Following Fock,[2] let us denote the operator $i\hbar c\sum_i \alpha_i \partial/\partial x_i$ by L, and its quantum-mechanical average by L_0. Then from Eq. (6)

$$-L_0 = -i\hbar c \int \sum_i \frac{\partial}{\partial x_i}\left[\psi^* \alpha_i \left(\sum_j x_j \frac{\partial \psi}{\partial x_j}\right)\right] d\tau + \langle \sum_j x_j F_j \rangle. \tag{7}$$

For the case of a hydrogen atom, for example, ψ vanishes at infinity, the first term on the right of (7) is zero, and we arrive at Fock's result. However, in some cases of physical interest other boundary conditions arise. Einbinder,[7] for example, has considered a system of free electrons enclosed in a finite volume. He has applied Fock's result to this case, arguing that the virial of the pressure p is $-3pv$, where v is the volume. In fact, however, Fock's result is not applicable to this problem, whereas Eq. (7) can be used. The boundary conditions in this case are the periodicity conditions, and one can evaluate the first term on the right of (7) using the explicit wave functions of Darwin,[8] representing an electron confined in a volume v. The result is simply $-3pv$, thus verifying Einbinder's form of the theorem.

In this case, then, it appears that the first term on the right of (7) can be interpreted as the contribution to the virial arising from the forces represented through the boundary conditions rather than through an explicit potential in the Dirac equation.

It would be necessary to consider the first term on the right of (7) when dealing with the problem of an electron in a spherical hole in the manner of Broch,[9] and it is also clear from (7) why the result of Rose and Welton[5] cannot be applied to continuum states.

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Relativistic Correction to the Lamb Shift*†

M. BARANGER‡ AND H. A. BETHE, *Cornell University, Ithaca, New York*

AND

R. P. FEYNMAN, *California Institute of Technology, Pasadena, California*

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The relativistic corrections to the Lamb shift, i.e., terms of order $\alpha(Z\alpha)^5 mc^2$, are calculated. For this purpose, the Lamb shift is separated into one term in which the Coulomb potential acts only once, and another term in which it acts two or more times (Sec. II). The one-potential term is shown to be equal to the expression calculated in previous papers except for corrections of order $\alpha(Z\alpha)^6$ (Sec. III), and a method is given by which these corrections could be evaluated if desired (Appendix). The many-potential term can be separated into a nonrelativistic part which is again equal to the term calculated in previous papers, and a relativistic term which can be calculated by considering the intermediate states as free (Sec. IV). The calculation of the latter term which, of course, involves the Coulomb potential exactly twice, is described in Sec. V. A correction to the vacuum polarization term which is of the same order, is evaluated in Sec. VI.

The result for the relativistic correction is 7.13 Mc/sec, and is in agreement with the result of Karplus, Klein, and Schwinger which was obtained by an independent method. The result for the complete Lamb shift has been given in a recent paper by Salpeter. The small remaining discrepancy of 0.6 Mc/sec between theory and experiment might be due to the next order relativistic correction which should be of order $\alpha(Z\alpha)^6 \ln(Z\alpha)$.

I. INTRODUCTION

SINCE the first accurate measurement by Lamb and Retherford[1] of the displacement of the $2S$ level of the hydrogen atom now known as the "Lamb shift," theoretical calculations, based on quantum electrodynamics and using the mass renormalization procedure, have been given by several authors.[2–6] A simple way to obtain their result is to start from the operator for the radiative corrections to scattering[7] [F II, Eq. (22)], or more exactly its limit for small momentum

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‡ Now at the California Institute of Technology, Pasadena, California.

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transfer [F II, Eq. (24)],[8]

$$\frac{e^2}{4\pi}\left[\frac{1}{2m}(qV-Vq)+\frac{4q^2}{3m^2}V\left(\ln\frac{m}{\lambda}-\frac{3}{8}\right)\right], \qquad (1)$$

where q is the 4-momentum transferred by the potential V and λ a fictitious "photon mass." The level shift is simply the expectation value of this operator for the state in consideration. But there is an infrared catastrophy, i.e., the result becomes infinite when λ goes to 0. For this reason, λ is not put equal to 0, but the result is added to the nonrelativistic Lamb shift,[2]

$$\frac{2e^2}{3\pi m^2}\sum_n|\mathbf{p}_{n0}|^2(E_n-E_0)\ln\frac{K}{|E_n-E_0|}, \qquad (2)$$

which is certainly correct for very small photon energy. $\mathbf{p}_{n0}$ is the matrix element of the electron momentum between the state under study ψ_0 and another state ψ_n of the hydrogen atom. E_0 and E_n are the energies of these two states. K is an ultraviolet cutoff. French[4] has shown that the correspondence relation,

$$\ln\lambda=\ln(2K)-\tfrac{5}{6}, \qquad (3)$$

should be used. It is then found that λ or K disappears from the total. To terms (1) and (2) should still be added the so-called "vacuum polarization" term, considered here in Sec. VI. The numerical work given by Bethe, Brown, and Stehn,[9] and corrected by Salpeter[10] for effect of the finite mass of the nucleus, gives the result 1051.0 Mc/sec for the $2S_{\frac{1}{2}}-2P_{\frac{1}{2}}$ transition in hydrogen.

In the meantime, experimental accuracy has been much improved. The recent experiments of Triebwasser, Dayhoff, and Lamb[11] yielded 1057.77±0.10 Mc/sec for hydrogen and 1059.00±0.10 Mc/sec for deuterium, thus leaving room for more accurate calculations.

It can be seen [reference 2, Eq. (11)], that the nonrelativistic term (2) is of order $\alpha(Z\alpha)^4\ln(Z\alpha)$. The expectation value of the operator (1), called in the following the "one-potential Lamb shift," is of order $\alpha(Z\alpha)^4$. The purpose of this paper is to evaluate all corrections of order $Z\alpha$ with respect to the main term, that is terms of order $\alpha(Z\alpha)^5\ln(Z\alpha)$ or $\alpha(Z\alpha)^5$. Among all terms in α^6, these can be characterized as the one-photon part of the Lamb shift, represented by the diagrams of Fig. 1.

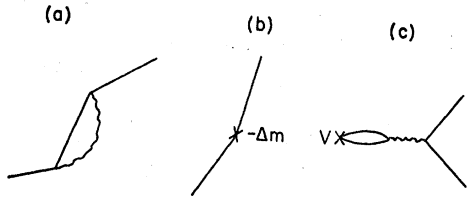

FIG. 1. Diagrams for the one-photon part of the Lamb shift. In these diagrams, the electrons should not be considered as free, but as bound electrons propagating in the field of the nucleus.

Terms of order $\alpha^2(Z\alpha)^4$, or two-photons parts, are not considered. Terms of order $\alpha(Z\alpha)^6$ or $\alpha(Z\alpha)^6\ln(Z\alpha)$ are neglected throughout. However, a method for evaluating some of them is given in the appendix.

The main part of the paper is concerned with diagrams 1(a) and 1(b). The vacuum polarization diagram 1(c) is evaluated at the end.

II. SEPARATION OF TERMS

The energy shift contributed by diagram 1(a) is easy to write down by a straightforward modification for a bound particle of the argument given in F II, p. 773.

$$\Delta E_a=e^2\int\bar\psi_0(\mathfrak{x}_2)\gamma_\mu K_+^V(\mathfrak{x}_2,\mathfrak{x}_1)\gamma_\mu\psi_0(\mathfrak{x}_1)\cdot\delta_+[(\mathfrak{x}_2-\mathfrak{x}_1)^2]d^3\mathbf{x}_2d^3\mathbf{x}_1d(t_2-t_1),$$

where $\psi_0(\mathfrak{x})$ is the electronic wave function, normalized by

$$\int\bar\psi_0(\mathfrak{x})\beta\psi_0(\mathfrak{x})d^3\mathbf{x}=\int\psi_0^*(\mathfrak{x})\psi_0(\mathfrak{x})d^3\mathbf{x}=1, \qquad (4)$$

$K_+^V(\mathfrak{x}_2,\mathfrak{x}_1)$ is the kernel for propagation of an electron bound in a hydrogen atom[12] [F I, Eq. (15)], δ_+ is defined in F II, and t is the time component of $\mathfrak{x}$.

Diagram 1(b) comes from the following cause: We use everywhere the propagation kernel K_{+1}^V, computed with the physical or measured mass m_1, and solution of the equation[13]

$$(i\nabla_2-V(\mathfrak{x}_2)-m_1)K_{+1}^V(\mathfrak{x}_2,\mathfrak{x}_1)=i\delta(\mathfrak{x}_2-\mathfrak{x}_1). \qquad (5)$$

But we really should use the kernel K_{+0}^V, computed with the bare mass $m_0=m_1-\Delta m$, and solution of the equation

$$(i\nabla_2-V(\mathfrak{x}_2)-m_1+\Delta m)K_{+0}^V(\mathfrak{x}_2,\mathfrak{x}_1)=i\delta(\mathfrak{x}_2-\mathfrak{x}_1).$$

A solution of the last equation correct to first order in Δm is

$$K_{+0}^V(\mathfrak{x}_2,\mathfrak{x}_1)=K_{+1}^V(\mathfrak{x}_2,\mathfrak{x}_1)+i\int K_{+1}^V(\mathfrak{x}_2,\mathfrak{x}_3)\Delta mK_{+1}^V(\mathfrak{x}_3,\mathfrak{x}_1)d^4\mathfrak{x}_3.$$

[8] Units and notations: $\hbar=c=1$. m is the electronic mass, e the positive quantum of charge in nonrationalized units. $e^2=\alpha=1/137$. $V=-Ze^2/r$ is the Coulomb potential energy. $\mathbf{A}$ is a 3-vector, of components A_i ($i=1,2,3$) and length A. $\mathfrak{A}$ is a 4-vector, of components A_μ ($\mu=0,1,2,3$). The summation convention is $A_\mu B_\mu=A_0B_0-A_1B_1-A_2B_2-A_3B_3$. The fundamental tensor $\delta_{\mu\nu}$ is such that $\delta_{\mu\nu}A_\nu=A_\mu$ and $\delta_{\mu\mu}=4$. The conventions for Dirac matrices are: $\gamma_i=\beta\alpha_i$, $\gamma_0=\beta$, $\boldsymbol{A}=\gamma_\mu A_\mu$, $V=\gamma_\mu V_\mu=-\beta Ze^2/r$. Dirac's equation reads $(p-m)\psi=V\psi$. $\bar\psi=\psi^*\beta$ is the relativistic adjoint of ψ, ψ^* being the ordinary Hermitian conjugate. λ is the infinitesimal mass of the photon.

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[10] E. E. Salpeter, Phys. Rev. **87**, 328 (1952); **89**, 92 (1953).

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[12] R. P. Feynman, Phys. Rev. **76**, 749 (1949), referred to as F I.

[13] Here ∇_μ stands for the vector $(\partial/\partial t, -\partial/\partial x, -\partial/\partial y, -\partial/\partial z)$.

The correction gives the following contribution to the transition amplitude from ψ_0 to ψ_0:

$$i\Delta m\int\bar{\psi}_0(\mathfrak{x})\psi_0(\mathfrak{x})d^4\mathfrak{x},$$

and therefore the following energy shift:

$$\Delta E_b=-\Delta m\int\bar{\psi}_0(\mathfrak{x})\psi_0(\mathfrak{x})d^3\mathbf{x}. \tag{6}$$

In this paper, the convergence factor defined in F II, p. 776, will be used. Therefore, we replace Δm by[14]

$$\Delta m=(e^2/2\pi)m[3\ln(\Lambda/m)+\tfrac{3}{4}]. \tag{7}$$

The most important part of the Lamb shift is therefore given by $\Delta E_a+\Delta E_b$. As usual, it is easier to work in momentum space. Define

$$\psi_0(\mathfrak{x})=\psi_0(\mathbf{x})\exp(-iE_0t), \tag{8}$$

$$\varphi_0(\mathbf{p})=(2\pi)^{-\frac{3}{2}}\int\psi_0(\mathbf{x})\exp(-i\mathbf{p}\cdot\mathbf{x})d^3\mathbf{x}, \tag{8a}$$

$$V(\mathbf{q})=(2\pi)^{-3}\int V(\mathbf{x})\exp(-i\mathbf{q}\cdot\mathbf{x})d^3\mathbf{x}$$
$$=-(Ze^2/2\pi^2)\beta/\mathbf{q}^2, \tag{8b}$$

$$K_+{}^V(\mathfrak{x}_2,\mathfrak{x}_1)=(2\pi)^{-4}\int K_+{}^V(\mathfrak{p}_2,\mathfrak{p}_1)$$
$$\times\exp(-i\mathfrak{p}_2\cdot\mathfrak{x}_2+i\mathfrak{p}_1\cdot\mathfrak{x}_1)d^4\mathfrak{p}_2d^4\mathfrak{p}_1, \tag{9}$$

$$K_+{}^V(\mathfrak{p}_2,\mathfrak{p}_1)=\delta(E_2-E_1)K_+{}^V(E;\mathbf{p}_2,\mathbf{p}_1), \tag{9a}$$

where E is the time component of $\mathfrak{p}$. This part of the Lamb shift then becomes

$$\Delta E_{a+b}=-(e^2/\pi)\int\bar{\varphi}_0(\mathbf{p}_2)\gamma_\mu K_+{}^V(E_0-\omega;\mathbf{p}_2-\mathbf{k},\mathbf{p}_1-\mathbf{k})$$
$$\cdot\gamma_\mu\varphi_0(\mathbf{p}_1)\mathfrak{k}^{-2}d^4\mathfrak{k}_Fd^3\mathbf{p}_2d^3\mathbf{p}_1 \tag{10}$$
$$-\Delta m\int\bar{\varphi}_0(\mathbf{p})\varphi_0(\mathbf{p})d^3\mathbf{p}.$$

ω is the time component of $\mathfrak{k}$. $d^4\mathfrak{k}_F$ means $d^4\mathfrak{k}/4\pi^2$ and will make it easier to use the formulas in the appendix of F II, where $d^4\mathfrak{k}$ is defined in this manner [see appendix of F II, second line after Eq. (1a)].

