifying in itself, but implies, as shown below, that the matrix elements x_{03}, x_{05} , etc., are small in comparison with x_{01} ; as we shall see, this makes it feasible to carry the energy calculation to higher order in \mathcal{H}' .

Numerical Values⁸

The first four eigenvalues of Eq. (11) with $\alpha = \infty$ are $\lambda_0 = 1.0605$, $\lambda_1 = 3.7998$, $\lambda_2 = 7.465$, and $\lambda_3 = 11.650$. The matrix elements of x and x^2 computed from the corresponding eigenfunctions are $x_{01} = 0.6007$, $x_{12} = -0.7336$, $x_{23} = 0.8384$, $x_{03} = -0.0320$, $(x^2)_{00} = 0.3618$, $(x^2)_{11} = 0.9016$, $(x^2)_{22} = 1.2426$, $(x^2)_{33} = 1.5522$, $(x^2)_{02} = -0.4674$, and $(x^2)_{13} = -0.6943$.

It would provide an enormous simplification for the higher-order calculations if we could assume, as is true in the harmonic oscillator case, that all matrix elements x_{nm} can be neglected unless $m = n \pm 1$. If this were a valid assumption, the sum rule (19) and the matrix product rule could be used to find several quantities that would be equal to unity. These quantities, and their numerically computed values, are

$$\begin{aligned} x_{01}^2/(x^2)_{00} &= 0.9973, & (x_{01}^2 + x_{12}^2)/(x^2)_{11} &= 0.9971, \\ (x_{12}^2 + x_{23}^2)/(x^2)_{22} &= 0.9988; \end{aligned} \quad (A)$$

$$\begin{aligned} (\lambda_1 - \lambda_0)x_{01}^2 &= 0.9884, \\ (\lambda_2 - \lambda_1)x_{12}^2 - (\lambda_1 - \lambda_0)x_{01}^2 &= 0.9841, \\ (\lambda_3 - \lambda_2)x_{23}^2 - (\lambda_2 - \lambda_1)x_{12}^2 &= 0.9692; \end{aligned} \quad (B)$$

$$x_{01}x_{12}/(x^2)_{02} = 0.9428, \quad x_{12}x_{23}/(x^2)_{13} = 0.8859. \quad (C)$$

It is interesting to note that the neglected terms in group (A) are of second order in the small matrix elements, and hence very small. The neglected terms in group (B) are also of second order, but are multiplied by the difference of two λ 's, and are therefore somewhat larger. In group (C), the neglected terms are of first order, so that the errors are largest in this case.

For finite α in Eq. (11), the eigenvalues and matrix elements approach more closely those of the harmonic oscillator, for which all x_{nm} with $m \neq n \pm 1$ are zero. We

⁷ The writer is indebted to D. Saxon of the National Bureau of Standards, Institute for Numerical Analysis, for arranging for these precise numerical computations, which provided a very important stimulus to the present work, and also to E. Osborne and P. Kaus who carried them out.

⁸ Values of λ_0 and λ_1 were first computed by W. E. Milne, Phys. Rev. **35**, 863 (1930). The other λ 's and all the u 's were computed at the Institute for Numerical Analysis (reference 7). The matrix elements were computed from the u 's by connecting on the asymptotic form beyond the range of the numerical computations.

expect, therefore, that the neglect of such matrix elements in the nonlinear case is an even better approximation for finite α than is indicated by the above numerical results for infinite α .

Higher-Order Terms in the Energy of a Particle

In everything that follows, we assume that all matrix elements x_{nm} or ϕ_{nm} with $m \neq n \pm 1$ can be neglected. This is expected to introduce errors of the general order of magnitude of one to ten percent. The perturbation treatment of \mathcal{H}' has been carried to third order; this is feasible because the sums over intermediate states are severely limited in scope by the assumed structure of the matrix elements.

For the lowest (vacuum) state of the lattice, the zero-order energy is $N\epsilon_0$, and the first-order energy is zero, as stated earlier. The second-order energy is

$$-\frac{\phi_{01}^4}{4(\epsilon_1 - \epsilon_0)} \sum_s \sum'_t A_{st}{}^2 = -\frac{\phi_{01}^4}{4(\epsilon_1 - \epsilon_0)} N(Q - P^2), \tag{20}$$

$$Q \equiv N^{-1} \sum_{\mathbf{k}} k^4 \xrightarrow{N \rightarrow \infty} \frac{19}{15} \frac{\pi^4}{l^4} = \frac{19}{15} P^2.$$

The third-order vacuum energy is

$$\frac{\phi_{01}^6}{4(\epsilon_1 - \epsilon_0)^2} \sum_s \sum'_t \sum''_q A_{st} A_{tq} A_{qs}$$

$$= \frac{\phi_{01}^6}{4(\epsilon_1 - \epsilon_0)^2} N(R - 3QP + 2P^3), \tag{21}$$

$$R \equiv N^{-1} \sum_{\mathbf{k}} k^6 \xrightarrow{N \rightarrow \infty} \frac{583}{315} P^3.$$

The dashes on the summations over t in Eqs. (20) and (21) mean that the terms $t=s$ are omitted, and the double dash on the summation over q in (21) means that the terms $q=t$ and $q=s$ are omitted.

The zero- and first-order energy of the first excited state of the lattice was obtained earlier: $(N-1)\epsilon_0 + \epsilon_1 + (K^2 - P)\phi_{01}^2$, and corresponds to the first-order perturbed eigenfunction (14). The second-order energy of this state is

$$-\frac{\phi_{01}^4}{2(\epsilon_1 - \epsilon_0)} (K^4 - 2K^2P - 2Q + 3P^2)$$

$$-\frac{\phi_{01}^2 \phi_{12}^2}{\epsilon_2 - \epsilon_0} (Q - P^2) - \frac{\phi_{01}^4}{4(\epsilon_1 - \epsilon_0)} N(Q - P^2). \tag{22}$$

Up to this point, all of the summations over s, t , etc. have been reducible to the form (5); that is, there have never been more than two \mathbf{k} vectors in a single exponent. In the calculation of the third-order energy of the excited state, however, there appear summations of

the form

$$\sum_s e^{i(\mathbf{k} + \mathbf{k}' + \mathbf{k}'' + \mathbf{K}) \cdot \mathbf{r}_s}. \tag{23}$$

The summation (23) is zero unless the sum of the four \mathbf{k} vectors in the exponent is equal to zero or to one of the 26 other vectors with Cartesian coordinates $0, \pm 2\pi/l$, in which case it is equal to N . These 26 other vectors, which can be reached by the sum of four \mathbf{k} vectors, cannot be reached by the sum of two \mathbf{k} vectors as in (5), since each \mathbf{k} vector is confined to the interior of a cube whose edges are at $\pm\pi/l$; that the edges and surfaces of the reciprocal lattice cube are not accessible to the individual \mathbf{k} vectors is most easily seen by considering the case in which L/l is an odd number, when each component of a \mathbf{k} vector can have values $0, \pm 2\pi/L, \pm 4\pi/L, \dots, \pm[(\pi/l) - (\pi/L)]$. In similar fashion, there are no contributions to (23) from \mathbf{k} -sum vectors with components $\pm 4\pi/l$, since these cannot be reached by the sum of four \mathbf{k} vectors.

