Evaluation of the Interaction Effect in *n-p* Capture

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It is found possible to establish by theoretical arguments that the recently measured thermal *n-p* capture cross section cannot all be ascribed to the neutron and proton magnetic moments, but that about 8 percent ±5 percent must be contributed by an "interaction" magnetic dipole moment.

I. INTRODUCTION

IT is generally recognized that when a nuclear system is in interaction with the electromagnetic field the nucleons do not merely act as individuals, possessing charges and magnetic moments, but rather their electromagnetic interaction is also somewhat influenced by their nuclear interaction. This influence has been named "interaction effect" (although also called "exchange effect"), because of its association with the mechanism of nuclear interaction, and it is expected to be important only for magnetic multipole radiation. Perhaps the best case from which to obtain some quantitative information about interaction effects is the well-known magnetic dipole transition in *n-p* capture. Here we would look for some "interaction" magnetic dipole moment, in addition to the neutron and proton moments. The thermal capture cross section is considered in this paper.

Until this year, considerations of *n-p* capture[1,2] were unable to draw conclusions from the experimental numbers which have been available. Accurate thermal *n-p* capture measurements have since been made,[3,4] supplementing those of Whitehouse and Graham,[5] and the *n-p* singlet effective range has been measured in precision scattering experiments.[6] With these new data the situation has been altered to the extent that the chief remaining uncertainties are now of theoretical origin.

The present paper was originally written as an analysis of the result of Hamermesh *et al.*[3] Harris *et al.*,[4] in a brief analysis of their own experiment, have since announced the existence of an interaction moment contribution in the capture cross section. Their analysis is based on the effective range discussion of Bethe and Longmire.[7] In actuality the *n-p* capture matrix element (assuming no interaction moment) receives such large contributions from regions within the range of nuclear forces that the precision to be expected of the Bethe-Longmire result is by no means obvious. The aim of the present paper is to present a careful study of the sensitivity of the *n-p* capture matrix element to whatever arbitrary assumptions one might make about the wave functions.

It is found that the limitations of our knowledge of the wave functions lead to an uncertainty of about $1\frac{1}{2}$ percent in the matrix element. In particular, it is very unlikely that the wave functions can be of such a sort that the matrix element is much bigger, for example, than for Hulthén functions. Thus, the existence and sign of the interaction effect are certain, although the magnitude is not known very well. The sign is that which was expected from studies[1,2] of the three-body moment anomaly, and the magnitude also is roughly as expected. Numerically, the effect is 8±5 percent of the cross section.

Interestingly, the new values for σ_c(exp) are just about 8 percent larger than the old values. This 8 percent is mostly a consequence of a recalibration[8] of the boron capture cross section, with which that of hydrogen is compared.

II. FORMAL ASPECTS OF THE CALCULATION

If there is no interaction moment contribution, the total *n-p* capture cross section is given by

$$\sigma_c=(\pi/2)(e^2/Mc^2)(\omega^3/k^3c^3)(\mu_N-\mu_P)^2\left(\int_0^\infty u_g u_s dr\right)^2. \quad (1)$$

Here k is the wave number for the incident system in center-of-mass coordinates, having the value 1.743×10^8 cm^{-1} at the standard neutron velocity, 2200 m/sec. For thermal energies the frequency of the emitted light is $\omega=\epsilon/\hbar$, ϵ being the deuteron binding energy. The wave functions u_g and u_s are the usual "radial" wave functions for the ground state and for the continuum singlet s state, respectively. Here u_g is normalized such that $\int_0^\infty(u_g^2+w_g^2)dr=1$, where w_g is the corresponding D state function. The function u_s is normalized so as to go asymptotically to $\sin(kr+\delta_s)$ as r becomes very large.