Expression (10) is exact, for diagrams 1a and 1b, i.e., for the part of the Lamb shift involving one virtual photon, exclusive of vacuum polarization (diagram 1c). The difficulty of the problem comes entirely from the lack of a convenient expression for $K_+{}^V$. In previous treatments, e.g., F II, the kernel $K_+{}^V$ was approximated in the relativistic region by an expression of first order in the Coulomb potential V, and in the nonrelativistic region by the ordinary Schrödinger kernel; these parts were then fitted together. Our first aim will now be to make a rigorous separation of (10) into one part which is of first order in V, and another part involving the potential twice or more often.

We can perform this separation most easily using operator notation. The kernel $K_+{}^V$ may be written as:

$$-iK_+{}^V=(\boldsymbol{p}-m)^{-1}+(\boldsymbol{p}-m)^{-1}\boldsymbol{V}(\boldsymbol{p}-m)^{-1}$$
$$+(\boldsymbol{p}-m)^{-1}\boldsymbol{V}(\boldsymbol{p}-m)^{-1}\boldsymbol{V}(\boldsymbol{p}-m)^{-1}+\cdots, \tag{11}$$

where $\boldsymbol{p}$ stands for the momentum operator $\boldsymbol{p}=i\boldsymbol{\nabla}$. Equation (11) may be formally simplified into

$$-iK_+{}^V=(\boldsymbol{p}-\boldsymbol{V}-m)^{-1}. \tag{12}$$

In (12) there occur the noncommuting operators $\boldsymbol{p}$ and $\boldsymbol{V}$; Eq. (11) defines the order in which these operators should be applied.

The expression (12) for $K_+{}^V$ follows, of course, naturally from (5). We shall use (12) essentially in one way only. We know that

$$-i(\boldsymbol{p}-\boldsymbol{V}-m)K_+{}^V=-iK_+{}^V(\boldsymbol{p}-\boldsymbol{V}-m)=1, \tag{13}$$

where 1 is the unit operator. This follows directly from (5), or from (11), by straightforward algebra.

The expression for ΔE_a is

$$\Delta E_a=-i(e^2/\pi)\int\bar{\varphi}_0\gamma_\mu\exp(-i\mathfrak{k}\cdot\mathfrak{x})$$
$$\cdot(\boldsymbol{p}-\boldsymbol{V}-m)^{-1}\gamma_\mu\exp(i\mathfrak{k}\cdot\mathfrak{x})\varphi_0\mathfrak{k}^{-2}d^4\mathfrak{k}. \tag{14}$$

The exponentials give the space-time variation for the emitted and absorbed photon which has been expanded into components of momentum $\mathfrak{k}$. The operator $\boldsymbol{p}=i\boldsymbol{\nabla}$ acts on everything that follows it, i.e., on φ_0, on V inasmuch as is required by the defining Eq. (11), and on $\exp(i\mathfrak{k}\cdot\mathfrak{x})$. The $\exp(i\mathfrak{k}\cdot\mathfrak{x})$ commutes with all quantities except $\boldsymbol{p}$, and $\exp(-i\mathfrak{k}\cdot\mathfrak{x})\boldsymbol{p}\exp(i\mathfrak{k}\cdot\mathfrak{x})=\boldsymbol{p}-\boldsymbol{k}$ so that (14) becomes[15]

$$\Delta E_a=-i(e^2/\pi)\int\bar{\varphi}_0\gamma_\mu(\boldsymbol{p}-\boldsymbol{k}-\boldsymbol{V}-m)^{-1}\gamma_\mu\varphi_0\mathfrak{k}^{-2}d^4\mathfrak{k}. \tag{15}$$

Now $\boldsymbol{p}=i\boldsymbol{\nabla}$ only acts on φ_0 and on V as required, no longer on $\exp(i\mathfrak{k}\cdot\mathfrak{x})$; $\mathfrak{k}$ is a c number. The function φ_0 satisfies the Dirac equation,

$$(\boldsymbol{p}-\boldsymbol{V}-m)\varphi_0=0. \tag{16}$$

We shall now rearrange the integrand in (15),

$$\bar{\varphi}_0\gamma_\mu(\boldsymbol{p}-\boldsymbol{k}-\boldsymbol{V}-m)^{-1}\gamma_\mu\varphi_0, \tag{15a}$$

[14] Λ and λ stand for λ and $\lambda_{\min}$ of F II, respectively.

[15] In any expression involving operators one can perform an equivalence transformation:

$$Sf(A,B,\cdots)S^{-1}=f(A',B',\cdots),$$

where $A'=SAS^{-1}$, etc. With $S=\exp(-i\mathfrak{k}\cdot\mathfrak{x})$, and $f=(\boldsymbol{p}-\boldsymbol{V}-m)^{-1}$ we get

$$\exp(-i\mathfrak{k}\cdot\mathfrak{x})(\boldsymbol{p}-\boldsymbol{V}-m)^{-1}\exp(i\mathfrak{k}\cdot\mathfrak{x})=(\boldsymbol{p}'-\boldsymbol{V}'-m)^{-1},$$
$$\boldsymbol{V}'=\exp(-i\mathfrak{k}\cdot\mathfrak{x})\boldsymbol{V}\exp(i\mathfrak{k}\cdot\mathfrak{x})=\boldsymbol{V},$$

and $\boldsymbol{p}'=\exp(-i\mathfrak{k}\cdot\mathfrak{x})\boldsymbol{p}\exp(i\mathfrak{k}\cdot\mathfrak{x})=\boldsymbol{p}-\boldsymbol{k}$.

by use of the following lemma:

$$A\equiv(\not p_b-m)\frac{\not p_b A+A\not p_a}{\not\mathfrak{p}_b^2-\not\mathfrak{p}_a^2}-\frac{\not p_b A+A\not p_a}{\not\mathfrak{p}_b^2-\not\mathfrak{p}_a^2}(\not p_a-m), \quad (17)$$

which is valid for any operator A which causes the momentum to change[16] from $\not\mathfrak{p}_a$ to $\not\mathfrak{p}_b$.

The identity (17) is verified by direct algebra using the fact that $\not p_a^2=\not\mathfrak{p}_a^2$ is free of Dirac operators. We apply (17) to re-express the left hand γ_μ in (15a) (which replaces A), with $\not\mathfrak{p}_b=\not\mathfrak{p}$, $\not\mathfrak{p}_a=\not\mathfrak{p}-\not\mathfrak{k}$ and obtain

$$\bar\varphi_0[(\not p-m)B_\mu-B_\mu(\not p-\not k-m)]\times(\not p-\not k-V-m)^{-1}\gamma_\mu\varphi_0, \quad (18)$$

with

$$B_\mu=\frac{\not p\gamma_\mu+\gamma_\mu(\not p-\not k)}{\not\mathfrak{p}^2-(\not\mathfrak{p}-\not\mathfrak{k})^2}=\frac{2p_\mu-\gamma_\mu\not k}{\not\mathfrak{p}^2-(\not\mathfrak{p}-\not\mathfrak{k})^2}. \quad (19)$$

(18) may be rewritten:

$$\bar\varphi_0[(\not p-V-m)B_\mu-B_\mu(\not p-\not k-V-m)]\times(\not p-\not k-V-m)^{-1}\gamma_\mu\varphi_0+\bar\varphi_0(VB_\mu-B_\mu V)(\not p-\not k-V-m)^{-1}\gamma_\mu\varphi_0. \quad (20)$$

The first term vanishes because φ_0 satisfies the Dirac equation (16), and the second term is simply

$$\mathrm{I}=-\bar\varphi_0 B_\mu\gamma_\mu\varphi_0. \quad (21)$$

In the last term of (20) we now transform the second γ_μ, again using the lemma (17). This time $\not\mathfrak{p}_a=\not\mathfrak{p}$ and $\not\mathfrak{p}_b=\not\mathfrak{p}-\not\mathfrak{k}$ and the last term of (20) becomes

$$\begin{aligned}\mathrm{II}+\mathrm{III}&=\bar\varphi_0(VB_\mu-B_\mu V)(\not p-\not k-V-m)^{-1}\cdot[(\not p-\not k-V-m)B_\mu^\dagger-B_\mu^\dagger(\not p-V-m)]\varphi_0 \quad (22)\\&+\bar\varphi_0(VB_\mu-B_\mu V)(\not p-\not k-V-m)^{-1}(VB_\mu^\dagger-B_\mu^\dagger V)\varphi_0,\end{aligned}$$

with

$$B_\mu^\dagger=\frac{(\not p-\not k)\gamma_\mu+\gamma_\mu\not p}{(\not\mathfrak{p}-\not\mathfrak{k})^2-\not\mathfrak{p}^2}=\frac{2p_\mu-\not k\gamma_\mu}{(\not\mathfrak{p}-\not\mathfrak{k})^2-\not\mathfrak{p}^2}. \quad (19a)$$

The first term of (22) gives

$$\mathrm{II}=\bar\varphi_0(VB_\mu-B_\mu V)B_\mu^\dagger\varphi_0 \quad (21a)$$

the second vanishes, and the last term is

$$\mathrm{III}=\bar\varphi_0(VB_\mu-B_\mu V)(\not p-\not k-V-m)^{-1}\times(VB_\mu^\dagger-B_\mu^\dagger V)\varphi_0. \quad (21b)$$

The term (21b) is now the only one which still contains the propagation kernel in the Coulomb field, K_+^V. Of the other two terms, (21) does not contain the potential at all but is a self-energy for a free particle which will largely cancel the mass renormalization Δm. (21a) is the one-potential part of the Lamb shift. In (21b), the potential acts at least twice, once with the emission of the virtual photon (term $VB^\dagger-B^\dagger V$) and once with its absorption. In addition, the potential may also act in the propagation kernel K_+^V. Thus in (21b) are contained all terms in which the potential acts twice or more, and they are cleanly separated from the terms involving only one or no potential, thus fulfilling the first point of our program.[17]

It is interesting to note that in (21b) the emission and absorption of a virtual photon is coupled with the action of the potential. This expresses the physical fact that emission and absorption can only take place when there is an external field acting on the electron.

It is convenient to introduce an abbreviation for the expression which occurs repeatedly, *viz*:

$$VB_\mu-B_\mu V=M_\mu, \quad (23)$$

and $M_\mu^\dagger$ correspondingly. Then we have explicitly:

$$M_\mu(\not\mathfrak{p}_f,\not\mathfrak{p}_i-\not\mathfrak{k})=V(\mathbf{p}_f-\mathbf{p}_i)\frac{2p_{i\mu}-\gamma_\mu\not k}{2\not\mathfrak{p}_i\cdot\not\mathfrak{k}-\not\mathfrak{k}^2}-\frac{2p_{f\mu}-\gamma_\mu\not k}{2\not\mathfrak{p}_f\cdot\not\mathfrak{k}-\not\mathfrak{k}^2}V(\mathbf{p}_f-\mathbf{p}_i), \quad (24)$$

and

$$\begin{aligned}M_\mu^\dagger(\not\mathfrak{p}_f-\not\mathfrak{k},\not\mathfrak{p}_i)&=V(\mathbf{p}_f-\mathbf{p}_i)\frac{2p_{i\mu}-\not k\gamma_\mu}{\not\mathfrak{k}^2-2\not\mathfrak{p}_i\cdot\not\mathfrak{k}}-\frac{2p_{f\mu}-\not k\gamma_\mu}{\not\mathfrak{k}^2-2\not\mathfrak{p}_f\cdot\not\mathfrak{k}}V(\mathbf{p}_f-\mathbf{p}_i) \quad (24a)\\&=\bar M_\mu(\not\mathfrak{p}_i,\not\mathfrak{p}_f-\not\mathfrak{k}),\end{aligned}$$

where $\bar M_\mu$ is the operator relativistic adjoint of M_μ.

$V(\mathbf{p}_f-\mathbf{p}_i)$ is the Fourier component of the Coulomb potential corresponding to a momentum change $\mathbf{p}_f-\mathbf{p}_i$. The operator $M_\mu^\dagger$ corresponds to the emission of a quantum of momentum $\not\mathfrak{k}$ and polarization μ; the electron has initially the momentum $\not\mathfrak{p}_i$ and after emission the momentum $\not\mathfrak{p}_f-\not\mathfrak{k}$, so that the total final momentum of electron plus quantum is $\not\mathfrak{p}_f$; the initial and final momentum of the electron are put in evidence as the arguments of $M_\mu^\dagger$. M_μ corresponds to the absorption of the same quantum; the momentum of the electron before absorption is $\not\mathfrak{p}_i-\not\mathfrak{k}$, after absorption $\not\mathfrak{p}_f$. We shall call $M_\mu^\dagger$ and M_μ "radiation operators" or, specifically, emission and absorption operators.[18]

We can now collect the various parts of the Lamb

[16] More generally, $\not\mathfrak{p}_b$ and $\not\mathfrak{p}_a$ can be thought of as operators ordered according to the rule [see R. P. Feynman, Phys. Rev. **84**, 108 (1951)] that $\not\mathfrak{p}_a$ precedes and $\not\mathfrak{p}_b$ follows the variables in A; the γ's operate in the order in which they are written.

[17] The same aim could have been achieved by using the more straightforward identity:

$$K_+^V=K_+^0-iK_+^0VK_+^0-K_+^0VK_+^VVK_+^0.$$

But our method presents certain advantages which will become evident later.