All terms that contain summations of the form (23), with four \mathbf{k} vectors in a single exponent, can be reduced to

$$Z \equiv N^{-4} \sum_s \sum'_t \sum_{\mathbf{k}} \sum_{\mathbf{k}'} \sum_{\mathbf{k}''} k^2 k'^2 k''^2 e^{i(\mathbf{k} + \mathbf{k}' + \mathbf{k}'' + \mathbf{K}) \cdot (\mathbf{r}_s - \mathbf{r}_t)}.$$

To evaluate Z , we first perform the summations over s and t , to give

$$Z = N^{-2} \sum_{\mathbf{k}} \sum_{\mathbf{k}'} \sum_{\mathbf{k}''} \sum_i k^2 k'^2 k''^2 \delta(\mathbf{k} + \mathbf{k}' + \mathbf{k}'' + \mathbf{K} - \boldsymbol{\kappa}_i) - P^3,$$

where the vectors $\boldsymbol{\kappa}_i$ are the 26 referred to above, and the δ symbol is unity if its argument is zero and is zero otherwise. Next we sum over \mathbf{k}'' for fixed values of \mathbf{k}, \mathbf{k}' , and \mathbf{K} , and note that any $\boldsymbol{\kappa}_i$ can only be reached with \mathbf{k}'' by starting from a value of $\boldsymbol{\gamma} \equiv \mathbf{k} + \mathbf{k}' + \mathbf{K}$ that lies within the cube of edge length $2\pi/l$ whose center is at that $\boldsymbol{\kappa}_i$. Thus all possible combinations of \mathbf{k}, \mathbf{k}' and \mathbf{K} contribute, each just once, and to various $\boldsymbol{\kappa}$'s. We thus get

$$Z + P^3 = N^{-2} \sum_{\mathbf{k}} \sum_{\mathbf{k}'} k^2 k'^2 (\boldsymbol{\kappa}_i - \mathbf{k} - \mathbf{k}' - \mathbf{K})^2,$$

where now $\boldsymbol{\kappa}_i$ is a discontinuous function of $\boldsymbol{\gamma}$.

We now take the limit $N \rightarrow \infty$ by replacing the \mathbf{k} summations by integrations: $N^{-1} \sum_{\mathbf{k}} [\] \rightarrow (l/2\pi)^3 \times \int [\] d\boldsymbol{\tau}_k$, where the integral is over a cube of edge positions $\pm\pi/l$. When the integrand is squared out, we obtain

$$Z + P^3 = K^2 P^2 + 2QP + W, \tag{24}$$

$$W \equiv (l/2\pi)^6 \int \int k^2 k'^2 [\boldsymbol{\kappa}^2 - 2\boldsymbol{\kappa} \cdot (\mathbf{k} + \mathbf{k}' + \mathbf{K})] d\boldsymbol{\tau}_k d\boldsymbol{\tau}_{k'},$$

where the subscript has been dropped from $\boldsymbol{\kappa}_i$. The vector $\boldsymbol{\kappa}$ is a discontinuous function of $\boldsymbol{\gamma} (= \mathbf{k} + \mathbf{k}' + \mathbf{K})$; the relation between x components of $\boldsymbol{\kappa}$ and $\boldsymbol{\gamma}$ is $\kappa_x = 2\pi/l$ for $\gamma_x > \pi/l$, $\kappa_x = 0$ for $-\pi/l < \gamma_x < \pi/l$, $\kappa_x = -2\pi/l$ for $\gamma_x < -\pi/l$, with corresponding relations for the other components. The evaluation of W is rather

tedious; it can be done either by breaking up the domain of integration according to the value of κ , or by representing the components of κ as discontinuous integrals. Both methods were used in order to have a check available; the result is

$$W = \frac{1}{30}K^6 - \frac{1}{6}K^4P - \frac{83}{30}K^2P^2 - \frac{997}{630}P^3. \quad (25)$$

The third-order energy of the excited state may now be calculated; it exceeds the third-order vacuum energy (21) by the amount

$$\begin{aligned} & \frac{\phi_{01}^6}{2(\epsilon_1 - \epsilon_0)^2} (K^6 - 3K^4P + 6K^2P^2 - 3K^2Q \\ & \quad - 3R + 12QP - 10P^3 + 2Z) \\ & + \frac{\phi_{01}^4\phi_{12}^2}{(\epsilon_1 - \epsilon_0)(\epsilon_2 - \epsilon_0)} [2(K^2 - P)(Q - P^2) \\ & \quad - 2Z + R - 3QP + 2P^3] \\ & + \frac{\phi_{01}^4\phi_{12}^2}{(\epsilon_2 - \epsilon_0)^2} [- (K^2 - P)(Q - P^2) + R \\ & \quad - 3QP + 2P^3] + \frac{\phi_{01}^2\phi_{12}^4}{(\epsilon_2 - \epsilon_0)^2} Z, \quad (26) \end{aligned}$$

where Z may be obtained from Eqs. (24) and (25). In order to obtain the particle energy through third order, we must add to this the energy through second order, which is the sum of (15) and (22) minus (20):

$$\begin{aligned} & (\epsilon_1 - \epsilon_0) + \phi_{01}^2(K^2 - P) - \frac{\phi_{01}^4}{2(\epsilon_1 - \epsilon_0)} \\ & \times (K^4 - 2K^2P - 2Q + 3P^2) - \frac{\phi_{01}^2\phi_{12}^2}{\epsilon_2 - \epsilon_0} (Q - P^2). \quad (27) \end{aligned}$$

Approximate Relativistic Relation Between Particle Energy and Momentum

The energy of the particle, through third order in \mathcal{H}' , is given by the sum of Eqs. (26) and (27), within the limits of the approximation that matrix elements ϕ_{nm} with $m \neq n \pm 1$ are neglected. The momentum of this particle is \mathbf{K} , provided \mathbf{K} is small in comparison with $1/l$. The relation between \mathbf{K} and the momentum is not affected by the higher-order corrections to Eq. (14); this is because \mathbf{p} commutes with \mathcal{H}_0 and \mathcal{H}' separately, so that only unperturbed energy eigenfunctions that are eigenfunctions of \mathbf{p} with the same value of \mathbf{K} as Eq. (14) can mix in with it.

Before considering the relation between the energy and momentum of this particle, it is instructive to see what this relation is in the linear case ($\alpha=0$). The solutions of Eq. (10) are the harmonic oscillator wave

functions; the eigenvalues and matrix elements are

$$\begin{aligned} \epsilon_n &= (n + \frac{1}{2})(P + \mu^2)^{\frac{1}{2}}, \\ \phi_{n, n+1} &= (P + \mu^2)^{-\frac{1}{2}} [\frac{1}{2}(n+1)]^{\frac{1}{2}}, \end{aligned}$$

with other matrix elements equal to zero. With these substitutions, the sum of Eqs. (26) and (27) becomes

$$\begin{aligned} & (\epsilon_1 - \epsilon_0) + \phi_{01}^2(K^2 - P) - \frac{\phi_{01}^4}{2(\epsilon_1 - \epsilon_0)} (K^2 - P)^2 \\ & + \frac{\phi_{01}^6}{2(\epsilon_1 - \epsilon_0)^2} (K^2 - P)^3 = (P + \mu^2)^{\frac{1}{2}} + \frac{(K^2 - P)}{2(P + \mu^2)^{\frac{1}{2}}} \\ & - \frac{(K^2 - P)^2}{8(P + \mu^2)^{\frac{3}{2}}} + \frac{(K^2 - P)^3}{16(P + \mu^2)^{5/2}}. \quad (28) \end{aligned}$$

The right side of Eq. (28) will immediately be recognized as the first four terms in the power series expansion of

$$[(P + \mu^2) + (K^2 - P)]^{\frac{1}{2}} = (\mu^2 + K^2)^{\frac{1}{2}}, \quad (29)$$

which is the relativistic expression for the energy of a particle of rest mass μ and momentum \mathbf{K} . This is hardly a surprising result, since the quantization of Eq. (1) with $\alpha=0$ is easily carried through exactly by using expansions of the field variables in plane waves, both in the continuum and lattice cases. The energy of a single particle is known to be given by Eq. (29), and the foregoing derivation of Eq. (28) might well be regarded as unnecessarily complicated.