The deuteron D state plays no role in Eq. (1). Indeed, the analysis which completely neglects the D state is almost exact, for the reduced amplitude which u_g possesses by virtue of the presence of the D state is

[1] N. Austern and R. G. Sachs, Phys. Rev. 81, 710 (1951).
[2] E. P. Gray, thesis, Cornell University, 1952 (unpublished).
[3] Hamermesh, Ringo, and Wexler, Phys Rev. 90, 603 (1953).
[4] Harris, Muehlhause, Rose, Schroeder, Thomas, and Wexler, Phys. Rev. 91, 125 (1953).
[5] W. J. Whitehouse and G. A. R. Graham, Can. J. Research A25, 261 (1947)
[6] R. K. Adair (to be published). Hafner, Hornyak, Falk, Snow, and Coor, Phys. Rev. 89, 204 (1953). The singlet range value which is used here is essentially that of Adair, and is fairly close to the *p-p* range.
[7] H. A. Bethe and C. Longmire, Phys. Rev. 77, 647 (1950).
[8] Unpublished Argonne National Laboratory data.

counterbalanced by a corresponding correction to the triplet effective range. However, there is some D state influence on the shape of u_g, even if not on the amplitude, and this enters in the subsequent discussion. The rudiments of effective range theory with tensor forces are presented in the Appendix.

It is convenient to define the functions U_g, W_g, U_s, $\mathcal{U}_g$, $\mathcal{W}_g$, $\mathcal{U}_s$. Here U_g, W_g, and U_s have the same shapes as u_g, w_g, and u_s, only being normalized differently. The relative normalization is given by $N_g=u_g/U_g=w_g/W_g$, so that N_g satisfies

$$N_g^{-2}=\int_0^\infty (U_g^2+W_g^2)dr. \tag{2}$$

The functions $\mathcal{U}_g$, $\mathcal{W}_g$, and $\mathcal{U}_s$ agree asymptotically with U_g, W_g, and U_s, but satisfy differential equations in which the nuclear interaction and centrifugal repulsion terms have been dropped. The normalizations of $\mathcal{U}_g$ and $\mathcal{U}_s$, hence of U_g, W_g, and U_s, are given by $\mathcal{U}_g(0)=\mathcal{U}_s(0)=1$. Specific forms for the asymptotic functions are

$$\mathcal{U}_g=e^{-\gamma r}, \tag{3a}$$

$$\mathcal{U}_s=\sin(kr+\delta_s)/\sin\delta_s\approx 1-r/a_s, \tag{3b}$$

$$\mathcal{W}_g=\eta e^{-\gamma r}. \tag{3c}$$

Here $\hbar^2\gamma^2/M=\epsilon$, the deuteron binding energy, and η is a parameter which is both defined and utilized only in the Appendix. In terms of U_g and U_s, Eq. (1) becomes

$$\sigma_c=(\pi/2k)(e^2/Mc^2)(\omega/c)^3(\mu_N-\mu_P)^2N_g^2a_s^2|\mathcal{M}|^2, \tag{4}$$

where

$$\mathcal{M}\equiv\int_0^\infty U_gU_sdr, \tag{5}$$

and a_s is the zero-energy singlet scattering length.

The calculation of N_g is given in terms of the effective range as

$$N_g^{-2}=1/2\gamma-\tfrac{1}{2}\rho_t(-\epsilon,-\epsilon),$$

where

$$\rho_t(E_1,E_2)=2\int_0^\infty (\mathcal{U}_{g1}\mathcal{U}_{g2}-U_{g1}U_{g2}-W_{g1}W_{g2})dr. \tag{A7}$$

As usual the quantity which is found experimentally is $\rho_t(0,-\epsilon)$, so some energy correction is needed in order to compute N_g. This correction is uncertain. It is customarily made with the aid of the "shape dependent parameter" of the effective range theory and amounts to about 1 percent of ρ_t. (Only in the Yukawa potential analysis has the correction been found to be much larger, and Biedenharn has shown this large result to be a spurious consequence of the neglect of tensor forces.[9]) Consistent with the spirit of the present calculation, which does not directly concern itself with the nuclear forces, it is only possible to ignore the energy variation of ρ_t and to tolerate the resulting uncertainty in N_g.

[9] Private communication from J. Blatt Doctoral Thesis, Massachusetts Institute of Technology (unpublished). Other calculations also suggest that tensor forces tend to reduce the Yukawa "shape dependent parameter," although they do not seem to show so complete a reduction as is stated above. For the present calculation, however, this is not a major source of error.