[18] The method given here for separating the electrodynamic shift into three terms such as (21), (21a), and (21b), has been used previously, in connection with the radiative corrections to hyperfine structure, by N. M. Kroll and F. Pollock, Phys. Rev. **86**, 876 (1952).

shift and get

$$\Delta E_{a+b}=\Delta E_2+\Delta E_1$$

$$\begin{aligned}\Delta E_2=&-(e^2/\pi)\int\bar{\varphi}_0(\mathbf{p}_2)M_\mu(\mathfrak{p}_2,\,\mathfrak{p}_2-\mathfrak{s}_2-\mathfrak{k})\\&\times K_+{}^V(E_0-\omega;\,\mathbf{p}_2-\mathbf{s}_2-\mathbf{k},\,\mathbf{p}_1+\mathbf{s}_1-\mathbf{k})\\&\times M_\mu{}^\dagger(\mathfrak{p}_1+\mathfrak{s}_1-\mathfrak{k},\,\mathfrak{p}_1)\\&\times\varphi_0(\mathbf{p}_1)\mathfrak{k}^{-2}d^4\mathfrak{k}_F d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{s}_1 d^3\mathbf{s}_2,\end{aligned}\tag{25}$$

$$\Delta E_1=\Delta E_{11}+\Delta E_{12}+\Delta E_{13}+\Delta E_{14},$$

$$\Delta E_{11}=\frac{e^2}{\pi i}\int\bar{\varphi}_0(\mathbf{p}_2)\frac{2p_{2\mu}-\gamma_\mu k}{\mathfrak{k}^2-2\mathfrak{p}_2\cdot\mathfrak{k}}V(\mathbf{p}_2-\mathbf{p}_1)\cdot\frac{2p_{1\mu}-k\gamma_\mu}{\mathfrak{k}^2-2\mathfrak{p}_1\cdot\mathfrak{k}}\varphi_0(\mathbf{p}_1)\mathfrak{k}^{-2}d^4\mathfrak{k}_F d^3\mathbf{p}_1 d^3\mathbf{p}_2,$$

$$\Delta E_{12}=-\frac{e^2}{\pi i}\int\bar{\varphi}_0(\mathbf{p}_2)V(\mathbf{p}_2-\mathbf{p}_1)\frac{2p_{1\mu}-\gamma_\mu k}{\mathfrak{k}^2-2\mathfrak{p}_1\cdot\mathfrak{k}}\cdot\frac{2p_{1\mu}-k\gamma_\mu}{\mathfrak{k}^2-2\mathfrak{p}_1\cdot\mathfrak{k}}\varphi_0(\mathbf{p}_1)\mathfrak{k}^{-2}d^4\mathfrak{k}_F d^3\mathbf{p}_1 d^3\mathbf{p}_2,\tag{26}$$

$$\Delta E_{13}=\frac{e^2}{\pi i}\int\bar{\varphi}_0(\mathbf{p})\frac{2p_\mu-\gamma_\mu k}{\mathfrak{k}^2-2\mathfrak{p}\cdot\mathfrak{k}}\gamma_\mu\varphi_0(\mathbf{p})\mathfrak{k}^{-2}d^4\mathfrak{k}_F d^3\mathbf{p},$$

$$\Delta E_{14}=-\Delta m\int\bar{\varphi}_0(\mathbf{p})\varphi_0(\mathbf{p})d^3\mathbf{p}.$$

In (25) and (26) the $\mathfrak{p}$'s have time component E_0 and the $\mathfrak{s}$'s have time component 0. $M_\mu(\mathfrak{p}_2,\,\mathfrak{p}_2-\mathfrak{s}_2-\mathfrak{k})$ is obtained from Eq. (24) by replacing $\mathfrak{p}_f$ by $\mathfrak{p}_2$ and $\mathfrak{p}_i$ by $\mathfrak{p}_2-\mathfrak{s}_2$; similarly $M_\mu{}^\dagger$ comes from (24a) replacing $\mathfrak{p}_i$ by $\mathfrak{p}_1$ and $\mathfrak{p}_f$ by $\mathfrak{p}_1+\mathfrak{s}_1$.

Of the terms in ΔE_1, Eq. (26), ΔE_{11} and ΔE_{12} arise from (21a) and thus contain one potential; ΔE_{13} arises from (21) and contains no potential at all; ΔE_{14} is the mass renormalization term, i.e., the last term of (10). The entire term ΔE_1 will be called "one-potential Lamb shift" and treated in Sec. III. ΔE_1 is identical with the term evaluated in references 3 to 6, except for the fact that, here, the momenta are not free particle ones: $\mathfrak{p}_1{}^2$, $\mathfrak{p}_2{}^2$, $\mathfrak{p}^2\neq m^2$, and $\mathfrak{p}\varphi_0\neq m\varphi_0$. ΔE_2 will be called "many-potential Lamb shift." It arises from (21b) and it contains the nonrelativistic Lamb shift (2); it will be treated in Secs. IV and V. It should be noted that Eqs. (25) and (26) are still exact, and approximations will only be made in the evaluation.

The poles that appear in the propagation factors $(\not{p}-m)^{-1}$ and $\mathfrak{k}^{-2}$ will be defined in the usual way by giving to the masses of electron and photon a small negative imaginary part. But, in Eqs. (17), (24), (24a) above, a new kind of pole, $(\mathfrak{k}^2-2\mathfrak{p}\cdot\mathfrak{k})^{-1}$, has been introduced, and has to be defined in some way. Since these poles are spurious, i.e., disappear when the two terms in the right side of Eq. (17) are added, it does not make any difference which way they are defined, provided it is done consistently. One could for instance take their principal values, but it seems easier to define them in the same way as the others, the ones that come in the propagation factors. Therefore when integrating over k_0 we shall take the contour of integration to go below the left-hand pole of $(\mathfrak{k}^2-2\mathfrak{p}\cdot\mathfrak{k})^{-1}$, and above the right-hand one, as usual. In subsequent integrations, in case a pole still appears, we shall assume that the absolute value of the time component of $\mathfrak{p}$, E_0, has a small negative imaginary part. These poles might then give imaginary contributions. But we expect the whole imaginary contribution from them to cancel at the end, since they are just artificial poles. Therefore we shall ignore it.

III. THE ONE-POTENTIAL LAMB SHIFT

The integration over $\mathfrak{k}$ in (26) can be performed by using the methods of the appendix of F II. First define I_1 and I_2, functions of $\mathbf{p}_1$ and $\mathbf{p}_2$, and I_3, function of $\mathbf{p}$, by

$$\Delta E_{11}=(e^2/\pi)\int\bar{\varphi}_0(\mathbf{p}_2)I_1\varphi_0(\mathbf{p}_1)d^3\mathbf{p}_1 d^3\mathbf{p}_2,$$

$$\Delta E_{12}=(e^2/\pi)\int\bar{\varphi}_0(\mathbf{p}_2)I_2\varphi_0(\mathbf{p}_1)d^3\mathbf{p}_1 d^3\mathbf{p}_2,$$

$$\Delta E_{13}+\Delta E_{14}=(e^2/\pi)\int\bar{\varphi}_0(\mathbf{p})I_3\varphi_0(\mathbf{p})d^3\mathbf{p}.$$

I_1, for instance, can be written

$$I_1=\frac{1}{i}\int\frac{2p_{2\mu}-\gamma_\mu k}{\mathfrak{k}^2-2\mathfrak{p}_2\cdot\mathfrak{k}}V\frac{2p_{1\mu}-k\gamma_\mu}{\mathfrak{k}^2-2\mathfrak{p}_1\cdot\mathfrak{k}}\frac{C(\mathfrak{k}^2)}{\mathfrak{k}^2}d^4\mathfrak{k}_F,$$

where $C(\mathfrak{k}^2)$ is a convergence factor, which will enable the integral to be performed at the limits of both small and large $\mathfrak{k}$. Specifically, we take

$$\mathfrak{k}^{-2}C(\mathfrak{k}^2)=(\mathfrak{k}^2-\lambda^2)^{-1}-(\mathfrak{k}^2-\Lambda^2)^{-1}=-\int_{\lambda^2}^{\Lambda^2}dL(\mathfrak{k}^2-L)^{-2}.$$

Next we define three expressions J_0, J_σ, and $J_{\sigma\tau}$ by

$$J_{(0;\sigma;\sigma\tau)}=\frac{1}{i}\int\frac{(1;\,k_\sigma;\,k_\sigma k_\tau)}{(\mathfrak{k}^2-2\mathfrak{p}_2\cdot\mathfrak{k})(\mathfrak{k}^2-2\mathfrak{p}_1\cdot\mathfrak{k})}\frac{C(\mathfrak{k}^2)}{\mathfrak{k}^2}d^4\mathfrak{k}_F.$$

These integrals are easily evaluated by the usual technique of combining denominators with auxiliary variables [F II, Eq. (14a)]. The result is

$$J_0=-\frac{1}{8}\int_0^1(dx/\mathfrak{p}_x{}^2)\ln(\mathfrak{p}_x{}^2/\lambda^2),$$

$$J_\sigma=-\frac{1}{4}\int_0^1(p_{x\sigma}/\mathfrak{p}_x{}^2)dx,$$

$$J_{\sigma\tau}=-\frac{1}{8}\int_0^1 dx\left[\frac{p_{x\sigma}p_{x\tau}}{\mathfrak{p}_x{}^2}-\frac{\delta_{\sigma\tau}}{2}\left(\ln\frac{\Lambda^2}{\mathfrak{p}_x{}^2}-\frac{1}{2}\right)\right],$$

where

$$\mathfrak{p}_x=x\mathfrak{p}_2+(1-x)\mathfrak{p}_1, \tag{27}$$

and it has been assumed that Λ and λ are, respectively, very large and very small, compared to all momenta involved. I_1 is given in terms of J_0, J_σ, and $J_{\sigma\tau}$ by

$$I_1=4p_{2\mu}Vp_{1\mu}J_0-2(p_{2\mu}V\gamma_\sigma\gamma_\mu+\gamma_\mu\gamma_\sigma Vp_{1\mu})J_\sigma +\gamma_\mu\gamma_\sigma V\gamma_\tau\gamma_\mu J_{\sigma\tau}.$$

After summing over repeated indices, and making some algebraic manipulation in order to bring $\not{p}_2$ to the left and $\not{p}_1$ to the right of V, we find

$$I_1=-\tfrac{3}{8}V-\tfrac{1}{2}\mathfrak{p}_2\cdot\mathfrak{p}_1V\int_0^1(dx/\mathfrak{p}_x^2)\ln(\mathfrak{p}_x^2/\lambda^2) +\tfrac{1}{4}V\int_0^1 dx\ln(\Lambda^2/\mathfrak{p}_x^2)+\tfrac{1}{2}\int_0^1(dx/\mathfrak{p}_x^2) \cdot\{[(1-x)\mathfrak{p}_1^2+x\mathfrak{p}_2^2+2\mathfrak{p}_2\cdot\mathfrak{p}_1]V+\not{p}_2V\not{p}_1-E_0V\not{p}_x\}. \tag{28}$$

I_2 is computed in a similar fashion. The J's are the same as for I_1, with the additional simplification that the two momenta are the same and that there is no auxiliary variable x.

$$I_2=-\frac{1}{i}V\int\frac{2p_\mu-\gamma_\mu\not{k}}{\mathfrak{k}^2-2\mathfrak{p}\cdot\mathfrak{k}}\frac{2p_\mu-\not{k}\gamma_\mu}{\mathfrak{k}^2-2\mathfrak{p}\cdot\mathfrak{k}}\frac{C(\mathfrak{k}^2)}{\mathfrak{k}^2}d^4\mathfrak{k}_F =V[\tfrac{1}{2}\ln(\mathfrak{p}^2/\lambda^2)-\ln(\Lambda^2/\mathfrak{p}^2)]. \tag{28a}$$

In I_2, $\mathfrak{p}$ has been written instead of $\mathfrak{p}_1$, since the result is symmetric between $\mathfrak{p}_1$ and $\mathfrak{p}_2$. One can, for instance, use $I_2/2$ with $\mathfrak{p}$ replaced by $\mathfrak{p}_1$, plus $I_2/2$ with $\mathfrak{p}$ replaced by $\mathfrak{p}_2$. For I_3 we find, after using Eq. (7),

$$I_3=\frac{1}{i}\int\frac{2p_\mu-\gamma_\mu\not{k}}{\mathfrak{k}^2-2\mathfrak{p}\cdot\mathfrak{k}}\gamma_\mu\frac{C(\mathfrak{k}^2)}{\mathfrak{k}^2}d^4\mathfrak{k}_F-\frac{\pi}{e^2}\Delta m =\tfrac{3}{4}[(\not{p}-m)(\ln(\Lambda^2/\mathfrak{p}^2)+\tfrac{1}{2})+m\ln(m^2/\mathfrak{p}^2)]. \tag{28b}$$

In I_3, we can replace $\not{p}-m$ by V, since it operates on φ_0, solution of Dirac's equation. When I_1, I_2, and I_3 are added together, we find that Λ disappears, as expected. After some rearranging, the result is

$$\Delta E_1=(e^2/\pi)\int\bar{\varphi}_0(\mathbf{p}_2)(I_a+\cdots+I_f)\varphi_0(\mathbf{p}_1)d^3\mathbf{p}_1d^3\mathbf{p}_2, \tag{29}$$

$$I_a=V\ln(\lambda/E_0)\left[\int_0^1(\mathfrak{p}_1\cdot\mathfrak{p}_2/\mathfrak{p}_x^2)dx-1\right], \tag{29a}$$

$$I_b=(V/2)\int_0^1(dx/\mathfrak{p}_x^2)[(1-x)\mathfrak{p}_1^2 +x\mathfrak{p}_2^2+2\mathfrak{p}_1\cdot\mathfrak{p}_2]-3V/2, \tag{29b}$$

$$I_c=(3V/8)\int_0^1 dx\ln[\mathfrak{p}_1^2\mathfrak{p}_2^2/(\mathfrak{p}_x^2)^2], \tag{29c}$$

$$I_d=-(V/2)\int_0^1 dx[(\mathfrak{p}_1\cdot\mathfrak{p}_2/\mathfrak{p}_x^2)-1]\ln(\mathfrak{p}_x^2/E_0^2), \tag{29d}$$

$$I_e=\tfrac{1}{2}\int_0^1(dx/\mathfrak{p}_x^2)(\not{p}_2V\not{p}_1-E_0V\not{p}_x), \tag{29e}$$

$$I_f=(3m/4)\ln(m^2/\mathfrak{p}^2)\delta^3(\mathbf{p}_2-\mathbf{p}_1)+3V/2. \tag{29f}$$

It will now be shown that, except for a non-gauge-invariant part discussed later and which is canceled by a similar term in the many-potential Lamb shift, the terms of order $\alpha(Z\alpha)^4$ in ΔE_1 are given by the expectation value of the operator (1). Moreover, as discussed in the appendix, there are no corrections of order $\alpha(Z\alpha)^5$ to that.[19] Therefore, to the approximation desired, the first-order Lamb shift is just equal to what has already been calculated in references 3 to 6.