It is nevertheless important to observe that Eq. (28) was derived by using a perturbation method that in the linear case can only be justified for large l (see the next to the last paragraph of Sec. II). In spite of this restriction, we presumably could, by continuing the development in powers of \mathcal{H}' , sum the series to obtain Eq. (29), which we would then be willing to accept regardless of whether or not l is large. We therefore argue that it is possible in this way to obtain a result that is valid even when the perturbation expansion from which it was derived cannot justifiably be used. We further argue that results like Eq. (29) that have a kind of relativistic character can be obtained in this way from a manifestly non-covariant theory.

We now proceed with the nonlinear case, and see how closely the sum of Eqs. (26) and (27) agrees with the left side of Eq. (28), when allowance is made in the latter for altered values of $(\epsilon_1 - \epsilon_0)$ and ϕ_{01} . It is apparent that the zero- and first-order parts of both are the same. The second-order part of Eq. (27) exceeds the second-order part of Eq. (28) by

$$\left[\frac{\phi_{01}^4}{\epsilon_1 - \epsilon_0} - \frac{\phi_{01}^2\phi_{12}^2}{\epsilon_2 - \epsilon_0} \right] (Q - P^2). \quad (30)$$

The third-order energy (26) exceeds the third-order

part of (28) by

$$\left[\frac{3\phi_{01}^6}{2(\epsilon_1 - \epsilon_0)^2} + \frac{2\phi_{01}^4\phi_{12}^2}{(\epsilon_1 - \epsilon_0)(\epsilon_2 - \epsilon_0)} + \frac{\phi_{01}^4\phi_{12}^2}{(\epsilon_2 - \epsilon_0)^2} \right] \times (K^2 - P)(Q - P^2) + \left[\frac{3\phi_{01}^6}{2(\epsilon_1 - \epsilon_0)^2} + \frac{\phi_{01}^4\phi_{12}^2}{(\epsilon_1 - \epsilon_0)(\epsilon_2 - \epsilon_0)} + \frac{\phi_{01}^4\phi_{12}^2}{(\epsilon_2 - \epsilon_0)^2} \right] (R - 3QP + 2P^3) + \left[\frac{\phi_{01}^6}{(\epsilon_1 - \epsilon_0)^2} - \frac{2\phi_{01}^4\phi_{12}^2}{(\epsilon_1 - \epsilon_0)(\epsilon_2 - \epsilon_0)} + \frac{\phi_{01}^2\phi_{12}^4}{(\epsilon_2 - \epsilon_0)^2} \right] Z. \quad (31)$$

In estimating the errors implied by Eqs. (30) and (31), we shall use for the ϵ 's and matrix elements the numerical values given previously for the case $\alpha = \infty$. Since the individual square brackets in Eqs. (30) and (31) all vanish in the linear case ($\alpha = 0$), this procedure is likely to overestimate the errors. We then find for the errors in the coefficients of K^4 , K^2 , and K^0 in the second-order part, 0, 0, and 19.4 percent, respectively. In similar fashion, we find for the errors in the coefficients of K^6 , K^4 , K^2 , and K^0 in the third-order part, 0.9, 1.5, 24.5, and 19.3 percent, respectively. Thus of the ten coefficients that can be compared with the left side of Eq. (28), five are in agreement, two show about one percent errors, and three show about twenty percent errors; these are believed to overestimate the errors that would be obtained with finite α .⁹

This encourages us to regard the series on the left side of Eq. (28) as a sufficiently accurate representation of the energy of the particle. If we also equate $2\phi_{01}^2(\epsilon_1 - \epsilon_0)$ to unity (the error here is 1.2 percent when $\alpha = \infty$), then these are the first four terms in the power series expansion of

$$\begin{aligned} [(\epsilon_1 - \epsilon_0)^2 + (K^2 - P)]^{\frac{1}{2}} &= (M^2 + K^2)^{\frac{1}{2}}, \\ M &\equiv [(\epsilon_1 - \epsilon_0)^2 - P]^{\frac{1}{2}}. \end{aligned} \quad (32)$$

We therefore interpret the quantity M in Eq. (32) as the rest mass of the particle. Moreover, following the lead provided by the discussion of the linear case above, we assume that Eq. (32) is valid for all α , and not only for large α where the perturbation calculation is justified.

Comparison of Eqs. (32) and (10) shows that M is a well-defined function of the three parameters of the theory: μ , α , and l . We have already observed that if $\alpha = 0$, $M = \mu$ regardless of the value of l [see Eq. (29)]. We can now see from the structure of Eq. (10) that the addition of any positive fourth-power potential to the harmonic potential increases the spacing of the energy levels, so that $(\epsilon_1 - \epsilon_0)$ is larger for finite α than it is for $\alpha = 0$. It is also apparent that for any fixed finite α , the importance of the fourth-power potential

⁹This agreement could be improved upon by altering the definition of M in Eq. (32) to compensate for the errors in the higher-order terms. This added complication hardly seems worthwhile at the present time.

increases as l decreases, and we have already seen (Sec. II) that the energy levels become infinite like $1/l$ as $l \rightarrow 0$. We therefore conclude that so long as $\alpha \neq 0$, M is larger than μ , and by an amount that becomes infinite as $l \rightarrow 0$. This is a physically satisfactory conclusion, since we expect the nonlinearity, which corresponds to an internal self-repulsion of the field, to increase the self-energy of any particle that is described by the field. It is also not surprising that this self-energy should become infinite as $l \rightarrow 0$, since this makes the particles more nearly like points.

Quantitative results can be expressed in terms of Eq. (11). We put $\beta \equiv 2^{\frac{1}{2}}(\pi^2 + \mu^2 l^2)/\alpha^{4/3}$ and $z(\beta) \equiv \lambda_1(\beta) - \lambda_0(\beta)$. For small values of β , $z(\beta)$ may be found by perturbing the solutions already obtained with $\beta = 0$:

$$\begin{aligned} z(\beta) &\cong \lambda_1(0) - \lambda_0(0) + \beta[(x^2)_{11} - (x^2)_{00}] \\ &= 2.7393 + 0.5398\beta. \end{aligned}$$

For large values of β , $z(\beta)$ may be found by perturbing the harmonic oscillator solutions which are valid for $\beta = \infty$:

$$z(\beta) \cong 2\beta^{\frac{1}{2}} + 3\beta^{-1} - 9\beta^{-5/2}.$$

For three intermediate values of β ($= 3, 6, 9$), numerical integrations of Eq. (11) were performed.¹⁰ It is then convenient to define two new variables:

$$\begin{aligned} y &\equiv \frac{z^2 - 4\beta}{2^{8/3}} = \frac{l^2}{\alpha^{4/3}}(M^2 - \mu^2), \\ x &\equiv \frac{2^{\frac{1}{2}}\pi^{\frac{1}{2}}}{\beta^{\frac{1}{2}}} = \frac{\alpha}{(1 + \mu^2 l^2/\pi^2)^{\frac{1}{2}}}. \end{aligned} \quad (33)$$

A curve of y against x is plotted in Fig. 1; from this, the value of M that corresponds to any combination of values of μ , α , and l is readily found.