Aside from the calculation of N_g, the principal use made of the effective ranges will be in obtaining limitations on the forms of the wave functions, U_g and U_s. In each case the technique is to assume a form for the wave function, leaving one parameter undetermined, then to choose this parameter so that the wave function leads to a correct value for the effective range. For U_s the procedure is especially simple:

$$\rho_s(0,0)=2\int_0^\infty (\mathcal{U}_s^2-U_s^2)dr. \tag{6}$$

For U_g the D state introduces some complication. We have

$$\rho_t(-\epsilon,-\epsilon)=2\int_0^\infty (\mathcal{U}_g^2-U_g^2-W_g^2)dr. \tag{A7}$$

Then

$$\int_0^\infty W_g^2dr=N_g^{-2}\int_0^\infty w_g^2dr=N_g^{-2}P_D, \tag{7}$$

where $P_D=\int_0^\infty w_g^2dr$ is the quantity which is commonly called the "percent of D state" in the deuteron. In terms of P_D we can define

$$\rho_t'\equiv\rho_t(-\epsilon,-\epsilon)+2N_g^{-2}P_D, \tag{8}$$

and base the further calculations on the relation

$$\rho_t'=2\int_0^\infty (\mathcal{U}_g^2-U_g^2)dr. \tag{9}$$

The quantity ρ_t', the effective range for the s state alone, is uncertain to just that extent that the percent D state is unknown. The uncertainty in the percent D state will lead to an uncertainty in the capture cross section which will be seen to be of about the same size as the other major uncertainties. A reasonable estimate for P_D is $P_D=0.04\pm0.02$.

A tabulation of relevant numbers is given in Table I.

TABLE I. Relevant data.

	Ref. No.
$\sigma_c=0.329\pm0.0046$ at 2200 m/sec	a
$\epsilon=2.225$ Mev, $\gamma=0.2315\times10^{13}$ cm^{-1}	b
$k=1.743\times10^{8}$ cm^{-1}	
$(\mu_N-\mu_P)=-4.7054$	c
$a_s=-23.67\times10^{13}$ cm	b
$\rho_s=2.6\pm0.2\times10^{-13}$ cm	d
$\rho_t(0,-\epsilon)=1.703(1\pm0.02)\times10^{-13}$ cm	b
$\rho_t'=1.80\pm0.05\times10^{-13}$ cm	

[a] See references 3 and 4.
[b] E. E. Salpeter, unpublished tabulation of low-energy data.
[c] J. E. Mack, Revs. Modern Phys. 22, 64 (1950).
[d] See reference 6.

TABLE II. Repulsive core parameters [case (b)].

R	ξ	ζ	$\mathfrak{M}$
0.3	1.635	1.552	4.032
0.6	2.396	2.480	4.036
0.9	4.525	5.50	4.045
1.1	11.10	23.4	4.022

III. DETAILED CALCULATIONS

It might be supposed impossible to estimate the overlap integral, $\mathfrak{M}=\int_0^\infty U_gU_sdr$, to the accuracy required for our application, i.e., about 2 percent. The corresponding integral for zero-range wave functions, $\mathfrak{M}_0\equiv\int_0^\infty \mathfrak{U}_g\mathfrak{U}_sdr$ is easily computed and is seen to receive a 30 percent contribution from distances $r\leqslant 2\times 10^{-13}$ cm. Even for wave functions with the correct ranges the contribution from $r\leqslant 2\times 10^{-13}$ cm is still about 10 percent. Nevertheless $\mathfrak{M}$ can be estimated to the requisite accuracy, a result which is only suggested by the analysis of Bethe and Longmire. A particularly satisfying result for the present purpose is that the upper limit on $\mathfrak{M}$ is especially sharp and plausible.