First, let us examine I_a. $\mathfrak{p}_x^2$ can be replaced by $x^2\mathfrak{p}_2^2+(1-x)^2\mathfrak{p}_1^2+2x(1-x)\mathfrak{p}_1\cdot\mathfrak{p}_2$ and $\mathfrak{p}_1\cdot\mathfrak{p}_2=\frac{1}{2}(\mathfrak{p}_2^2+\mathfrak{p}_1^2-\mathfrak{q}^2)$, where $\mathfrak{q}=\mathfrak{p}_2-\mathfrak{p}_1$, so that

$$I_a=V\ln\frac{\lambda}{E_0}\int_0^1 dx \times\frac{[2x(1-x)-1]\mathfrak{q}^2+(1-2x)(\mathfrak{p}_2^2-\mathfrak{p}_1^2)}{2\mathfrak{p}_x^2}. \tag{30}$$

Since the most important momenta in the hydrogen atom are of order αm, the main contribution can be obtained by replacing $\mathfrak{p}_x^2=E_0^2-\mathbf{p}_x^2$ by E_0^2. Also $E_0=m(1-\text{order }\alpha^2)$ can be replaced by m. After integration over x, we obtain

$$I_a\simeq\tfrac{1}{3}(\mathfrak{q}^2/m^2)V\ln(m/\lambda). \tag{30a}$$

It is shown in the appendix that the error is actually of order $\alpha(Z\alpha)^6\ln Z\alpha$. The same reasoning can be applied to

$$I_b=\frac{V}{2}\int_0^1 dx\frac{[3x(1-x)-1]\mathfrak{q}^2+(1-2x)(\mathfrak{p}_2^2-\mathfrak{p}_1^2)}{\mathfrak{p}_x^2} \tag{31}$$

$$\simeq-\tfrac{1}{4}(\mathfrak{q}^2/m^2)V. \tag{31a}$$

For I_c, we write

$$I_c=\frac{3V}{8}\int_0^1 dx\ln\frac{(E_0^2-\mathbf{p}_1^2)(E_0^2-\mathbf{p}_2^2)}{(E_0^2-\mathbf{p}_x^2)^2}, \tag{32}$$

and we can keep only the first term of an expansion in powers of $\mathbf{p}^2/E_0^2$:

$$I_c\simeq(3V/8)\int_0^1 dx(2\mathbf{p}_x^2-\mathbf{p}_1^2-\mathbf{p}_2^2)/E_0^2.$$

After replacing E_0 by m, and integrating over x, we obtain

$$I_c\simeq\tfrac{1}{8}(\mathfrak{q}^2/m^2)V. \tag{32a}$$

[19] This fact was first pointed out to the authors by N. M. Kroll.

That this is true including order $\alpha(Z\alpha)^5$, and also that I_d does not contribute in the same order, is shown in the appendix.

I_e can be transformed by writing its numerator

$$(\not p_2-m)V(\not p_1-m)+m[(\not p_2-m)V+V(\not p_1-m)] -E_0V(\not p_x-m)+m^2V-E_0mV,$$

and using Dirac's equation to replace the $\not p-m$ operators by V operators:

$$\begin{aligned}\Delta E_{1e}&=(e^2/\pi)\int\bar\varphi_0(\mathbf{p}_2)I_e\varphi_0(\mathbf{p}_1)d^3\mathbf{p}_2d^3\mathbf{p}_1\\&=(e^2/2\pi)\int d^3\mathbf{p}_fd^3\mathbf{p}_id^3\mathbf{p}_2d^3\mathbf{p}_1\int_0^1(dx/\mathfrak{p}_x^2)\\&\quad\cdot\bar\varphi_0(\mathbf{p}_f)[\beta V(\mathbf{p}_f-\mathbf{p}_2)V(\mathbf{p}_2-\mathbf{p}_1)V(\mathbf{p}_1-\mathbf{p}_i)\\&\quad+(m-E_0x\beta)V(\mathbf{p}_f-\mathbf{p}_2)V(\mathbf{p}_2-\mathbf{p}_1)\delta^3(\mathbf{p}_1-\mathbf{p}_i)\\&\quad+(m-E_0(1-x)\beta)V(\mathbf{p}_2-\mathbf{p}_1)V(\mathbf{p}_1-\mathbf{p}_i)\delta^3(\mathbf{p}_f-\mathbf{p}_2)\\&\quad+m(\beta m-E_0)V(\mathbf{p}_2-\mathbf{p}_1)\delta^3(\mathbf{p}_f-\mathbf{p}_2)\delta^3(\mathbf{p}_1-\mathbf{p}_i)]\varphi_0(\mathbf{p}_i).\end{aligned}\tag{33}$$

The term with $3V$'s is of order $\alpha(Z\alpha)^6\ \ln Z\alpha$, since it involves essentially $\alpha(Z\alpha)^3$ times the expectation value of r^{-3}, with a cutoff for radii of the order of the Compton wavelength. For the other terms, we shall use again the approximation, justified in the appendix, of replacing $\mathfrak{p}_x^2$ by m^2. For brevity, we write the result in operator form:

$$\Delta E_{1e}\simeq(e^2/2\pi m^2)\langle\varphi_0^*|\,(2m\beta-E_0)V^2 +m(m-\beta E_0)V\,|\,\varphi_0\rangle.\tag{33a}$$

This can be transformed, by use of the equations

$$\begin{aligned}\langle\varphi_0^*|\beta(\boldsymbol\alpha\cdot\mathbf{p}V-V\boldsymbol\alpha\cdot\mathbf{p})|\,\varphi_0\rangle&=-\langle\varphi_0^*|\,\boldsymbol\alpha\cdot\mathbf{p}\beta V+\beta V\boldsymbol\alpha\cdot\mathbf{p}|\,\varphi_0\rangle\\&=-\langle\varphi_0^*|\,(E_0-\beta m-V)\beta V+\beta V(E_0-\beta m-V)\,|\,\varphi_0\rangle\\&=2\langle\varphi_0^*|\,\beta V^2+(m-\beta E_0)V\,|\,\varphi_0\rangle.\end{aligned}$$

Therefore

$$\Delta E_{1e}\simeq(e^2/2\pi m^2)\langle\varphi_0^*|\,(m\beta-E_0)V^2 +(m/2)\beta(\boldsymbol\alpha\cdot\mathbf{p}V-V\boldsymbol\alpha\cdot\mathbf{p})\,|\,\varphi_0\rangle.$$

$(m\beta-E_0)V^2$ is negligible, within the accuracy desired, so that, going back to the momentum representation,

$$I_e\simeq(1/4m)\boldsymbol\alpha\cdot\mathbf{q}V(\mathbf{q})=(1/8m)(\boldsymbol qV-V\boldsymbol q).\tag{34}$$

Finally, I_f can be replaced by

$$I_f=\tfrac{3}{4}m\delta^3(\mathbf{p}_2-\mathbf{p}_1)\int_0^1dx\left[\frac{2(\not p-m)}{m}-\frac{\mathfrak{p}^2-m^2}{\mathfrak{p}^2x+m^2(1-x)}\right].$$

By repeated use of the formula

$$\mathfrak{p}^2-m^2=(\not p-m)^2+2m(\not p-m),$$

the integrand in I_f can be written

$$\begin{aligned}&\frac{2x(\not p-m)(\mathfrak{p}^2-m^2)}{m[\mathfrak{p}^2x+m^2(1-x)]}-\frac{(\not p-m)^2}{\mathfrak{p}^2x+m^2(1-x)}\\&\quad=\frac{2x[2m(\not p-m)^2+(\not p-m)^3]}{m[\mathfrak{p}^2x+m^2(1-x)]}-\frac{(\not p-m)^2}{\mathfrak{p}^2x+m^2(1-x)}.\end{aligned}$$

Finally

$$I_f=\frac{3m}{4}\delta^3(\mathbf{p}_2-\mathbf{p}_1)\int_0^1\left[\frac{(4x-1)(\not p-m)^2}{\mathfrak{p}^2x+m^2(1-x)} +\frac{2x(\not p-m)^3}{m[\mathfrak{p}^2x+m^2(1-x)]}\right]dx.\tag{35}$$

The first term in the bracket of (35) can be approximated by replacing its denominator by m^2. The error involved in this and in neglecting the second term is of order $\alpha(Z\alpha)^6\ \ln Z\alpha$, as shown in the appendix. Therefore, using Dirac's equation for the first term,

$$\Delta E_{1f}\simeq(3e^2/4\pi m)\langle\varphi_0^*|\,\beta V^2\,|\,\varphi_0\rangle.\tag{36}$$

However, such a term cannot contribute on physical grounds, because its value depends on the gauge and would be changed if a constant were added to the potential. Indeed, it will be shown that this term is canceled by an equal and opposite one arising in the many-potential Lamb shift. This term being omitted, the first-order Lamb shift (29) is equal, within the accuracy desired, to the expectation value of e^2/π times the sum of the operators (30a), (31a), (32a), (34), that is exactly the expectation value of (1), so that no corrections of order $\alpha(Z\alpha)^5$ have been found here.

IV. THE MANY-POTENTIAL LAMB SHIFT

The many-potential Lamb shift is expression (25), where $\mathfrak{M}$ and $\mathfrak{M}^\dagger$ are given by (24) and (24a). It is interesting to note that the transversality condition

$$k_\mu M_\mu{}^\dagger=k_\mu M_\mu=0\tag{37}$$

is satisfied, so that the time component of a radiation operator can be written in terms of its space components by

$$M_0=\mathbf{k}\cdot\mathbf{M}/\omega.\tag{38}$$

The emission operator (24a) can be transformed by using the identity

$$\gamma_0\not k\gamma_\mu=2\omega\gamma_\mu-2\delta_{0\mu}\not k+\not k\gamma_\mu\gamma_0\tag{39}$$

in its first term. One obtains

$$\begin{aligned}&M_\mu{}^\dagger=M_\mu{}^{\dagger\mathrm{I}}+M_\mu{}^{\dagger\mathrm{II}},\\&M_\mu{}^{\dagger\mathrm{I}}=[2p_{i\mu}/(\mathfrak{k}^2-2\mathfrak{p}_i\cdot\mathfrak{k})-2p_{f\mu}/(\mathfrak{k}^2-2\mathfrak{p}_f\cdot\mathfrak{k})\\&\qquad+\not k\gamma_\mu((\mathfrak{k}^2-2\mathfrak{p}_f\cdot\mathfrak{k})^{-1}\\&\qquad-(\mathfrak{k}^2-2\mathfrak{p}_i\cdot\mathfrak{k})^{-1})]\gamma_0V(\mathbf{p}_f-\mathbf{p}_i),\\&M_\mu{}^{\dagger\mathrm{II}}=2[(\delta_{0\mu}\not k-\omega\gamma_\mu)/(\mathfrak{k}^2-2\mathfrak{p}_i\mathfrak{k})]V(\mathbf{p}_f-\mathbf{p}_i).\end{aligned}\tag{40}$$

The first two terms of $M_\mu^{\dagger I}$ are essentially the Schrödinger expression for the current, the last term is related to the contribution of the magnetic moment to the current. In a similar fashion we get

$$\begin{aligned} M_\mu &= M_\mu^{I} + M_\mu^{II}, \\ M_\mu^{I} &= \gamma_0[-2p_{i\mu}/(\mathfrak{k}^2-2\mathfrak{p}_i\cdot\mathfrak{k}) + 2p_{f\mu}/(\mathfrak{k}^2-2\mathfrak{p}_f\cdot\mathfrak{k}) \\ &\quad -\gamma_\mu \boldsymbol{k}((\mathfrak{k}^2-2\mathfrak{p}_f\cdot\mathfrak{k})^{-1} \\ &\quad -(\mathfrak{k}^2-2\mathfrak{p}_i\cdot\mathfrak{k})^{-1})]V(\mathbf{p}_f-\mathbf{p}_i), \\ M_\mu^{II} &= 2[(\delta_{0\mu}\boldsymbol{k}-\omega\gamma_\mu)/(\mathfrak{k}^2-2\mathfrak{p}_f\cdot\mathfrak{k})]V(\mathbf{p}_f-\mathbf{p}_i). \end{aligned} \tag{40a}$$

It is easy to see that $M_\mu^{\dagger II}$ and M_μ^{II} taken alone satisfy also the transversality condition (37) or (38). The same is therefore true of $M_\mu^{\dagger I}$ and M_μ^{I}.

One advantage of the separation into two parts is that both $\mathfrak{M}^{I}$ and $\mathfrak{M}^{II}$ reduce to very simple (but very different) expressions in the nonrelativistic limit. We have for the spatial components of $\mathfrak{M}^{I}$:

$$\begin{aligned} &M_j^{I} \approx (p_{ij}-p_{fj})V(\mathbf{p}_f-\mathbf{p}_i)/m\omega \equiv R_j \quad (j=1,2,3), \\ &M_j^{\dagger I} \approx (p_{fj}-p_{ij})V(\mathbf{p}_f-\mathbf{p}_i)/m\omega \equiv R_j^* \\ &\qquad\qquad \text{(Hermitian conjugate)}, \end{aligned} \tag{41}$$

and the zero-component can be calculated by using the transversality condition,

$$R_0 = \mathbf{k}\cdot\mathbf{R}/\omega. \tag{42}$$

Similarly, the nonrelativistic limit for $\mathfrak{M}^{II}$ is simply

$$M_j^{II} \approx M_j^{\dagger II} \approx m^{-1}\gamma_j V(\mathbf{p}_f-\mathbf{p}_i) \equiv Q_j, \tag{43}$$

where again the transversality condition can be used to obtain Q_0.