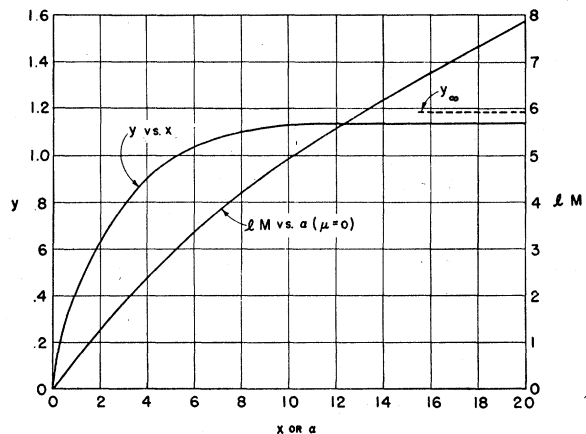


FIG. 1. Plots of quantities related to the rest mass of a free particle; see Eq. (33) *et seq.*

¹⁰The writer is indebted to D. L. Judd of the Radiation Laboratory, University of California, for arranging for these numerical computations, and also to J. Killeen and the differential analyzer group who carried them out.

Since for finite α the field rest mass μ must always be less than the particle rest mass M , it might be argued that μ does not fulfill any useful purpose in the present theory. Then one free parameter can be eliminated by setting $\mu=0$, which means that the entire rest mass of the particle arises from the nonlinearity. From Eqs. (33), we see that for $\mu=0$, $x=\alpha$, and $lM=\alpha^2 y^{\frac{1}{2}}$, so that it is possible to plot lM against α directly; this is also done in Fig. 1. If, for example, we choose $l=1/M$, so that l is equal to the reduced Compton wavelength of the particle, Fig. 1 shows that $\alpha \cong 1.6$.

Higher Energy Eigenvalues

We have interpreted the first two eigenvalues of Eq. (10) in terms of the particle rest mass, through Eq. (32). A question then arises as to the role played by the higher energy eigenvalues. It might at first be thought that they correspond to particles of higher rest mass. To investigate this idea, consider the analog of Eq. (15) for the excitation energy through first order of the N -fold degenerate state in which a single lattice-point oscillator has the eigenvalue ϵ_n : $\epsilon_n - \epsilon_0 - P\phi_{0n}^2 + K^2\phi_{0n}^2$. We have already seen that all matrix elements ϕ_{0n} are very small unless $n=1$, so that there is a wide discrepancy between the approximate rest mass $\epsilon_n - \epsilon_0$ and the approximate kinetic mass $1/(2\phi_{0n}^2)$; the momentum is of course still given by \mathbf{K} . It is not reasonable, therefore, to interpret the higher energy eigenvalues in terms of particles of higher rest mass.

A clue as to the part they play in the theory can be obtained by comparison with the linear case. Here the energy levels are equally spaced, so that there is degeneracy between any higher eigenvalue and multiply excited lower eigenvalues; for example, the eigenvalue ϵ_n is degenerate with excitation of n lattice points to the eigenvalue ϵ_1 . Now n -fold excitation to ϵ_1 corresponds physically to the presence of n particles, so that all the other modes of excitation that are degenerate with this must be thought of as part of the description of the n -particle system, at least in the case of large l , where the perturbation theory may be used in the linear case.

We believe that a similar interpretation of the higher eigenvalues is required in the nonlinear case as well, even though the system is no longer degenerate. We shall see in the next section that such states form part of the description of the scattering of one particle by another, and are essential if the result is to have the proper behavior when $\alpha=0$ (linear case).

IV. SCATTERING OF ONE FREE PARTICLE BY ANOTHER

We have seen that a single free particle of momentum \mathbf{K} can be described to first order by the wave function (14). We therefore expect that two particles with momenta \mathbf{K}_1 and \mathbf{K}_2 can be described, at least approxi-

mately, by the wave function

$$\Psi_{11}(\mathbf{K}_1, \mathbf{K}_2) = N^{-1} \sum_s \sum_t' e^{i(\mathbf{K}_1 \cdot \mathbf{r}_s + \mathbf{K}_2 \cdot \mathbf{r}_t)} U_{11}(s, t),$$

$$U_{11}(s, t) \equiv u_1(\phi_s) u_1(\phi_t) \prod_q'' u_0(\phi_q); \quad (34)$$

the dash on the second summation means that the terms $t=s$ is omitted, and the double dash on the product means that the terms $q=t$ and $q=s$ are omitted. Note that since the states $\Psi_{11}(\mathbf{K}_1, \mathbf{K}_2)$ and $\Psi_{11}(\mathbf{K}_2, \mathbf{K}_1)$ are identical and the state $\mathbf{K}_1 = \mathbf{K}_2$ is permitted, the particles obey Einstein-Bose statistics.

An important difference between Eqs. (14) and (34) is that (14) diagonalizes the perturbation \mathcal{H}' to first order while (34) does not. This is because there is first-order interaction with all the states $\Psi_{11}(\mathbf{K}_1', \mathbf{K}_2')$ for which $\mathbf{K}_1' + \mathbf{K}_2' = \mathbf{K}_1 + \mathbf{K}_2$, and also with the state

$$\Psi_2(\mathbf{K}_1 + \mathbf{K}_2) = N^{-\frac{1}{2}} \sum_s e^{i(\mathbf{K}_1 + \mathbf{K}_2) \cdot \mathbf{r}_s} U_2(s),$$

$$U_2(s) \equiv u_2(\phi_s) \prod_t' u_0(\phi_t). \quad (35)$$

Because of the commutativity of \mathfrak{p} and \mathcal{H}' , the total \mathbf{K} vector must be the same for all combining states.¹¹ We now proceed to make a first-order diagonalization of \mathcal{H} , using an arbitrary linear combination of the above states:

$$\Psi = \sum_{\mathbf{k}} B(\mathbf{k}) \Psi_{11}(\mathbf{k}, \mathbf{K} - \mathbf{k}) + A \Psi_2(\mathbf{K}). \quad (36)$$

If the wave function (36) were an exact eigenfunction of \mathcal{H} , we would have $(\mathcal{H} - E)\Psi = 0$, where E is the total first-order energy of the initial collision state with momenta \mathbf{k}_0 and $\mathbf{K} - \mathbf{k}_0$:

$$E = (N-2)\epsilon_0 + 2\epsilon_1 + [\kappa_0^2 + (\mathbf{K} - \mathbf{k}_0)^2 - 2P]\phi_{01}^2.$$

Since it is not a solution, the best we can do is require that when $(\mathcal{H} - E)\Psi$ is analyzed into states that correspond to the excitation of 1, 2, \dots lattice points by various amounts, the coefficients of Ψ_{11} and Ψ_2 are zero. This means that

$$\int \dots \int \Psi_{11}^*(\mathbf{k}', \mathbf{K} - \mathbf{k}') (\mathcal{H} - E) \Psi d\phi_1 \dots d\phi_N = 0,$$

$$\int \dots \int \Psi_2^*(\mathbf{K}) (\mathcal{H} - E) \Psi d\phi_1 \dots d\phi_N = 0, \quad (37)$$

where \mathbf{k}' is arbitrary. The first of Eqs. (37) gives

$$[B(\mathbf{k}') + B(\mathbf{K} - \mathbf{k}')][\kappa'^2 + (\mathbf{K} - \mathbf{k}')^2 - \kappa_0^2$$

$$- (\mathbf{K} - \mathbf{k}_0)^2]\phi_{01}^2 - 2\phi_{01}^2 N^{-1} \sum_{\mathbf{k}} B(\mathbf{k})$$

$$\times [\kappa'^2 + (\mathbf{K} - \mathbf{k}')^2 + \kappa^2 + (\mathbf{K} - \mathbf{k})^2 - \kappa_0^2$$

$$- (\mathbf{K} - \mathbf{k}_0)^2 - 2P] + A\phi_{01}\phi_{12} N^{-\frac{1}{2}}$$

$$\times [\kappa'^2 + (\mathbf{K} - \mathbf{k}')^2 - 2P] = 0, \quad (38)$$

¹¹ There would also be interaction with the vacuum state (12) if $\mathbf{K}_1 + \mathbf{K}_2$ were exactly zero, that is, if the lattice were at rest in the center-of-mass coordinate system. The extra term introduced does not seem physically plausible, so we avoid it by assuming that $\mathbf{K}_1 + \mathbf{K}_2 \neq 0$.