A variational calculation for $\mathfrak{M}$ would be very pleasant, as the extrema found in such a way would be reliable. The effective ranges would appear as conditions on the variations. Unfortunately, we also want to limit the possible wave functions to the class of, what might be called, "reasonable" functions. This class seems not to have any very sharp mathematical definition. Essentially, a "reasonable" wave function is one for which the associated nuclear potential is reasonable. We can require, for example, that $U_g(0)=U_s(0)=0$; that U_g and U_s everywhere be bounded between zero and the asymptotic functions $\mathfrak{U}_g$ and $\mathfrak{U}_s$; that they go over monotonically to the asymptotic forms; that the associated potential have not too long a range. For a problem as ill-defined as this one, an empirical approach seems advised, even if the reliability of the answer is left uncertain.

Nearly all the functions which will be tested will satisfy all of the conditions just listed. The functions are made to satisfy Eqs. (6) and (9) and their overlap $\mathfrak{M}$ computed in each case. The choice of functions tested is designed to explore systematically the range between extreme localization at $r=0$, and extreme repulsion from $r=0$. Our requirement that the functions fit the effective ranges establishes a familiar connection. Namely, functions which are strongly localized at $r=0$ must become asymptotic more slowly than functions which are small near $r=0$, i.e., they must arise from "long-tailed" nuclear potentials.

TABLE III Damping parameters ζ and ξ as functions of α and β.

α	1	0.7	0.5	0.3	0
ζ	1.340	0.987	0.694	0.351	impossible
β	1	0.7	0.5	0.3	0
ξ	1.207	0.942	0.718	0.469	0

The cases which follow below are all fitted to $\rho_s=2.6\times 10^{-13}$ cm and $\rho_t'=1.8\times 10^{-13}$ cm, but with the effects of differential variations of the effective ranges being indicated in several typical cases. All the numbers below have dimensions 10^{-13} cm or 10^{13} cm^{-1}.

(a) *U_s and U_g of Hulthên shape*

$$U_s=1-r/a_s-e^{-\xi r},$$
$$U_g=e^{-\gamma r}-e^{-\zeta r},$$
$$\xi=1.207,\quad \zeta=1.340,$$
$$\mathfrak{M}=4.031[1-0.036d\rho_s-0.102d\rho_t'].$$

(b) *Hard core types*

$$U_g=e^{-\gamma r}[1-e^{-\xi(r-R)}],\quad \text{for } r>R$$
$$=0,\quad \text{for } r<R$$
$$U_s=(1-r/a_s)[1-e^{-\xi(r-R)}],\quad \text{for } r>R$$
$$=0,\quad \text{for } r<R.$$

TABLE IV. $\mathfrak{M}$ as a function of α and β.

$\beta\diagdown\alpha$	1	0.7	0.5	0.3
1	4.031	3.987	3.908	3.643
0.7	4.046	4.021	3.958	3.712
0.5	4.050	4.040	3.990	3.759
0.3	4.072	4.080	4.041	3.828
0	4.334	4.364	4.340	4.146

The values of ξ, ζ, and $\mathfrak{M}$ for this case are listed in Table II as functions of R.

For the case $R=0.6$ we also have

$$\mathfrak{M}=4.036[1-0.093d\rho_t'-0.042d\rho_s].$$

(c) *"Maximum" hard core*

$$U_g=e^{-\gamma r},\quad \text{for } r>b_t$$
$$=0,\quad \text{for } r<b_t$$
$$U_s=(1-r/a_s),\quad \text{for } r>b_s$$
$$=0,\quad \text{for } r<b_s.$$

Interestingly, the quantities b_s and b_t are roughly equal.

$$b_s=1.235,\quad b_t=1.161.$$

To first approximation $\mathfrak{M}$ is independent of b_t here, hence of ρ_t'.

$$\mathfrak{M}=3.99[1-0.089d\rho_s].$$

(d) *U_g Hulthên—U_s maximum core*

$$\mathfrak{M}=3.85.$$

(e) *U_g maximum core—U_s Hulthên*

$$\mathfrak{M}=3.93.$$

(f) *Long-tailed functions*

The class of functions considered here has the property that the functions become asymptotic rather more slowly than the ones considered above, so they must be closer to asymptotic in the region near $r=0$. An interesting analytical realization of the class envisaged can be achieved with a modified Hulthén type, where we do not require that $U_g(0)$ and $U_s(0)$ be zero. Thus, we now consider

$$U_g=e^{-\gamma r}-\alpha e^{-\zeta r},$$

$$U_s=1-r/a_s-\beta e^{-\xi r},$$

and $\mathfrak{M}$ can be studied as a function of α and β. Values of the damping parameters ζ and ξ, and the overlap integral $\mathfrak{M}$ are given in Tables III and IV.