The major obstacle to the evaluation of (25) is still the lack of a convenient expression for K_+^V. However, the splitting of the radiation operator into $\mathfrak{M}^{I}$ and $\mathfrak{M}^{II}$ will be very helpful: In fact, only in the part $\langle\mathfrak{M}^{I}K_+^V\mathfrak{M}^{I}\rangle$ will it be necessary to use the kernel in the Coulomb field, K_+^V; in all other contributions, it will be sufficient to replace K_+^V by the propagation kernel for free electrons, K_+^0. Moreover, it will be shown that the main contribution to $\langle\mathfrak{M}^{I}K_+^V\mathfrak{M}^{I}\rangle$ arises from intermediate states of the electron with nonrelativistic energy so that both K_+^V and $\mathfrak{M}^{I}$ can be replaced by their nonrelativistic approximations which are simple. We thus get a separation into two main parts; in the first, the radiation operator is very simple and relativity can be neglected but the effect of the Coulomb field on the electron in the intermediate state must be taken into account. This part reduces essentially to the nonrelativistic Lamb shift; in the second part, the radiation operator is more complicated and must be treated relativistically, but the electrons may be considered as free in the intermediate state.

This section will serve two purposes. The first is to show that the separation into these two parts, nonrelativistic Coulomb and relativistic-free, is actually possible. In particular, it will be shown (in Subsection A) that $\langle\mathfrak{M}^{II}K_+^V\mathfrak{M}^{II}\rangle$ can be evaluated with free intermediate states, and it will be indicated (in *D*) that the same is true of the mixed term $\langle\mathfrak{M}^{I}K_+^V\mathfrak{M}^{II}\rangle$. It will further be shown (in *D*) that even in $\langle\mathfrak{M}^{I}K_+^V\mathfrak{M}^{I}\rangle$ the Coulomb potential needs to be taken into account only in nonrelativistic intermediate states, and that it is never required to take into account Coulomb potential and relativity simultaneously.

The second purpose is to evaluate those parts of the many-potential Lamb shift which are of order $\alpha(Z\alpha)^4$. They arise from $\langle\mathfrak{M}^{II}K_+^V\mathfrak{M}^{II}\rangle$ (Subsection A) and from $\langle\mathfrak{M}^{I}K_+^V\mathfrak{M}^{I}\rangle$ (Subsection B), but not from the mixed term. The $\mathfrak{M}^{II}$ contribution will cancel the non-gauge-invariant term (36) from the one-potential Lamb shift, the $\mathfrak{M}^{I}$ contribution provides the "nonrelativistic Lamb shift." In Subsection C, the relativistic correction to the Lamb shift is finally separated explicitly from the nonrelativistic terms: It is shown that the many-potential Lamb shift can be separated into terms of order $\alpha(Z\alpha)^4$ and terms of order $\alpha(Z\alpha)^5$; the former are the nonrelativistic terms evaluated in Subsections A and B; the latter represent the desired relativistic corrections and can be calculated with free intermediate states, and will be actually evaluated in Sec. V.

For our investigations, it will obviously be necessary to examine the propagation kernel K_+^V. For this purpose, we note that K_+^V can be built up of the wave functions φ_n of the electron in the Coulomb field. For instance, in the nonrelativistic case, F I Eq. (3) gives

$$\begin{aligned} K_{NR}^V(\mathfrak{x}_2,\mathfrak{x}_1) &= \textstyle\sum_n \psi_n(\mathbf{x}_2)\psi_n^*(\mathbf{x}_1)\exp[-iE_n(t_2-t_1)] \\ &\qquad\qquad \text{for } t_2-t_1>0, \\ &= 0 \qquad\qquad \text{for } t_2-t_1<0, \end{aligned} \tag{44}$$

the summation being extended over all stationary states of the aton, with energies E_n. Going to momentum space,

$$\begin{aligned} &K_{NR}^V(E_0-\omega;\,\mathbf{p}_4,\,\mathbf{p}_3) \\ &= (2\pi)^{-3}\int \exp[-i\mathbf{p}_4\cdot\mathbf{x}_2+i\mathbf{p}_3\cdot\mathbf{x}_1+i(E_0-\omega)(t_2-t_1)] \\ &\qquad\qquad \cdot K_{NR}^V(\mathfrak{x}_2,\mathfrak{x}_1)d^3\mathbf{x}_2 d^3\mathbf{x}_1 d(t_2-t_1) \\ &= -i\textstyle\sum_n \varphi_n(\mathbf{p}_4)\varphi_n^*(\mathbf{p}_3)(E_n-E_0+\omega)^{-1}, \end{aligned} \tag{45}$$

where φ_n is a momentum wave function and E_n is considered as having a small negative imaginary part. We shall refer to the states n as "intermediate states."

A. The Contribution $\langle\mathfrak{M}^{II}K_+^V\mathfrak{M}^{II}\rangle$

We shall first merely examine the intermediate states for the radiation operator $\mathfrak{M}^{II}$; and for this purpose we shall replace $\mathfrak{M}^{II}$ by the approximation Q_j, Eq. (43), with $j=1, 2, 3$. We shall disregard for the moment Q_0 and we shall use the nonrelativistic expression (45) for K_+^V. Later on (Eq. (50) ff.) we shall actually *calculate* $\langle\mathfrak{M}^{II}K_+^V\mathfrak{M}^{II}\rangle$, to order $\alpha(Z\alpha)^4$, and this will be done with the full expression for $\mathfrak{M}^{II}$ and with the relativistic

propagation kernel K_+^0. Finally, we shall prove that the use of K_+^0 rather than K_+^V is justified.

With the approximations mentioned, the contribution of $\mathfrak{M}^{II}$, or rather $\mathbf{Q}$, to (25) becomes

$$\{M^{II}\}=\frac{ie^2}{\pi^2m^2}\sum_n\int\frac{V(\mathbf{s}_1)V(\mathbf{s}_2)k^2dkd\omega d^3\mathbf{p}_1d^3\mathbf{p}_2d^3\mathbf{s}_1d^3\mathbf{s}_2}{(\omega^2-k^2-\lambda^2)(E_n-E_0+\omega)}$$
$$\cdot\bar{\varphi}_0(\mathbf{p}_2)\gamma_\mu\varphi_n(\mathbf{p}_2-\mathbf{s}_2-\mathbf{k})\bar{\varphi}_n(\mathbf{p}_1+\mathbf{s}_1-\mathbf{k})\gamma_\mu\varphi_0(\mathbf{p}_1). \quad (46)$$

The integration over ω is carried out by giving a small negative imaginary part to both λ, the mass of the photon, and E_n, the energy of the intermediate state. Then

$$\{M^{II}\}=\frac{e^2}{\pi m^2}\sum_n\int\frac{V(\mathbf{s}_1)V(\mathbf{s}_2)k^2dkd^3\mathbf{p}_1d^3\mathbf{p}_2d^3\mathbf{s}_1d^3\mathbf{s}_2}{(k^2+\lambda^2)^{\frac{1}{2}}[E_n-E_0+(k^2+\lambda^2)^{\frac{1}{2}}]}$$
$$\cdot\bar{\varphi}_0(\mathbf{p}_2)\gamma_\mu\varphi_n(\mathbf{p}_2-\mathbf{s}_2-\mathbf{k})\bar{\varphi}_n(\mathbf{p}_1+\mathbf{s}_1-\mathbf{k})\gamma_\mu\varphi_0(\mathbf{p}_1). \quad (47)$$

Clearly, λ can be replaced by zero, and the main contribution comes from large values of k. This in turn means that the momentum of the electron in the intermediate state is essentially $-\mathbf{k}$ because $\mathbf{k}$ is in general much larger than those values of $\mathbf{s}$ and $\mathbf{p}$ for which φ_0 and $V(\mathbf{s})$ are large; these are both[20] of order $Z\alpha m$. The energy E_n is then nearly $(m^2+k^2)^{\frac{1}{2}}$.

The matrix element

$$(\bar{\varphi}_n(\mathbf{p}_1+\mathbf{s}_1-\mathbf{k})\gamma_\mu\varphi_0(\mathbf{p}_1))$$

is then of order

$$(k_\mu/m)(\bar{\varphi}_n\varphi_0)$$

if $k\ll m$, and of order $(\bar{\varphi}_n\varphi_0)$ if $k>m$. Since

$$\sum_n\varphi_n(\mathbf{p}_2-\mathbf{s}_2-\mathbf{k})\bar{\varphi}_n(\mathbf{p}_1+\mathbf{s}_1-\mathbf{k})$$
$$=\delta(\mathbf{p}_1+\mathbf{s}_1+\mathbf{s}_2-\mathbf{p}_2), \quad (47a)$$

we get approximately

$$\{M^{II}\}\sim(e^2/\pi m^2)\left[\int_0^m(k^2/m^2)dk+\int_m^\infty dk\right]$$
$$\cdot\int V(\mathbf{s}_1)V(\mathbf{q}-\mathbf{s}_1)\bar{\varphi}_0(\mathbf{p}_2)\varphi_0(\mathbf{p}_1)d^3\mathbf{p}_1d^3\mathbf{p}_2d^3\mathbf{s}_1, \quad (48)$$

with $\mathbf{q}=\mathbf{p}_2-\mathbf{p}_1=\mathbf{s}_1+\mathbf{s}_2$. The main contribution comes clearly from the relativistic region.[21]

Having now shown that the important intermediate states are all of high energy, we may in first approximation regard them as free. (See below for further proof.)

[20] What matters is actually

$$\chi(\mathbf{p}_3)=\int\varphi_0(\mathbf{p}_1)V(\mathbf{p}_3-\mathbf{p}_1)d^3\mathbf{p}_1,$$

with $\mathbf{p}_3=\mathbf{p}_1+\mathbf{s}_1$; and this can be shown to be large only for $\mathbf{p}_3$ of order $Z\alpha m$.

[21] In fact, the integral (48) diverges for large k. However, we shall show presently that when the full expression (40) for M^{II} is used rather than (43), the integrand is sufficiently reduced for large k to make the integral converge.

This enables us to go back to the full expression (40) for $\mathfrak{M}^{II}$ (including the component $\mu=0$), and to evaluate its contribution to (25) by the techniques of F II. To simplify this evaluation, $\mathfrak{p}_i$ and $\mathfrak{p}_f$ in the denominators of (40) and (40a) are replaced by the momentum vector for an electron at rest, $\mathfrak{p}=(m, 0, 0, 0)$, (but in the arguments of $V(\mathbf{p}_f-\mathbf{p}_i)$, the correct momenta are retained); the error due to this approximation will be discussed below. We use the usual expression for the propagation kernel of free electrons:

$$K_+^0(E_0-\omega;\ \mathbf{p}_2-\mathbf{s}_2-\mathbf{k},\ \mathbf{p}_1+\mathbf{s}_1-\mathbf{k})$$
$$=\frac{i\delta^3(\mathbf{p}_1+\mathbf{s}_1+\mathbf{s}_2-\mathbf{p}_2)}{\mathfrak{r}-\boldsymbol{k}-m}, \quad (48a)$$

where $\mathfrak{r}$ denotes the common value of the vectors $(E_0,\ \mathbf{p}_1+\mathbf{s}_1)$ and $(E_0,\ \mathbf{p}_2-\mathbf{s}_2)$ that is the intermediate 4-momentum. We replace E_0 simply by m, and call the intermediate 3-momentum $\mathbf{s}$. Therefore

$$\mathbf{p}_1+\mathbf{s}_1=\mathbf{p}_2-\mathbf{s}_2=\mathbf{s}, \quad (48b)$$

$$\mathfrak{r}\doteq(m,\ \mathbf{s}). \quad (48c)$$

Then, the contribution of $\mathfrak{M}^{II}$ becomes

$$\{\mathfrak{M}^{II}\}\equiv\langle\mathfrak{M}^{II}K_+^0\mathfrak{M}^{II}\rangle=(e^2/\pi)\int d^3\mathbf{p}_1d^3\mathbf{p}_2d^3\mathbf{s}$$
$$\cdot\bar{\varphi}_0(\mathbf{p}_2)V(\mathbf{p}_2-\mathbf{s})N(\mathfrak{p},\ \mathbf{s})V(\mathbf{s}-\mathbf{p}_1)\varphi_0(\mathbf{p}_1), \quad (49)$$

with

$$N(\mathfrak{p},\ \mathbf{s})$$
$$=\frac{4}{i}\int\frac{(\delta_{0\mu}\boldsymbol{k}-\omega\gamma_\mu)(\mathfrak{r}-\boldsymbol{k}+m)(\delta_{0\mu}\boldsymbol{k}-\omega\gamma_\mu)}{(\mathfrak{k}^2-2\mathfrak{p}\cdot\mathfrak{k})^2(\mathfrak{k}^2-2\mathfrak{r}\cdot\mathfrak{k}-s^2)\mathfrak{k}^2}d^4\mathfrak{k}_F. \quad (49a)$$

The integral in (49a) is to be evaluated with $\mathfrak{p}=(m, 0)$.

The integral (49a) can be evaluated for any $\mathbf{s}$, using the methods of Sec. V. The result can be expanded in a power series in $\mathbf{s}$, starting with a constant term and continuing with a term proportional to s^2. If the constant term is inserted into (49), one obtains a result of order $\alpha(Z\alpha)^4$, while the term in s^2 (and the higher terms) give results of order $\alpha(Z\alpha)^5$ or less. Moreover, the latter terms get their main contribution from *large* values of s, of order m, while the constant term comes mostly from small values of s, of order αZm. Therefore the exact momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ contained in φ_0 and $\bar{\varphi}_0$ will matter for the constant term but not for the higher ones; for the latter terms, only the integral $\int\varphi_0(\mathbf{p}_1)d^3\mathbf{p}_1$ matters. This is the approximation which will be made in Sec. V, and this approximation is thus shown to be sufficient for all but the constant term.