and the second gives

$$A\{\epsilon_2 + \epsilon_0 - 2\epsilon_1 - [\kappa_0^2 + (\mathbf{K} - \kappa_0)^2 - 2P]\phi_{01}^2\} + \phi_{01}\phi_{12}N^{-\frac{1}{2}}\sum_{\kappa} B(\kappa)[\kappa^2 + (\mathbf{K} - \kappa)^2 - 2P] = 0. \quad (39)$$

Elimination of A between Eqs. (38) and (39) yields a set of simultaneous algebraic equations for the quantities $B(\kappa')$ with all possible values for κ' :

$$\begin{aligned} & [B(\kappa') + B(\mathbf{K} - \kappa')][\kappa'^2 + (\mathbf{K} - \kappa')^2 - \kappa_0^2 - (\mathbf{K} - \kappa_0)^2] \\ & = 2N^{-1}\sum_{\kappa} B(\kappa)\{[\kappa'^2 + (\mathbf{K} - \kappa')^2 + \kappa^2 + (\mathbf{K} - \kappa)^2 \\ & \quad - \kappa_0^2 - (\mathbf{K} - \kappa_0)^2 - 2P] - (2\Delta E)^{-1} \\ & \quad \times \phi_{12}^2[\kappa'^2 + (\mathbf{K} - \kappa')^2 - 2P][\kappa^2 + (\mathbf{K} - \kappa)^2 - 2P]\}, \quad (40) \\ \Delta E & \equiv [\kappa_0^2 + (\mathbf{K} - \kappa_0)^2 - 2P]\phi_{01}^2 + 2\epsilon_1 - (\epsilon_2 + \epsilon_0). \end{aligned}$$

Equation (40) is a Tamm-Dancoff-type approximation to the field equation, in which only the states corresponding to two singly excited lattice points and to one doubly excited lattice point are included. We do not solve Eq. (40) exactly, but rather find the first Born approximation to the solution, by putting

$$B(\kappa) = \delta_{\kappa, \kappa_0} + b(\kappa), \quad b(\kappa) \ll 1, \quad (41)$$

and neglecting $b(\kappa)$ in comparison with unity. Substitution of Eq. (41) into Eq. (40) gives, for κ' different from κ_0 and $\mathbf{K} - \kappa_0$:

$$\begin{aligned} b(\kappa') + b(\mathbf{K} - \kappa') & = \frac{2[\kappa'^2 + (\mathbf{K} - \kappa')^2 - 2P]}{N[\kappa'^2 + (\mathbf{K} - \kappa')^2 - \kappa_0^2 - (\mathbf{K} - \kappa_0)^2]} \\ & \times \left\{ 1 - \frac{[\kappa_0^2 + (\mathbf{K} - \kappa_0)^2 - 2P]\phi_{01}^2}{2\Delta E} \right\}. \quad (42) \end{aligned}$$

The two-particle part of Ψ may now be found by rewriting the first term on the right side of Eq. (36) as

$$\frac{1}{2}\sum_{\kappa} [B(\kappa) + B(\mathbf{K} - \kappa)]\Psi_{11}(\kappa, \mathbf{K} - \kappa)$$

and substituting from Eqs. (41) and (42):

$$\begin{aligned} \Psi_{11}(\kappa_0, \mathbf{K} - \kappa_0) + N^{-1}\sum_{\kappa} \frac{[\kappa^2 + (\mathbf{K} - \kappa)^2 - 2P]}{[\kappa^2 + (\mathbf{K} - \kappa)^2 - \kappa_0^2 - (\mathbf{K} - \kappa_0)^2]} \\ \times \left\{ 1 - \frac{[\kappa_0^2 + (\mathbf{K} - \kappa_0)^2 - 2P]\phi_{01}^2}{2\Delta E} \right\} \cdot \Psi_{11}(\kappa, \mathbf{K} - \kappa). \quad (43) \end{aligned}$$

The physical significance of Eq. (43) can be seen by substituting for Ψ_{11} from Eq. (34), and interpreting the coefficient of $U_{11}(s, t)$ as the probability amplitude for finding the two particles at \mathbf{r}_s and \mathbf{r}_t . We take the limit $N \rightarrow \infty$ by replacing the summation over κ by an integral, as in Sec. III:

$$N^{-1}\sum_{\kappa} [\] \rightarrow (l/2\pi)^3 \int [\] d\tau.$$

As with the usual Born approximation, the requirement that the scattered wave be outgoing means that the path of integration must pass below the singularity in the integrand when the integral over the magnitude κ of the momentum is carried out. The square of the amplitude of the outgoing wave gives the differential scattering cross section. Since the summand in Eq. (43) is independent of the directions of the momenta in the center-of-mass coordinate system, the scattering is spherically symmetric in this coordinate system. The total cross section, when allowance is made for the identity of the two particles in the normalization of the incident plane wave, is

$$\sigma = \frac{l^6}{\pi} (K_0^2 + \frac{1}{4}K^2 - P)^2 \left\{ 1 - \frac{(K_0^2 + \frac{1}{4}K^2 - P)\phi_{12}^2}{\Delta E} \right\}^2, \quad (44)$$

$$\Delta E = 2(K_0^2 + \frac{1}{4}K^2 - P)\phi_{01}^2 + 2\epsilon_1 - (\epsilon_2 + \epsilon_0),$$

where $\mathbf{K}_0 \equiv \frac{1}{2}\mathbf{K} - \kappa_0$ is the momentum of one of the particles with respect to the center of mass.¹²

We note first that for the linear case, which is included in the present calculation because all of the degenerate states are retained, $2\epsilon_1 = \epsilon_2 + \epsilon_0$ and $\phi_{12}^2 = 2\phi_{01}^2$, so that the scattering cross section is zero. This is as it should be in a theory that admits of a superposition principle.

In the nonlinear case, the two terms in the curly bracket of Eq. (44) do not cancel. The first term can be shown to arise from the fact that there is no term $U(s, s)$ is the summation of Eq. (34); the product of plane waves is not quite complete, and the "hole" when s and t are equal results in scattering. The second term arises from the second-order transition through the state $\Psi_2(\mathbf{K})$, as is evident from the fact that it contains ϕ_{12} and ϵ_2 . Since this calculation makes use of the perturbation theory, it is only valid when $\alpha \gg 1$. Then the second term is smaller than the first term by a factor of order $1/\alpha^{4/3}$. A further analysis shows that the first term corresponds to a repulsive and the second to an attractive interaction. The second term can in principle give rise to a resonance when $\Delta E \cong 0$, that is, when the total energy of the colliding particles equals that of the state $\Psi_2(\mathbf{K})$. This resonance is masked by the P term in the kinetic energy of the colliding particles, which cannot be treated correctly by a calculation of this order (see Sec. III).

Since the total momentum \mathbf{K} of the two colliding particles appears in the cross section, the result is not Galilean-invariant.¹³ However, as might be expected, the lack of Galilean invariance appears only when the wavelengths of the particles are comparable with the lattice constant, and then the interpretation in terms of momentum breaks down. For long wave lengths, K_0^2 and K^2 can be neglected in comparison with P , and

¹² Equation (44) in the case $\mathbf{K} \cong 0$ was first derived by D. R. Yennie, who kindly informed the writer of this calculation.