For the case $\beta=1$, $\alpha=0.5$, we also have

$$\mathfrak{M}=3.908[1-0.145d\rho_t'-0.037d\rho_s].$$

(g) *U_g long tail—U_s max core*

The case considered here is $\alpha=0.5$.

$$\mathfrak{M}=3.66.$$

(h) *U_g max core—U_s long tail*

Here $\beta=0.5$, $\mathfrak{M}=3.89$.

It was found above that it is only with difficulty, for the extreme cases of the long tailed functions, that $\mathfrak{M}$ can be made much bigger than about 4.05. The long-tailed functions, $\beta=0$ and 0.3 and $\alpha=0.3$, are limiting cases which are of interest chiefly because they show how unreasonable the wave functions have to become before $\mathfrak{M}$ can be made very large. In these cases the wave functions actually approach asymptotic so slowly that they would imply that a significant part of the n-p interaction would have a range of the order of or greater than the deuteron radius.

The reliability of the upper limit of $\mathfrak{M}$ can be made more plausible by considering a reordering of $\mathfrak{M}$, according to the method of Bethe and Longmire.[7] They find

$$\mathfrak{M}=\mathfrak{M}_0-\tfrac{1}{4}(\rho_s+\rho_t')+C, \qquad (10)$$

where

$$C\equiv\tfrac{1}{2}\int_0^\infty[(\mathcal{U}_g-\mathcal{U}_s)^2-(U_g-U_s)^2]dr. \qquad (11)$$

Their approximation for $\mathfrak{M}$ is obtained if $C=0$. Now C goes positive only if the first term of the integrand dominates the second. But also, to make C positive and large, the dominating must occur in a region where the first term is large enough to contribute strongly. Thus, it is interesting to study the function

$$C_0(R)\equiv\tfrac{1}{2}\int_0^R(\mathcal{U}_g-\mathcal{U}_s)^2dr. \qquad (12)$$

This function is shown in the accompanying graph (Fig. 1). Clearly, to obtain $C\gtrsim 0.15$, say, it is necessary that U_g and/or U_s differ considerably from asymptotic for distances $r\gtrsim 3\times10^{-13}$ cm. Thus, the long tailed functions do provide a satisfactory study of large $\mathfrak{M}$.

IV. CONCLUSIONS

The theoretical value for σ_c is obtained merely by substituting $\mathfrak{M}$ into Eq. (4). Suppose we take $\mathfrak{M}$ for the Hulthén case. Then

$$\sigma_c(\text{theor})=0.303\text{ b}.$$

Experimentally, σ_c is about

$$\sigma_c(\text{exp})=0.330\text{ b}.$$

The difference between these two numbers, 0.027 b, is attributed to the interaction moment. It is 8 percent of 0.330 b.

Now the work in the preceding section shows that it takes quite unreasonable wave functions to make $\mathfrak{M}$ much larger than the value obtained for the Hulthén

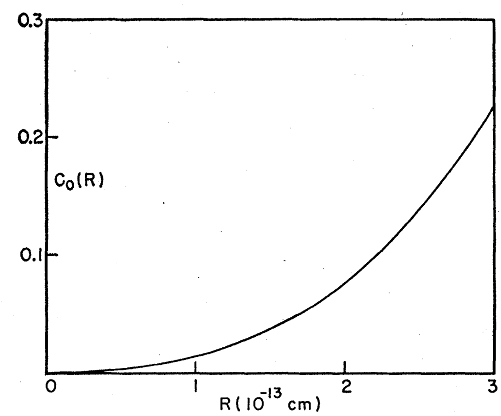

FIG. 1. The function $C_0(R)$.