We shall therefore need to evaluate here only the constant term, whose contribution to (49) will be denoted by $\{\mathfrak{M}^{II}\}_0$. This term is obtained by setting $\mathbf{s}=0$ in (49a), which simplifies this expression greatly and

makes it into

$$N(\mathfrak{p},0)$$
$$=\frac{4}{i}\int\frac{(\delta_{0\mu}\boldsymbol{k}-\omega\gamma_\mu)(\boldsymbol{p}-\boldsymbol{k}+m)(\delta_{0\mu}\boldsymbol{k}-\omega\gamma_\mu)}{(\mathfrak{k}^2-2\mathfrak{p}\cdot\mathfrak{k})^3\mathfrak{k}^2}d^4\mathfrak{k}_F$$
$$=4(\gamma_\sigma\delta_{0\mu}-\delta_{0\sigma}\gamma_\mu)(\boldsymbol{p}+m)(\gamma_\tau\delta_{0\mu}-\delta_{0\tau}\gamma_\mu)J^0{}_{\sigma\tau} \quad (49b)$$
$$-4(\gamma_\sigma\delta_{0\mu}-\delta_{0\sigma}\gamma_\mu)\gamma_\omega(\gamma_\tau\delta_{0\mu}-\delta_{0\tau}\gamma_\mu)J^0{}_{\sigma\tau\omega},$$

where $J^0{}_{\sigma\tau}$ and $J^0{}_{\sigma\tau\omega}$ are two tensors given by the following integrals:

$$J^0{}_{(\sigma\tau;\sigma\tau\omega)}$$
$$=-i\int(k_\sigma k_\tau;\, k_\sigma k_\tau k_\omega)(\mathfrak{k}^2-2\mathfrak{p}\cdot\mathfrak{k})^{-3}\mathfrak{k}^{-2}d^4\mathfrak{k}_F. \quad (49c)$$

Integrals of this sort will be calculated later, in Sec. V, and $J^0{}_{(\sigma\tau;\sigma\tau\omega)}$ is seen to be a particular case of $J_{(\sigma\tau;\sigma\tau\omega)}$ given by Eq. (75), where $\mathbf{s}$ is set equal to 0 and therefore $\mathfrak{r}$ replaced by $\mathfrak{p}$. From formulas (75a; 75b) for $J_{(\sigma\tau;\sigma\tau\omega)}$, one obtains immediately

$$J^0{}_{\sigma\tau}=\tfrac{1}{8}m^{-2}(\delta_{0\sigma}\delta_{0\tau}-\tfrac{1}{2}\delta_{\sigma\tau}),$$
$$J^0{}_{\sigma\tau\omega}=\tfrac{1}{16}m^{-1}[\delta_{0\sigma}\delta_{0\tau}\delta_{0\omega}$$
$$-\tfrac{1}{2}(\delta_{0\sigma}\delta_{\tau\omega}+\delta_{0\tau}\delta_{\sigma\omega}+\delta_{0\omega}\delta_{\sigma\tau})]. \quad (49d)$$

Carrying these values back in (49b) and taking the expectation value for the state of rest of the electron, which in this case amounts simply to replacing γ_0 by 1, one finds

$$N(\mathfrak{p},0)=-3/4m. \quad (50)$$

Thus the result for $\{\mathfrak{M}^{\mathrm{II}}\}_0$ is

$$\{\mathfrak{M}^{\mathrm{II}}\}_0=-(3\alpha/4\pi m)$$
$$\cdot\int\bar{\varphi}_0(\mathbf{p}_2)\varphi_0(\mathbf{p}_1)V(\mathbf{p}_2-\mathbf{s})V(\mathbf{s}-\mathbf{p}_1)d^3\mathbf{p}_1d^3\mathbf{p}_2d^3\mathbf{s}. \quad (51)$$

To evaluate (51) we note that

$$\int d^3\mathbf{s}V(\mathbf{p}_2-\mathbf{s})V(\mathbf{s}-\mathbf{p}_1)=(2\pi)^{-3}\int V^2(\mathbf{x})e^{-i\mathbf{q}\cdot\mathbf{x}}d^3\mathbf{x}, \quad (52)$$

which is the Fourier component $\mathbf{q}$ of the *square* of the spatial potential where $\mathbf{q}=\mathbf{p}_2-\mathbf{p}_1=\mathbf{s}_1+\mathbf{s}_2$. Changing (51) back to an integral in coordinate space, one obtains easily

$$\{\mathfrak{M}^{\mathrm{II}}\}_0=-(3\alpha/4\pi m)\int\bar{\psi}_0(\mathbf{x})\psi_0(\mathbf{x})V^2(\mathbf{x})d^3\mathbf{x}$$
$$=-(3\alpha/4\pi m)\langle\varphi_0{}^*|\beta V^2|\varphi_0\rangle. \quad (53)$$

This is a complete evaluation of $\{\mathfrak{M}^{\mathrm{II}}\}_0$, subject to the justification of our approximations which we shall give presently. It will be noted that our result, (53), cancels exactly the term (36) from the one-potential Lamb shift and thus removes the lack of gauge invariance which that term introduced.

We have made two errors in this evaluation of $\mathfrak{M}^{\mathrm{II}}$, namely (a) the replacement of $\mathfrak{p}_i$ and $\mathfrak{p}_f$ by $\mathfrak{p}$, corresponding to a free electron at rest, in the evaluation of the integral over $\mathfrak{k}$, and (b) the neglect of the potential in the intermediate states. Concerning (a), we note that the functions $\varphi_0(\mathbf{p}_1)$ and $V(\mathbf{s}_1)$ are large only if $\mathbf{p}_1$ and $\mathbf{s}_1$ are of order $Z\alpha$. (See also footnote 20.) Therefore $\mathfrak{p}_i$ and $\mathfrak{p}_f$ in (40) differ from $\mathfrak{p}$ by an amount of order $Z\alpha$ in the spatial components, and $(Z\alpha)^2$ in the time component. Clearly, the relative error introduced by this can only be of order:

(Correction to spatial momentum/
undisturbed momentum in intermediate state)^2

or

(Correction to energy/
unperturbed energy in intermediate state),

both of which are of order $(Z\alpha)^2/k^2$. Since the contribution of a given k to (49) goes as k^2dk for $k<m$, the error will be proportional to

$$(Z\alpha)^2\int_0^m dk=\mathcal{O}[(Z\alpha)^2]. \quad (54)$$

Since the main term (53) is of order $\alpha(Z\alpha)^4$, an error of relative order $(Z\alpha)^2$ will be of absolute order $\alpha(Z\alpha)^6$ which has been consistently neglected in this paper.

To evaluate the error due to the neglect of the potential energy in the intermediate state, we imagine that one extra potential $V(\mathbf{s}_3)$ acts between $\mathfrak{M}^{\mathrm{II}\dagger}$ and $\mathfrak{M}^{\mathrm{II}}$. This introduces an extra factor of order

$$\int d^3\mathbf{s}_3Ze^2/s_3{}^2k, \quad (55)$$

where k represents the additional energy denominator. Now $\mathbf{s}_3$ must again be of order αZ, in order that $\mathbf{p}_1+\mathbf{s}_1$ and $\mathbf{p}_2-\mathbf{s}_2$ be of that order, and therefore we get a correction of order

$$(Z\alpha)^2\int^m kdk\sim(Z\alpha)^2, \quad (56)$$

which is again negligible.

We have thus shown that, in the evaluation of $\mathfrak{M}^{\mathrm{II}}$, the intermediate state may indeed be regarded as free, which justifies the result (53).

B. The Contribution $\langle\mathfrak{M}^{\mathrm{I}}K_+{}^{V}\mathfrak{M}^{\mathrm{I}\dagger}\rangle$

The contribution of $\mathfrak{M}^{\mathrm{I}}$ from *relativistic* intermediate states is only of order $\alpha(Z\alpha)^5$. That $\mathfrak{M}^{\mathrm{I}}$ for relativistic states is less important than $\mathfrak{M}^{\mathrm{II}}$, is easily seen for the main part, $\mathfrak{R}$, of $\mathfrak{M}^{\mathrm{I}}$ as follows: If we choose ω of order m in (41), then $\mathbf{R}$ differs from $\mathbf{Q}$, Eq. (43), by a factor

$(\mathbf{p}_f-\mathbf{p}_i)/m$ which is small (of order $Z\alpha$). The second part of $\mathfrak{M}^{\mathrm{I}}$, Eq. (40), is similarly small.

On the other hand, $\mathbf{R}$ contains the factor ω in its denominator and can therefore be expected to be large in the *nonrelativistic* region. Here the potential must be taken into account in the intermediate states φ_n. However, for all intermediate states for which V matters, nonrelativistic wave functions may be used, and moreover, $\mathfrak{M}^{\mathrm{I}}$ may be replaced by its nonrelativistic limit, $\mathfrak{R}$. The validity of this approximation will be proved in Subsection D.

We can make the further approximation of neglecting the recoil of the electron due to the emission of the virtual quantum. That is, we shall take the momentum of the electron in the intermediate state to be $\mathbf{p}_1+\mathbf{s}_1$ or $\mathbf{p}_2-\mathbf{s}_2$, rather than $\mathbf{p}_1+\mathbf{s}_1-\mathbf{k}$ or $\mathbf{p}_2-\mathbf{s}_2-\mathbf{k}$. This amounts to leaving out the retardation factor, $e^{i\mathbf{k}\cdot\mathbf{x}}$, in the spatial integral. The justification for this is that the momentum of the quantum, $\mathbf{k}$, is generally small compared with the other part of the electron momentum, $\mathbf{s}=\mathbf{p}_1+\mathbf{s}_1$. To see this, consider Eq. (60a), below: For a given energy E_n of the intermediate state, the main contribution comes from quanta whose energy k is of the order of E_n-E_0. But for nonrelativistic states, E_n-E_0 is of the order of magnitude of $s^2/2m$, where s is the predominant electron momentum in the intermediate state n. The quantum momentum k is therefore of the order of $s^2/2m$, which is small compared with the electron momentum s for nonrelativistic states, for which $s\ll m$ by definition. We are therefore justified in neglecting retardation, along with relativity, in our present first approximation.

Using the nonrelativistic kernel in the Coulomb field, (45), and neglecting retardation, the contribution of $\mathfrak{R}$, Eq. (41), to the many-potential Lamb shift (25) becomes:

$$\begin{aligned}\langle\mathfrak{R}K_{NR}{}^{V}\mathfrak{R}\rangle=&-\frac{e^2}{\pi i}\sum_n\int\frac{d^4\mathfrak{k}_F d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{s}_1 d^3\mathbf{s}_2}{(\mathfrak{k}^2-\lambda^2)(\omega+E_n-E_0)}\\&\cdot\varphi_0^*(\mathbf{p}_2)R_\mu(\mathbf{s}_2)\varphi_n(\mathbf{p}_2-\mathbf{s}_2)\\&\cdot\varphi_n^*(\mathbf{p}_1+\mathbf{s}_1)R_\mu^*(\mathbf{s}_1)\varphi_0(\mathbf{p}_1)\end{aligned}\tag{57}$$

$$=-\frac{e^2}{\pi i}\int\frac{d^4\mathfrak{k}_F}{\mathfrak{k}^2-\lambda^2}\sum_n\frac{\sum_\mu|\langle\varphi_n|R_\mu|\varphi_0\rangle|^2}{\omega+E_n-E_0}.\tag{57a}$$

According to (42), we can replace $\sum_\mu|\langle\varphi_n|R_\mu|\varphi_0\rangle|^2$ by

$$\omega^{-2}|\langle\varphi_n|\mathbf{R}|\varphi_0\rangle\cdot\mathbf{k}|^2-|\langle\varphi_n|\mathbf{R}|\varphi_0\rangle|^2.\tag{57b}$$

The integral on the angles of $\mathbf{k}$ will introduce a factor $\frac{1}{3}$ in the first term.

$$\langle\mathfrak{R}K_{NR}{}^{V}\mathfrak{R}\rangle=\frac{e^2}{\pi^2 i}\int_0^\infty k^2dk\int_{-\infty}^{+\infty}\frac{d\omega}{\mathfrak{k}^2-\lambda^2}\times\sum_n\frac{|\langle\varphi_n|\mathbf{R}|\varphi_0\rangle|^2}{\omega+E_n-E_0}\left(1-\frac{1}{3}\frac{k^2}{\omega^2}\right).\tag{58}$$

$\langle\varphi_n|\mathbf{R}|\varphi_0\rangle$ is found from (41):

$$\begin{aligned}\langle\varphi_n|\mathbf{R}|\varphi_0\rangle&=(m\omega)^{-1}\langle\varphi_n|\mathbf{p}V-V\mathbf{p}|\varphi_0\rangle\\&=(m\omega)^{-1}\langle\varphi_n|\mathbf{p}H-H\mathbf{p}|\varphi_0\rangle\\&=-(m\omega)^{-1}(E_n-E_0)\langle\varphi_n|\mathbf{p}|\varphi_0\rangle,\end{aligned}\tag{59}$$

H being the Hamiltonian,

$$H=\mathbf{p}^2/2m+V.$$

$$\langle\mathfrak{R}K_{NR}{}^{V}\mathfrak{R}\rangle=(e^2/i\pi^2m^2)\sum_n(E_n-E_0)^2|\langle\varphi_n|\mathbf{p}|\varphi_0\rangle|^2\cdot\int_0^\infty k^2dk\int_{-\infty}^{+\infty}\frac{(1-k^2/3\omega^2)d\omega}{\omega^2(\omega^2-k^2-\lambda^2)(\omega+E_n-E_0)}.\tag{60}$$