¹³ A further lack of Galilean invariance is present in that the case $\mathbf{K} = 0$ is exceptional (see reference 11).

the total cross section for large α is

$$\sigma \cong l^6 P^2 / \pi = \pi^3 l^2. \quad (45)$$

The spherical symmetry of the scattering implies that the interaction is of very short range, and the proportionality of Eq. (45) with l^2 is consistent with this since the smallest length that can occur in the theory is l . This appearance of an essentially point repulsion between particles is in agreement with the interpretation of the ϕ^4 term in the continuum field Hamiltonian as a point-contact repulsion.¹⁴

V. INTERACTION WITH NUCLEONS

Nucleons as Classical Sources

Nucleons may be introduced into the theory as classical sources for the quantized field.¹ We assume that the source-field (nucleon-meson) coupling is linear,¹⁵ and add to the continuum Hamiltonian (1) the interaction

$$H_i = - \int g(\mathbf{r}) \phi d\tau, \quad (46)$$

where $g(\mathbf{r})$ is proportional to the density of nucleons. Neutrons and protons are coupled to mesons in the same way in the present treatment, and their distribution is assumed to be constant in time. In analogy with Eq. (4), we define the lattice source density

$$g_s \equiv l^3 \int f(\mathbf{r} - \mathbf{r}_s) g(\mathbf{r}) d\tau,$$

where the coefficient has been chosen so that

$$\sum_s g_s = \int g(\mathbf{r}) d\tau.$$

We then write for the lattice interaction Hamiltonian

$$\mathcal{H}_i = -l^{-3} \sum_s g_s \phi_s, \quad (47)$$

which approaches H_i as $l \rightarrow 0$.

Since \mathcal{H}_i is a sum of terms each of which involves a single lattice point, it can be included with \mathcal{H}_0 as part of the unperturbed Hamiltonian. This means that arbitrarily strong couplings can be included within the framework of the present theory. Equation (10) then becomes

$$\left[-\frac{1}{2} \frac{d^2}{d\phi_s^2} + \frac{1}{2} (P + \mu^2) \phi_s^2 + \frac{\alpha^2}{4l^3} \phi_s^4 - \frac{g_s}{l^3} \phi_s \right] u_{sn}(\phi_s) = \epsilon_{sn} u_{sn}(\phi_s), \quad (48)$$

¹⁴ See I; the scattering predicted by the usual quantization of Eq. (1), treating the ϕ^4 term as a perturbation ($\alpha \ll 1$), is also spherically symmetric and the total cross section is $9\alpha^3/16\pi\mu^2$ for low-energy particles.

¹⁵ Other possibilities are discussed by L. I. Schiff, Phys. Rev. 84, 10 (1951); 86, 856 (1952); Berger, Foldy, and Osborn, Phys. Rev. 87, 1061 (1952).

where the subscript s must be retained in case g_s in Eq. (47) is not the same for all lattice points. It is apparent that the energy eigenvalues of Eq. (48) form a discrete set extending from a smallest value (which is negative if $|g_s|$ is sufficiently large) to $+\infty$. The eigenfunctions u_{sn} with given s form a complete *orthonormal* set, but do not have definite parities if $g_s \neq 0$.

Nucleon Isobaric State

Suppose that a single nucleon is present at the point \mathbf{r}_0 , so that $g_0 = g$, and $g_s = 0$ for $s \neq 0$. We assume that α is large so that \mathcal{H}' can be treated as a perturbation. Then the vacuum state is that in which all lattice points are in their lowest states, and has the zero-order energy $(N-1)\epsilon_0(0) + \epsilon_0(g)$, where $\epsilon_n(g)$ is the $(n+1)$ th energy level of Eq. (48) when $g_s = g$. The first excited state has a zero-order energy that exceeds the vacuum energy by $\epsilon_1(g) - \epsilon_0(g)$ or $\epsilon_1(0) - \epsilon_0(0)$, according as the nucleon lattice point is or is not the one that is excited. As discussed in Sec. III, the latter excitation energy is related to the rest mass of a free particle; it is then natural to interpret the former excitation energy in terms of an isobaric state of the nucleon. If $\epsilon_1(g) - \epsilon_0(g)$ is larger than $\epsilon_1(0) - \epsilon_0(0)$, as is actually the case, the isobaric state will be degenerate with the state of a nucleon plus a free meson with an appropriate amount of kinetic energy. Then since these two states are coupled together by \mathcal{H}' , the isobaric state will be short-lived and manifest itself mainly as a resonance in the meson-nucleon scattering, as is discussed in the next subsection.

If we apply to Eq. (48) the change of variables that takes Eq. (10) into Eq. (11), we get

$$\frac{d^2 u_n}{dx^2} + (\lambda_n - \beta x^2 - x^4 + \gamma x) u_n = 0, \quad (49)$$

$$\beta \equiv 2^{\frac{2}{3}} (\pi^2 + \mu^2 l^2) / \alpha^{4/3}, \quad \gamma \equiv 2^{\frac{2}{3}} g / \alpha.$$

For α large compared to unity but not necessarily large compared to g , the βx^2 term can be neglected, and the effect of the γx term by itself can be investigated. A combination of perturbation and variational methods can be used to show that, for small γ ,

$$\lambda_0 \cong 1.0605 - 0.1318\gamma^2, \quad (50)$$

$$\lambda_1 \cong 3.7998 - 0.0152\gamma^2.$$

For large γ , the "potential" $x^4 - \gamma x$ can be approximated by a parabola, and a WKB calculation shows that

$$\lambda_0 \cong -3(\gamma/4)^{4/3} + 6^{1/3}(\gamma/4)^{\frac{1}{3}}, \quad (51)$$

$$\lambda_1 \cong -3(\gamma/4)^{4/3} + (54)^{1/3}(\gamma/4)^{\frac{1}{3}}.$$

For three intermediate values of γ ($= 1, 2, 3$), numerical calculations were carried out at Berkeley.¹⁰ Figure 2 is a plot of $[\lambda_1(\gamma) - \lambda_0(\gamma)] - [\lambda_1(0) - \lambda_0(0)]$ against γ for $\beta = 0$, which shows that the isobaric state is in fact unstable in this case.

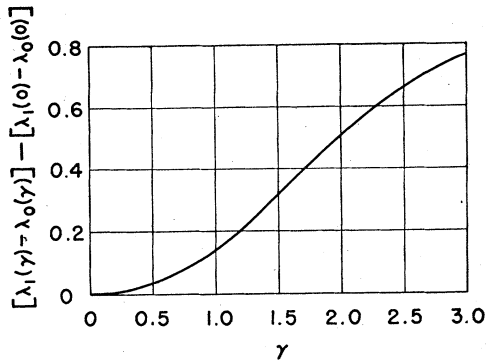


FIG. 2. The ordinate is proportional to the isobaric state energy, and the abscissa to the nucleon source strength.

Meson-Nucleon Scattering

The cross section for meson-nucleon scattering can be calculated in exact analogy with the meson-meson scattering calculation of Sec. IV. In place of Eq. (36) we adopt the following Tamm-Dancoff-type wave function:

$$\Psi = \sum_{\mathbf{k}} B(\mathbf{K})\Psi_1'(\mathbf{K}) + AU_1(0),$$

where $\Psi_1'(\mathbf{K})$ is given by Eq. (14) except that the nucleon lattice point ($s=0$) is excluded from the summation. Note that because of the presence of the nucleon, K is no longer a strict constant of the motion, so that states with different total \mathbf{K} vectors can mix together. In solving the resulting equations by the Born approximation, we put $B(\mathbf{K}) = \delta_{\mathbf{k}, \mathbf{k}_0} + b(\mathbf{K})$, where $b(\mathbf{K}) \ll 1$ and \mathbf{K}_0 is the incident meson momentum.