case. To fit experiment, $\mathfrak{M}$ would actually have to be as large as $\mathfrak{M}$("exp")$=4.20$, well above the range of values given by reasonable functions. For the Hulthén case, $\mathfrak{M}$ lies near the upper end of the range of reasonable values. Although certain of the repulsive core cases do give much smaller $\mathfrak{M}$ values, these are mostly interesting as indicating that U_g and U_s are probably not radically different in shape, as we do expect the interaction effect to be fairly small. (Thus, the calculation suggests as an incidental result that the assumption of a large respulsive core interaction in one of the s states probably requires a similar core for the other.) Altogether, from the set of calculations which are listed,[12] it seems plausible to claim about ±1–2 percent as the accuracy with which $\mathfrak{M}$ can be predicted, exclusive of variations of the effective ranges, and to take the Hulthén value as central.

The various errors which must be considered are (a) σ_c(exp) is ±1 percent; (b) $\mathfrak{M}$ is uncertain to ±1–2

[12] Additional calculations along the lines of those reported here were performed by W. A. Newcomb and E. E. Salpeter; see Salpeter, Phys. Rev. **82**, 60 (1951). Using the measured singlet effective range, their formulas for Yukawa, Gauss, and exponential potentials give results in complete agreement with the above material.

percent, by virtue of our not knowing the wave functions, so σ_c(theor) is uncertain to about ± 3 percent; (c) because ρ_s is uncertain by $\pm 0.2\times 10^{-13}$ cm, $\mathfrak{M}$ is uncertain by about 1 percent, so σ_c(theor) by ± 2 percent; (d) because P_D is uncertain by about ± 2 percent, ρ_t' is uncertain by $\pm 0.05\times 10^{-13}$ cm, so σ_c(theoret.) by about ± 2 percent; (e) ρ_t itself is uncertain, leading to about a ± 2 percent uncertainty in σ_c(theor). It is apparent that these uncertainties can be made to add up, and even to cancel the interaction moment effect, but the errors seem genuinely independent, the square root of the sum of their squares being 5 percent, so the effect would seem to be quite real.

With regard to the uncertainty in ρ_t, listed above as (e), we have both the energy correction from $\rho_t(0,-\epsilon)$ to $\rho_t(-\epsilon,-\epsilon)$, and also the experimental uncertainty in $\rho_t(0,-\epsilon)$. These enter into σ_c(theoret.) both through N_g^{-2} and through $\mathfrak{M}$, these two appearances partially canceling. For the above this cancellation has been considered as about 50 percent complete.

I am grateful to R. Margulies for some assistance with the numerical work and to J. Blatt, E. E. Salpeter, and P. Morrison for useful discussions.*

APPENDIX

Effective Range Theory with Tensor Forces

Two eigenwaves[13] of the scattering matrix are α, γ and we label the S and D functions as $U^{\alpha,\gamma}$ and $W^{\alpha,\gamma}$. These eigenwaves have the property that U^α and W^α have the same phase shift δ^α, similarly for state γ. Define $\eta^{\alpha,\gamma}$ as $\eta^{\alpha,\gamma}=\lim_{r\to\infty}(W^{\alpha,\gamma}/U^{\alpha,\gamma})$. For state α we assume little D admixture, so this is the state which scatters strongly. We know η^α is very small, and η^α goes to zero as the energy goes to zero. Since $\eta^\alpha\eta^\gamma=-1$, state γ is mostly D wave, and scatters weakly. State γ can thus be ignored. The following discussion only applies to state α, so the superscript can be dropped.

Following the method of Bethe,[14] the coupled Schrödinger equations are written for two different energies, multiplied by appropriate functions, and added judiciously.

$$U'\left\{\frac{\partial^2 U}{\partial r^2}+\frac{M}{\hbar^2}[(E+V_c)U+2\sqrt{2}V_T W]\right\}=0,$$

$$W'\left\{\frac{\partial^2 W}{\partial r^2}-\frac{6W}{r^2}+\frac{M}{\hbar^2}[(E+V_c-2V_T)W+2\sqrt{2}V_T U]\right\}=0,$$

$$U\left\{\frac{\partial^2 U'}{\partial r^2}+\frac{M}{\hbar^2}[(E'+V_c)U'+2\sqrt{2}V_T W']\right\}=0,$$

$$W\left\{\frac{\partial^2 W'}{\partial r^2}-\frac{6W'}{r^2}+\frac{M}{\hbar^2}[(E'+V_c-2V_T)W'+2\sqrt{2}V_T U']\right\}=0.$$