The integration over ω is performed by giving a small negative imaginary part to both λ, the mass of the photon, and E_n, the energy of the intermediate state. The imaginary part of $\langle\mathfrak{R}K_{NR}{}^{V}\mathfrak{R}\rangle$ gives the lifetime of the state φ_0 for decay into φ_n by real photon emission, if there is a state $E_n<E_0$ which combines optically with state φ_0. Only the real part is wanted here. Integration[22] over ω gives for the k integral

$$\pi i\int_0^\infty k^2\omega_0^{-3}(\omega_0+E_n-E_0)^{-1}(1-k^2/3\omega_0^2)dk,\tag{60a}$$

where $\omega_0=(k^2+\lambda^2)^{\frac{1}{2}}$. The further integration over k yields, in the limit λ very small:

$$\langle\mathfrak{R}K_{NR}{}^{V}\mathfrak{R}\rangle=\frac{2e^2}{3\pi m^2}\sum_n|\langle\varphi_n|\mathbf{p}|\varphi_0\rangle|^2\times(E_n-E_0)\left(\ln\frac{\lambda}{2|E_n-E_0|}+\frac{5}{6}\right),\tag{61}$$

[22] The pole at $\omega=0$ is slightly disturbing. However, if the complete denominator in (40) had been used, the $2m\omega$ of the denominator in (41) would be replaced by

$$2\mathfrak{p}\cdot\mathfrak{k}-\mathfrak{k}^2=2E_0\omega-2\mathbf{p}\cdot\mathbf{k}-\omega^2+\mathbf{k}^2,$$

resulting in two poles, located at

$$\omega=E_0\pm(\mathbf{k}^2-2\mathbf{p}\cdot\mathbf{k}+E_0^2)^{\frac{1}{2}}.$$

According to the convention made at the end of Sec. II, we are supposed to integrate on a contour passing below the left-hand pole (corresponding to the minus sign) and above the right-hand one. When we make our nonrelativistic approximation of considering $\mathbf{p}$, $\mathbf{k}$, and ω small compared to E_0, the left-hand pole is the one that tends toward $\omega=0$, while we neglect the contribution from the right-hand one. This shows that the pole $\omega=0$, arising from the first factor ω^2 in the denominator of (60), must be considered as lying above the contour of integration. Therefore, if the integral is carried out around the negative-imaginary half-plane (which is convenient anyway, since it also avoids getting a contribution from $\omega=E_0-E_n$), no contribution arises from this pole.

There remains the last term in (60), $k^2/3\omega^2$, arising from the elimination of the longitudinal waves by means of (42). The nature of the pole there can be understood if one uses for $M_\mu{}^{\mathrm{I}}$, (40), an approximation slightly better than $\mathfrak{R}$, (41), namely

$$(p_{i\mu}/\mathfrak{p}_i\cdot\mathfrak{k}-p_{f\mu}/\mathfrak{p}_f\cdot\mathfrak{k})V(\mathbf{p}_f-\mathbf{p}_i).\quad(\mu=0,1,2,3).$$

This satisfies the transversality condition (37), and the 0 component is

$$E_0(\mathbf{p}_i-\mathbf{p}_f)\cdot\mathbf{k}(E_0\omega-\mathbf{p}_i\cdot\mathbf{k})^{-1}(E_0\omega-\mathbf{p}_f\cdot\mathbf{k})^{-1}V(\mathbf{p}_f-\mathbf{p}_i),$$

which reduces to $\mathbf{k}\cdot\mathbf{R}/\omega$ for $\mathbf{p}$ very small. But in this form it is evident that the pole in the last term of (60) is of the same nature as the one treated in the first part of this footnote, and therefore gives no contribution for the contour chosen.

that is, precisely the nonrelativistic Lamb shift (2), the relationship between K and λ being given by (3).

C. Explicit Separation of Relativistic Lamb Shift

We have now evaluated the contributions of order $\alpha(Z\alpha)^4$, arising from the interactions $\mathfrak{R}$ and $\mathfrak{M}^{II}$. We shall now consider the *difference* between the full contribution from $\mathfrak{M}$, Eq. (40), and the contributions from $\mathfrak{R}$ and $\mathfrak{M}^{II}$ as calculated in Subsections A and B. This difference is of order $\alpha(Z\alpha)^5$ because it consists of the relativistic contributions to $\mathfrak{M}^{I}$ and $\mathfrak{M}^{II}$ which were shown to be of this order in Subsections B and A, and of the interference term between $\mathfrak{M}^{I}$ and $\mathfrak{M}^{II}$ which will be discussed at the end of D. Both types of terms arise mainly from intermediate states of relativistic energy. For such states, the influence of the Coulomb potential may be neglected, being of relative order $Z\alpha$ and hence absolute order $\alpha(Z\alpha)^6$. (For further discussion, see Subsection D.) It is therefore permissible to calculate the *difference*

$$D\equiv\langle\mathfrak{M}K_+{}^V\mathfrak{M}\rangle-\langle\mathfrak{R}K_+{}^V\mathfrak{R}\rangle-\langle\mathfrak{M}^{II}K_+{}^V\mathfrak{M}^{II}\rangle_0, \quad (62)$$

by replacing the kernel K^V by K^0, the propagation kernel for free electrons, thus:

$$D=\langle\mathfrak{M}K_+{}^0\mathfrak{M}\rangle-\langle\mathfrak{R}K_+{}^0\mathfrak{R}\rangle-\langle\mathfrak{M}^{II}K_+{}^0\mathfrak{M}^{II}\rangle_0. \quad (62a)$$

Thus, at last, we have realized the purpose of our separation of the Lamb shift into various parts: The difference D can be calculated using the usual methods of F II.

Our aim is, of course, to calculate the total many-potential Lamb shift, which may be written:

$$\langle\mathfrak{M}K_+{}^V\mathfrak{M}\rangle=\langle\mathfrak{R}K_+{}^V\mathfrak{R}\rangle+\langle\mathfrak{M}^{II}K_+{}^V\mathfrak{M}^{II}\rangle_0+D. \quad (62b)$$

Of the terms on the right-hand side, D will be calculated later on the basis of Eq. (62a). The second term on the right has been evaluated in (53); the subscript 0 in this term denotes, as it did in (51) to (53), that only the leading term (first term), (50), of the expansion of (49a) in powers of s^2 is being considered. The first term in (62b), however, has not been actually evaluated in Subsection B because we have used there nonrelativistic rather than relativistic intermediate states in the Coulomb field. However, we shall show in Subsection D that the Coulomb field and the relativity correction need not be taken into account *simultaneously* for the same intermediate state; therefore we may write

$$\langle\mathfrak{R}K_+{}^V\mathfrak{R}\rangle=\langle\mathfrak{R}K_{NR}{}^V\mathfrak{R}\rangle+\langle\mathfrak{R}K_+{}^0\mathfrak{R}\rangle-\langle\mathfrak{R}K_{NR}{}^0\mathfrak{R}\rangle. \quad (63)$$

Here $\langle\mathfrak{R}K_{NR}{}^V\mathfrak{R}\rangle$ denotes the contribution of $\mathfrak{R}$ as evaluated in Subsection B, i.e., with the intermediate states treated in the Coulomb field but nonrelativistically. $\langle\mathfrak{R}K_+{}^0\mathfrak{R}\rangle$ is evaluated with free relativistic intermediate states, and $\langle\mathfrak{R}K_{NR}{}^0\mathfrak{R}\rangle$ with free *nonrelativistic* intermediate states. Inserting (63) and (62a) into (62b), the term $\langle\mathfrak{R}K_+{}^0\mathfrak{R}\rangle$ cancels[23] and we obtain

$$\langle\mathfrak{M}K_+{}^V\mathfrak{M}\rangle=\langle\mathfrak{M}^{II}K_+{}^V\mathfrak{M}^{II}\rangle_0+\langle\mathfrak{R}K_{NR}{}^V\mathfrak{R}\rangle+D', \quad (64)$$

$$D'=\langle\mathfrak{M}K_+{}^0\mathfrak{M}\rangle-\langle\mathfrak{R}K_{NR}{}^0\mathfrak{R}\rangle-\langle\mathfrak{M}^{II}K_+{}^0\mathfrak{M}^{II}\rangle_0 \equiv A-B-C. \quad (65)$$

The first term in (64) cancels (36), as was remarked after (53); the second is the ordinary nonrelativistic Lamb shift (61); both terms are strictly of order $\alpha(Z\alpha)^4$ and do not contain any component of order $\alpha(Z\alpha)^5$. Therefore D' *represents the relativistic correction which is now completely separated from the nonrelativistic parts.* From our previous discussion it follows that D' is of order $\alpha(Z\alpha)^5$. It can be evaluated by taking exclusively free intermediate states, i.e., the relativistic correction has been reduced to an expression in which the potential acts twice and only twice, once with the emission and once with the absorption of the virtual quantum.

Since D' arises entirely from high-energy intermediate states, a further simplification is possible, namely D' can be evaluated with the initial and final state of the electron assumed to be free states of momentum zero. Indeed, using the same argument as for error (a) in Subsection A, we find that the error due to this assumption is expected to be only of relative order $(Z\alpha)^2$.

We shall understand from now on that all three terms in (65) are to be evaluated with zero momentum in initial and final state. The first term in (65), A, will be evaluated in Sec. V. In this section, we shall still calculate the last two terms in (65). The contribution of $\mathfrak{M}^{II}$, with initial and final momentum set equal to zero, is obtained from (51) by considering $\mathbf{p}_1$ and $\mathbf{p}_2$ small compared to $\mathbf{s}$. Now, from the definition (8a),

$$\int\varphi_0(\mathbf{p}_1)d^3\mathbf{p}_1=(2\pi)^{\frac{3}{2}}\psi_0(0), \quad (66)$$

i.e., the wave function in coordinate space at the origin, and from (8b),

$$V(\pm\mathbf{s})=-Ze^2/2\pi^2s^2. \quad (66a)$$

Thus the approximation to (51) for $\mathbf{p}_1=\mathbf{p}_2=0$ becomes

$$C=-6\alpha(Z\alpha)^2(\pi m)^{-1}\psi_0{}^2(0)\int ds/s^2. \quad (67)$$

This integral clearly diverges at $s=0$, but this is just what will be needed to cancel a similar divergence in term A (see Sec. V).

Next we must calculate B, i.e., the contribution of $\mathfrak{R}$, evaluated with free nonrelativistic intermediate states and with initial and final state having momentum zero. To do this, we replace in Eq. (57) (E_n-E_0) by $s^2/2m$; then the summation over n can be carried out and gives $\delta(\mathbf{p}_1+\mathbf{s}_1-\mathbf{p}_2+\mathbf{s}_2)$, which permits evaluation

[23] This is fortunate because this term would diverge for large energy of the intermediate state.

of $\int d^3\mathbf{s}_2$. Further, since $\mathbf{p}_2 \approx \mathbf{p}_1 \approx 0$, we may set $\mathbf{s}_2 = -\mathbf{s}_1$ in R_μ; we also drop the subscript in $\mathbf{s}_1$. From (41), $-\mathbf{R} = \mathbf{R}^* = \mathbf{s}V(\mathbf{s})/(m\omega)$, and we obtain:

$$B = \frac{e^2}{i\pi m^2}\left|\int \varphi_0(\mathbf{p})d^3\mathbf{p}\right|^2 \int \frac{V^2(\mathbf{s})[s^2-(\mathbf{s}\cdot\mathbf{k})^2/\omega^2]d^4\mathfrak{f}_F d^3\mathbf{s}}{\omega^2(\omega+s^2/2m)(\mathfrak{f}^2-\lambda^2)}$$

$$= \frac{8\alpha(Z\alpha)^2}{\pi}|\psi_0(0)|^2\int_0^\infty \frac{ds}{s^2}\frac{4}{3m}\left(\ln\frac{m\lambda}{s^2}+\frac{5}{6}\right). \tag{68}$$

To obtain the last line, the integration over ω and $\mathbf{k}$ has been carried out in analogy with that leading from (60) to (61). The integral (68), just as (67), diverges for $s=0$, but just as in that case, this divergent term will exactly cancel a similar divergence in contribution A of (65).

D. Further Discussion of Approximations

We shall now show (1) that the potential has been taken into account in all intermediate states for which this is required, and (2) that it is not necessary for any intermediate state to consider potential and relativity simultaneously.

We have already shown in Subsection A that the potential in the intermediate states need not be taken into account for $\langle \mathfrak{M}^{II} K_+{}^V \mathfrak{M}^{II}\rangle$. The mixed terms, $\langle \mathfrak{M}^{II} K_+{}^V \mathfrak{M}^{I}\rangle$ and $\langle \mathfrak{M}^{I} K_+{}^V \mathfrak{M}^{II}\rangle$ will be discussed briefly at the end of this Subsection. We shall therefore now take up once more the term $\langle \mathfrak{M}^{I} K_+{}^V \mathfrak{M}^{I}\rangle$ (see also Subsection B) and shall examine in detail how strongly the contribution of intermediate states of various energies is influenced by the Coulomb potential in the intermediate state.

As has been shown already above, Eq. (57), for any given nonrelativistic intermediate state E_n, the main contribution comes from quanta k which are of order $E_n - E_0$ and are therefore small compared to the predominant *momentum* in the intermediate state; since $\mathbf{p}_1$ is mostly of order $Z\alpha$, it follows that $\mathbf{p}_n$ is essentially equal to $\mathbf{s}_1$, i.e., the momentum in the intermediate state comes mainly from the potential acting in R_μ.