The scattering is spherically symmetric, and the total cross section is

$$\sigma = \frac{l^6}{4\pi} (K_0^2 - P)^2 \left\{ 1 - \frac{(K_0^2 - P)\phi_{01}'^2}{\Delta E} \right\}^2, \tag{52}$$

$$\Delta E = (K_0^2 - P)\phi_{01}'^2 + (\epsilon_1 - \epsilon_0) - (\epsilon_1' - \epsilon_0').$$

The primes in Eq. (52) refer to quantities calculated at the nucleon lattice point. We note first that for the linear case, the energy levels and matrix elements are not affected by the presence of the nucleon, so that the two terms in the curly bracket cancel. Such a vanishing cross section is to be expected since, as is well known, the presence of a fixed source does not affect the quantization of a real linear field.¹⁶ The same cancellation occurs, as it must, in the nonlinear case when g is set equal to zero.

In the nonlinear case with $g \neq 0$, the two terms in the curly bracket of Eq. (52) do not cancel. In analogy with the meson-meson scattering calculation (Sec. IV), the first term arises from the "hole" in the normal structure of the lattice at the point \mathbf{r}_0 , while the second term comes from the second-order transition through

¹⁶ See for example G. Wentzel, *Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1949), p. 47.

the state $U_1(0)$. Again the first term corresponds to a repulsive and the second to an attractive interaction. The second term should give rise to a resonance when $\Delta E \cong 0$, that is, when the total energy of the incident meson equals that of the isobaric state $U_1(0)$; however, the resonance is masked by the P term in the kinetic energy of the incident meson, which requires a higher-order calculation for its correct treatment. When the perturbation theory is taken seriously ($\alpha \gg 1$), the second term is small in comparison with the first term, and the total cross section for scattering of a slow meson by a nucleon is

$$\sigma \cong l^6 P^2 / 4\pi = \pi^3 l^2 / 4. \tag{53}$$

The spherical symmetry of the scattering and the proportionality of Eq. (53) with l^2 imply an essentially point interaction.

Interaction Between Two Nucleons

The zero-order vacuum energy of the lattice with two nucleons present is $(N-2)\epsilon_0(0) + 2\epsilon_0(g)$, regardless of where the nucleons are located provided that they do not occupy the same lattice point. There is a first-order change in the energy since the parity is not a good quantum number for the nucleon lattice points, and also a second-order change in the energy which arises from states in which one or another lattice point is excited. Both of these are position-dependent, but are small if the perturbation treatment of \mathcal{H}' is justified.

There still remains a zero-order point interaction between the two nucleons, since the vacuum energy is altered when the two nucleons occupy the same lattice point. This interaction may be thought of as having a range of order l and a value

$$[\epsilon_0(2g) + \epsilon_0(0)] - 2\epsilon_0(g),$$

which is negative. With the notation of Eq. (49), the quantity $2\lambda_0(\gamma) - [\lambda_0(0) + \lambda_0(2\gamma)]$ is proportional to the negative of the interaction energy. It is plotted against γ for $\beta=0$ in Fig. 3, which shows that the interaction is attractive in this case.

Saturation of Nuclear Forces

Thus far in this section we have considered situations in which one or two nucleons are introduced as classical

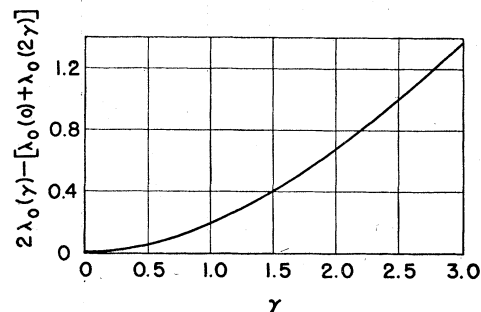


FIG. 3. The ordinate is proportional to the strength of the nucleon-nucleon interaction, and the abscissa to the nucleon source strength.

sources localized at as many lattice points. In considering many nucleons assembled to form a nucleus, we could continue in this way, or we could spread the sources out over more lattice points than there are nucleons. It seems more reasonable to follow the former course, according to which an assembly of n_N nucleons has a zero-order energy which is proportional to $(N - n_N)\lambda_0(0) + n_N\lambda_0(\gamma)$, regardless of how they are located so long as they are not crowded together onto fewer lattice points than there are nucleons.

If now the nucleon density is increased, so that the n_N nucleons are crowded into n_l lattice points ($n_l \leq n_N$), the energy will be proportional to

$$(N - n_l)\lambda_0(0) + n_l\lambda_0(\zeta\gamma), \quad \zeta \equiv n_N/n_l. \quad (54)$$

The potential energy per nucleon is proportional to the difference between Eq. (54) and the value of Eq. (54) with $\zeta=1$, divided by n_N :

$$\frac{1}{\zeta} [\lambda_0(\zeta\gamma) - \lambda_0(0)] - [\lambda_0(\gamma) - \lambda_0(0)]. \quad (55)$$

For small values of $\zeta\gamma$, Eq. (50) shows that this potential energy per nucleon is negative (attractive) and a linear function of the nucleon density ζ . For large values of $\zeta\gamma$, Eq. (51) shows that Eq. (55) is a linear function of $-\zeta^{\frac{1}{3}}$. Figure 4 is a plot of $-\lambda_0(\gamma) - \lambda_0(0)$ against γ , that can be used to evaluate Eq. (55) in the general case.

The stability of nuclear matter and the saturation of nuclear forces can now be discussed qualitatively, as was done earlier with the help of classical field theory.¹⁷ For lack of a more consistent mode of description, we say that the kinetic energy per nucleon is proportional to the $\frac{2}{3}$ power of the density or of ζ ; this regards the nucleons as described by wave functions in a con-

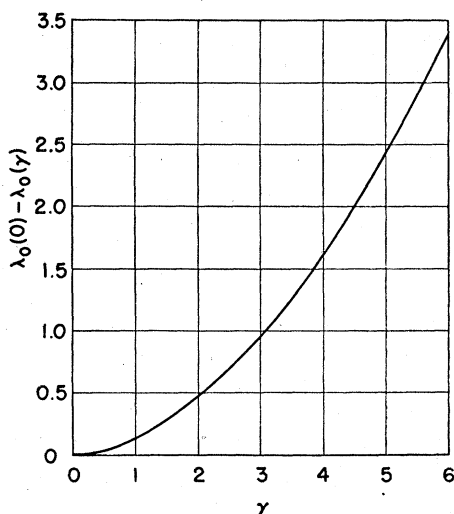


FIG. 4. Curve from which Eq. (55), which is related to the saturation of nuclear forces, can be computed.

¹⁷ See I, Sec. II.

tinuum, not a lattice space. Then stability is attained only when the density becomes so great that the $\frac{1}{3}$ -power law for the potential energy dominates. This evidently occurs only for nucleon densities somewhat greater than $1/l^3$. In this domain, the lattice-space quantization and the classical theory¹⁷ give similar results for any power of nonlinearity (not only for the fourth power considered above), since the derivation of the leading terms in Eq. (51) is based on the WKB method and hence is essentially classical.

It is also worth noting that a qualitatively similar result would have been obtained if we had permitted each nucleon to be spread out initially over several lattice points. Collapse would again occur down to a density somewhat greater than $1/l^3$.