Therefore,

$$U'\frac{\partial^2 U}{\partial r^2}+W'\frac{\partial^2 W}{\partial r^2}-U\frac{\partial^2 U'}{\partial r^2}-W\frac{\partial^2 W'}{\partial r^2}=(k'^2-k^2)(UU'+WW'). \quad \text{(A1)}$$

Now consider the comparison functions $\mathfrak{U}$, $\mathfrak{W}$.

$$U\underset{r=\infty}{\longrightarrow}\mathfrak{U},\quad W\underset{r=\infty}{\longrightarrow}\mathfrak{W}.$$

Including normalizations, the explicit forms for $\mathfrak{U}$ and $\mathfrak{W}$ are

$$\mathfrak{U}=\sin(kr+\delta)/\sin\delta,\quad \mathfrak{W}=\eta\sin(kr+\delta)/\sin\delta. \quad \text{(A2)}$$

Then the equation corresponding to (A1) is

$$\mathfrak{U}'\frac{\partial^2\mathfrak{U}}{\partial r^2}+\mathfrak{W}'\frac{\partial^2\mathfrak{W}}{\partial r^2}-\mathfrak{U}\frac{\partial^2\mathfrak{U}'}{\partial r^2}-\mathfrak{W}\frac{\partial^2\mathfrak{W}'}{\partial r^2}=(k'^2-k^2)(\mathfrak{U}\mathfrak{U}'+\mathfrak{W}\mathfrak{W}'). \quad \text{(A3)}$$

Subtraction of (A1) from (A3) and integration yields the result

$$(k'^2-k^2)\int_0^\infty(\mathfrak{U}\mathfrak{U}'+\mathfrak{W}\mathfrak{W}'-UU'-WW')dr=-\left[\mathfrak{U}'\frac{\partial\mathfrak{U}}{\partial r}+\mathfrak{W}'\frac{\partial\mathfrak{W}}{\partial r}-\mathfrak{U}\frac{\partial\mathfrak{U}'}{\partial r}-\mathfrak{W}\frac{\partial\mathfrak{W}'}{\partial r}\right]_{r=0}=-(1+\eta\eta')(k\cot\delta-k'\cot\delta').$$

Thus,

$$k\cot\delta-k'\cot\delta'=\frac{k^2-k'^2}{1+\eta\eta'}\int_0^\infty[(1+\eta\eta')\mathfrak{U}\mathfrak{U}'-UU'-WW']dr. \quad \text{(A4)}$$

Suppose E' is taken to be $E'=0$. Then $\eta'=0$, and

$$k\cot\delta=-1/a+\tfrac{1}{2}k^2\rho(0,E), \quad \text{(A5)}$$

where

$$\rho(0,E)=2\int_0^\infty(\mathfrak{U}\mathfrak{U}_0-UU_0-WW_0), \quad \text{(A6)}$$

and a is the scattering length.

Thus, η drops out when the effective range theory is referred to zero energy. Inasmuch as only the effective range $\rho(0,E)$ has been defined so far, we are still free to extend the definition and to set

$$\rho(E_1,E_2)\equiv 2\int_0^\infty dr(\mathfrak{U}_1\mathfrak{U}_2-U_1U_2-W_1W_2). \quad \text{(A7)}$$

This is not the integral which appears in (A4), but it is just as well not to have $\rho(E_1,E_2)$ cluttered up with η's.

* *Note added in proof.*—One further measurement of σ_c(exp) has since been published. G. von Dardel and A. W. Waltner [Phys. Rev. **91**, 1284 (1953)] give the value 0.321 ± 0.005 barns, which implies a 6 percent interaction moment contribution.

[13] For details of this formalism, consult F. Rohrlich and J. Eisenstein, Phys. Rev. **75**, 705 (1949); or J. Schwinger, lectures on nuclear physics.

[14] H. A. Bethe, Phys. Rev. **76**, 38 (1949).