In the classical theory, the interaction between two nucleons embedded in nuclear matter was found to be weaker than that in empty space.¹⁸ The quantity plotted in Fig. 3, which is proportional to the negative of the interaction energy in empty space, must be replaced in nuclear matter by $2\lambda_0(\zeta\gamma + \gamma) - [\lambda_0(\zeta\gamma) + \lambda_0(\zeta\gamma + 2\gamma)]$. This quantity decreases as the curvature of a plot of $-\lambda_0(\zeta\gamma)$ against $\zeta\gamma$ decreases, that is, as ζ increases, since the curve in Fig. 4 straightens out as the abscissa increases. Thus, as in the classical theory, the two-nucleon interaction decreases as the density of nucleons increases.

Interaction of Mesons with Nuclear Matter

In the last subsection, the ground-state energy of a distribution of classical sources was considered. A study of the first excited state of this system yields information on the interaction of mesons with nuclear matter, and provides an extension of the meson-nucleon scattering calculation that was given earlier in this section.

It is apparent that for an arbitrary distribution of sources, \mathbf{K} is not even approximately a constant of the motion, so that it is disadvantageous to use wave functions of the type (14). Instead, we take for our Tamm-Dancoff-type wave function

$$\Psi = \sum_s b(s) U_1(s),$$

and require that

$$\int \cdots \int U_1^*(t) (\mathcal{H} - E) \Psi d\phi_1 \cdots d\phi_N = 0$$

for all t . This leads to the following equation:

$$\begin{aligned} \{ (\epsilon_{t1} - \epsilon_{t0}) - P(\phi_t)_{01}^2 + [(\phi_t)_{11} - (\phi_t)_{00}] \\ \times \sum_s' A_{st}(\phi_s)_{00} - E_0 \} b(t) \\ + (\phi_t)_{01} \sum_s' A_{st}(\phi_s)_{01} b(s) = 0, \quad (56) \end{aligned}$$

$$E_0 \equiv E - \sum_s \epsilon_{s0} - \frac{1}{2} \sum_s \sum_q' A_{sq}(\phi_s)_{00}(\phi_q)_{00}.$$

¹⁸ See I, Sec. IX.

E_0 is the amount by which the energy E of the first excited state under consideration exceeds the first-order ground-state energy.

If the source strength g_s is different from zero at several lattice points, Eq. (56) will be very difficult to solve. However, if g_s changes little between neighboring lattice points, we can assume that there exist solutions for which $b(s)$ is also slowly varying. It is then convenient to define a function $\chi(\mathbf{r}_s) \equiv (\phi_s)_{01} b(s)$, and make use of the easily established relation

$$\sum_s A_{st} \chi(\mathbf{r}_t) \xrightarrow{l \rightarrow 0} -[\nabla^2 \chi(\mathbf{r})]_{\mathbf{r}=\mathbf{r}_s}. \quad (57)$$

This does not mean that we actually take the limit $l \rightarrow 0$, but rather that we assume that $\chi(\mathbf{r}_t)$ varies sufficiently slowly from one lattice point to the next so that we can regard it as a continuous function of the lattice-point coordinates. We are thus led to the approximate wave equation

$$\begin{aligned} -(\phi_t)_{01} \nabla^2 \chi(\mathbf{r}_t) + V(\mathbf{r}_t) \chi(\mathbf{r}_t) &= E_0 \chi(\mathbf{r}_t), \\ V(\mathbf{r}_t) &\equiv (\epsilon_{t1} - \epsilon_{t0}) - P(\phi_t)_{01}^2 \\ &\quad + [(\phi_t)_{11} - (\phi_t)_{00}] \sum_s' A_{st} (\phi_s)_{00}. \end{aligned} \quad (58)$$

A simple example of Eq. (58) is that in which all g_s are zero (single free meson). Then $(\phi_t)_{11} = (\phi_t)_{00} = 0$, and $V(\mathbf{r}_t) = \epsilon_{t1} - \epsilon_{t0} - P\phi_{01}^2 = \text{constant}$. There are plane wave solutions $\chi(\mathbf{r}_t) = C \exp i(\mathbf{K} \cdot \mathbf{r}_t)$, where

$$E_0 = \epsilon_1 - \epsilon_0 + (K^2 - P)\phi_{01}^2,$$

in agreement with Eqs. (14) and (15). The point at which the limiting procedure (57) shows up as an approximation is that in the present analysis \mathbf{K} can be any vector, whereas for a correct solution \mathbf{K} must be one of the reciprocal lattice vectors. So long as \mathbf{K} is small in comparison with $1/l$, χ is actually slowly varying as assumed in Eq. (57), and it makes little difference whether or not \mathbf{K}_{\perp} is precisely a reciprocal lattice vector.

Suppose now that g_s is different from zero only within a restricted region of space, which corresponds to a nucleus. Then outside the nucleus, χ may be built up out of solutions of the free-particle wave equation with definite values for the energy and the magnitude \mathbf{K} of the propagation vector. Inside the nucleus, Eq. (58) is the nonrelativistic Schrödinger equation for the motion of a particle with potential energy $V(\mathbf{r}_t)$, position-dependent rest mass $\frac{1}{2}(\phi_t)_{01}^2 \cong \epsilon_{t1} - \epsilon_{t0}$, and total energy E_0 . For large α , the dominant part of $V(\mathbf{r}_t)$ is $\epsilon_{t1} - \epsilon_{t0}$, which is larger inside the nucleus than it is outside (see Fig. 2). Thus the nucleus acts as a repulsive potential for mesons incident from without. This confirms the result obtained earlier from classical field theory.¹⁹ As pointed out there, a consequence of this result is that production of mesons in heavy nuclei should be pri-

marily a surface effect, since the wave function of the created meson does not penetrate far into the nucleus.

VI. CONCLUDING REMARKS

The method of quantization employed in this paper is a natural one to use whenever the gradient terms in the field Hamiltonian can be treated as a perturbation and there is no immediate requirement of covariance. Then the continuum solutions are expected to be limiting cases of the lattice solutions as $l \rightarrow 0$, and the fact that solutions cease to exist in this limit must be regarded as a strong indication that they do not exist at all for the continuum field Hamiltonian. This has indeed been the presumption that underlies most of the recent work in quantum field theory; meaningful results can be obtained only by a renormalization procedure, using covariance as a guide. In the present case, covariance cannot be used in this way; nevertheless, a quasi-relativistic solution for the motion of a single free particle has been obtained.

It is tempting to regard the introduction of the lattice constant l as analogous to the renormalization procedure, in that it enables one to extract finite results from an otherwise divergent theory. It would then follow that l is a physically real quantity, and not just a mathematical artifice that is to be eliminated at as early a stage as possible. Unfortunately for this point of view, Fig. 1 shows that for $\mu=0$ and M equal to the rest mass of a meson, either l must be larger than is physically reasonable or α must be so small that the perturbation theory cannot be employed. This last may not be an obstacle if the results of Secs. IV and V can be given the quasi-relativistic form of the results of Sec III.

It is possible in principle to apply lattice-space quantization to coupled meson-nucleon fields. The gradient term in the Dirac Hamiltonian for the nucleon field is then included as part of the perturbation, and the nucleon rest mass and coupling term (when the latter is of point type) are included in the unperturbed Hamiltonian. The neutral scalar nonlinear meson theory with scalar coupling differs in only minor respects from the classical source theory considered in Sec. V. The principal difference is that g_s is limited to an integer multiple m of some basic value, where m ranges from -4 to $+4$; the number 4 appears because there are two kinds of Dirac particles (neutrons and protons) each with two spin states. For pseudoscalar particles with pseudoscalar coupling, the calculation is far more complicated because the different components of the Dirac fields are coupled together. It is then necessary to solve a set of simultaneous differential equations at each lattice point. Some aspects of this situation are being explored.

The writer takes pleasure in expressing his appreciation to Dr. D. R. Yennie for several helpful and illuminating conversations.

¹⁹ See I, Sec. XI.