The relative effect of the potential V on the wave function φ_n of the intermediate state is measured by the well-known quantity Ze^2/hv which, in our units, is $Z\alpha/p_n \approx Z\alpha/s$. The contribution of $\mathfrak{R}$ for *free* intermediate states is given by (68); and since $\psi^2(0) \approx (Z\alpha)^3$ (see end of Sec. V), the order of magnitude of (68) is given by

$$B \sim Z^5\alpha^6 \int ds/s^2. \tag{69}$$

The Coulomb effect on the intermediate state wave function gives, as we have seen, a correction of relative order $Z\alpha/s$ which therefore amounts to

$$B_c \sim Z^6\alpha^7 \int ds/s^3. \tag{70}$$

We are considering in this paper contributions of the order $Z^5\alpha^6$. In order to get a contribution of this order or greater from (70), we must consider values of s smaller than $(Z\alpha)^{\frac{1}{2}}$, or excitation energies less than $\frac{1}{2}Z\alpha m = 137$ Rydberg, which is intermediate between the Rydberg energy $\frac{1}{2}(Z\alpha)^2 m$ and the rest mass of the electron. We thus find that, in our order of approximation, the Coulomb potential needs to be taken into account only for intermediate states of less energy than $137Z$ Rydberg; in particular, it does not need to be considered for states of relativistic energy.

This consideration already shows that it is presumably unnecessary to apply a relativity correction simultaneously with a Coulomb correction. To put this argument in quantitative form, we note that the relativity correction for an intermediate state of momentum $\mathbf{p}_n$ is of relative order $(p_n/m)^2$, or in our case about s^2/m^2. Therefore the relativity correction to the Coulomb effect (70) is

$$B_{cr} \sim Z^6\alpha^7 \int ds/s \sim Z^6\alpha^7 \ln Z\alpha, \tag{71}$$

which is negligible in our approximation.

In Subsection B, *retardation* was left out along with relativity. We must therefore show that also retardation does not need to be considered simultaneously with the potential. Retardation changes the momentum of the electron by $k \sim s^2/2m$, and since the "unretarded" electron momentum is $\mathbf{s}$, the relative change is of order s/m. Since the vector $\mathbf{k}$ may have any direction relative to $\mathbf{s}$, the effect of retardation will be of order $k^2/s^2 \sim (s/m)^2$. This is exactly the same order as the relativistic correction, and is therefore negligible for all those intermediate states for which the potential must be taken into account.

We now examine whether it was justified to replace $\mathfrak{M}^I$ by $\mathfrak{R}$ for the evaluation of the effect of the Coulomb potential in the intermediate state. The difference $\mathfrak{M}^I - \mathfrak{R}$ is of order

$$k\gamma_\mu(\mathfrak{p}_f - \mathfrak{p}_i)\cdot\mathfrak{f}V/(2m^2\omega^2), \tag{71a}$$

and its ratio to R_μ is therefore about

$$k\gamma_\mu k/(m\omega) \sim k_\mu/m, \tag{72}$$

because $k \approx \omega$. Since we have already seen that the important values of k are of order $E_n - E_0 \sim s^2$, the contribution of $\mathfrak{M}^I - \mathfrak{R}$, for intermediate states for which the Coulomb forces matter, is of the same order as the relativistic correction to $\mathfrak{R}$ and hence negligible.

Finally, it can be shown by direct evaluation that the cross term between $\mathfrak{M}^I$ and $\mathfrak{M}^{II}$ does not give a contribution for low-energy intermediate states. This is

due to the fact that the matrix elements of $\mathfrak{M}^{II}$ to non-relativistic states are extremely small. The cross term gives altogether a result of order α^6Z^5. The potential correction to the cross term can again be shown to be of order $Z^6\alpha^7 \log Z\alpha$.

V. RELATIVISTIC CORRECTIONS TO THE MANY-POTENTIAL LAMB SHIFT

According to (65) and other discussions in the preceding section, the corrections of order $\alpha(Z\alpha)^5$ to the Lamb shift are given by $A-B-C$. B is given by (68) and C by (67). A is given by (25) where the electron is assumed at rest initially and finally, E_0 is replaced by m, and the intermediate state is taken to be a free state, that is $K_+{}^V$ is replaced by the free kernel (48a)

$$i(r-k-m)^{-1}\delta^3(\mathbf{s}_2+\mathbf{s}_1),$$

where $\mathfrak{r}$ is defined by (48c). Therefore

$$A=8\pi^3|\psi(0)|^2\frac{e^2}{\pi i}\int d^3\mathbf{s}\left\langle\int M_\mu\frac{1}{r-k-m}M_\mu{}^\dagger\frac{d^4\mathfrak{f}_F}{\mathfrak{f}^2-\lambda^2}\right\rangle_0,$$

where ${}_0\langle\cdots\rangle_0$ means that the expectation value is taken for the state of rest of the electron. Using for $M_\mu{}^\dagger$ expression (24a), with $\mathfrak{p}_i=(m,0)=\mathfrak{p}$, $\mathfrak{p}_f=\mathfrak{r}$, and using for M_μ expression (24) with $\mathfrak{p}_i=\mathfrak{r}$, $\mathfrak{p}_f=\mathfrak{p}$, this can also be written

$$A=8\alpha(Z\alpha)^2\pi^{-1}|\psi(0)|^2\int_0^\infty ds(T_1+T_2+T_3)/s^2, \tag{73}$$

$$T_1=\left\langle\frac{1}{i}\int\frac{(2p_\mu-\gamma_\mu k)\beta(r-k+m)\beta(2p_\mu-k\gamma_\mu)}{(\mathfrak{f}^2-2\mathfrak{p}\cdot\mathfrak{f})^2(\mathfrak{f}^2-2\mathfrak{r}\cdot\mathfrak{f}-s^2)(\mathfrak{f}^2-\lambda^2)}d^4\mathfrak{f}_F\right\rangle_0,$$

$$T_3=\left\langle\frac{1}{i}\int\frac{\beta(2r_\mu-\gamma_\mu k)(\mathfrak{r}-k+m)(2r_\mu-k\gamma_\mu)\beta}{(\mathfrak{f}^2-2\mathfrak{r}\cdot\mathfrak{f})^2(\mathfrak{f}^2-2\mathfrak{r}\cdot\mathfrak{f}-s^2)(\mathfrak{f}^2-\lambda^2)}d^4\mathfrak{f}_F\right\rangle_0,$$

$$T_2=\left\langle-\frac{1}{i}\int\frac{\beta(2r_\mu-\gamma_\mu k)(r-k+m)\beta(2p_\mu-k\gamma_\mu)+(2p_\mu-\gamma_\mu k)\beta(r-k+m)(2r_\mu-k\gamma_\mu)\beta}{(\mathfrak{f}^2-2\mathfrak{p}\cdot\mathfrak{f})(\mathfrak{f}^2-2\mathfrak{r}\cdot\mathfrak{f})(\mathfrak{f}^2-2\mathfrak{r}\cdot\mathfrak{f}-s^2)(\mathfrak{f}^2-\lambda^2)}d^4\mathfrak{f}_F\right\rangle_0.$$

All poles are to be resolved by giving m and λ small negative imaginary parts. The integrals can be evaluated by using the methods in the appendix of F II. As an illustration, we consider T_1 in some detail.

$$\begin{aligned}T_1={}_0\langle&4m^2\beta(r+m)\beta J_0-2[\gamma_\mu\gamma_\sigma\beta(r+m)\beta p_\mu\\&+p_\mu\beta(r+m)\beta\gamma_\sigma\gamma_\mu+2m^2\beta\gamma_\sigma\beta]J_\sigma+[2\gamma_\mu\gamma_\sigma\beta\gamma_\tau\beta p_\mu\\&+2p_\mu\beta\gamma_\tau\beta\gamma_\sigma\gamma_\mu+\gamma_\mu\gamma_\sigma\beta(r+m)\beta\gamma_\tau\gamma_\mu]J_{\sigma\tau}\\&-\gamma_\mu\gamma_\sigma\beta\gamma_\tau\beta\gamma_\omega\gamma_\mu J_{\sigma\tau\omega}\rangle_0,\end{aligned} \tag{74}$$

with the definitions

$$J_{(0;\sigma;\sigma\tau;\sigma\tau\omega)}=\frac{1}{i}\int\frac{(1;k_\sigma;k_\sigma k_\tau;k_\sigma k_\tau k_\omega)d^4\mathfrak{f}_F}{(\mathfrak{f}^2-2\mathfrak{p}\cdot\mathfrak{f})^2(\mathfrak{f}^2-2\mathfrak{r}\cdot\mathfrak{f}-s^2)(\mathfrak{f}^2-\lambda^2)}$$

$$=\frac{6}{i}\int_0^1(1-y)dy\int_0^1 x^2dx$$

$$\times\int\frac{(1;k_\sigma;k_\sigma k_\tau;k_\sigma k_\tau k_\omega)d^4\mathfrak{f}_F}{[\mathfrak{f}^2-2x\mathfrak{p}_y\cdot\mathfrak{f}-xys^2-(1-x)\lambda^2]^4}, \tag{75}$$

$$\mathfrak{p}_y=y\mathfrak{r}+(1-y)\mathfrak{p}=(m,y\mathbf{s}),$$

$$\mathfrak{p}_y{}^2=m^2-y^2s^2.$$

The integral over $\mathfrak{f}$ is performed using formula (13a) of the appendix of F II, and also the following one:

$$\frac{24}{i}\int\frac{k_\sigma k_\tau k_\omega d^4\mathfrak{f}_F}{(\mathfrak{f}^2-2\mathfrak{p}\cdot\mathfrak{f}-\Delta)^4}=\frac{p_\sigma p_\tau p_\omega}{(\mathfrak{p}^2+\Delta)^2}-\frac{1}{2}\frac{p_\sigma\delta_{\tau\omega}+p_\tau\delta_{\sigma\omega}+p_\omega\delta_{\sigma\tau}}{\mathfrak{p}^2+\Delta}.$$

The result is

$$J_0=\frac{1}{4}\int_0^1(1-y)dy\int_0^1 dx\frac{x^2}{[x^2\mathfrak{p}_y{}^2+xys^2+(1-x)\lambda^2]^2},$$

$$J_\sigma=\frac{1}{4}\int_0^1(1-y)dy\int_0^1 dx\frac{xp_{y\sigma}}{(x\mathfrak{p}_y{}^2+ys^2)^2},$$

$$J_{\sigma\tau}=\frac{1}{4}\int_0^1(1-y)dy\int_0^1 dx\left[\frac{x^2p_{y\sigma}p_{y\tau}}{(x\mathfrak{p}_y{}^2+ys^2)^2}-\frac{1}{2}\frac{x\delta_{\sigma\tau}}{x\mathfrak{p}_y{}^2+ys^2}\right], \tag{75a}$$

$$J_{\sigma\tau\omega}=\frac{1}{4}\int_0^1(1-y)dy\int_0^1 dx\left[\frac{x^3p_{y\sigma}p_{y\tau}p_{y\omega}}{(x\mathfrak{p}_y{}^2+ys^2)^2}-\frac{x^2}{2}\frac{p_{y\sigma}\delta_{\tau\omega}+p_{y\tau}\delta_{\sigma\omega}+p_{y\omega}\delta_{\sigma\tau}}{x\mathfrak{p}_y{}^2+ys^2}\right]. \tag{75b}$$

λ has been put equal to 0 whenever this does not introduce a divergence. For all J's, except J_0, the integral

over x is easy to perform. For J_0, we can first compute an easier integral, where we replace $\mathfrak{p}_y^2$ by m^2:

$$\int_0^1 x^2dx\int_0^1\frac{(1-y)dy}{[x^2m^2+xys^2+(1-x)\lambda^2]^2}$$
$$=-\frac{1}{m^4\sigma^2}\left[\ln\frac{\lambda}{m}+\frac{1+\sigma^2}{\sigma^2}\ln(1+\sigma^2)-\ln\sigma^2\right], \quad (76)$$

where

$$\sigma=s/m, \quad (77)$$

and $\lambda/m\ll 1$ has been assumed. In the difference between this and J_0,

$$\int_0^1(1-y)dy\int_0^1 x^2dx[\{x^2\mathfrak{p}_y^2+xys^2+(1-x)\lambda^2\}^{-2}$$
$$-\{x^2m^2+xys^2+(1-x)\lambda^2\}^{-2}],$$

λ can be put equal to 0, yielding

$$\int_0^1(1-y)dy(ys^2)^{-1}[(\mathfrak{p}_y^2+ys^2)^{-1}-(m^2+ys^2)^{-1}]$$
$$=\int_0^1 y(1-y)dy(\mathfrak{p}_y^2+ys^2)^{-1}(m^2+ys^2)^{-1}. \quad (78)$$

The sum of (76) and (78) is $4J_0$. The J's have the coefficients appearing in formula (74), and the expectation value for the rest state of the electron has to be taken. For this, the following formulas are useful, in addition to those in the appendix of F II:

$${}_0\langle\beta M\beta\rangle_0={}_0\langle M\beta\rangle_0={}_0\langle\beta M\rangle_0={}_0\langle M\rangle_0,$$

M being any operator.

$$\boldsymbol{p}=\beta m,$$
$${}_0\langle\boldsymbol{p}\rangle_0={}_0\langle\boldsymbol{p}_y\rangle_0={}_0\langle\boldsymbol{r}\rangle_0=m,$$
$$\mathfrak{p}\cdot\mathfrak{p}_y=\mathfrak{p}\cdot\mathfrak{r}=m^2,$$
$$\mathfrak{r}\cdot\mathfrak{p}_y=m^2-ys^2,$$
$$\boldsymbol{ABC}+\boldsymbol{CBA}=2(\mathfrak{A}\cdot\mathfrak{B}\boldsymbol{C}+\mathfrak{B}\cdot\mathfrak{C}\boldsymbol{A}-\mathfrak{A}\cdot\mathfrak{C}\boldsymbol{B}),$$
$$\boldsymbol{ABA}=2\mathfrak{A}\cdot\mathfrak{B}\boldsymbol{A}-\mathfrak{A}^2\boldsymbol{B}.$$

Take for instance the coefficient of J_0:

$${}_0\langle 4m^2\beta(\boldsymbol{r}+m)\beta\rangle_0=4m^2\,{}_0\langle\boldsymbol{r}+m\rangle_0=8m^3.$$

Coefficient of $J_\sigma/p_{y\sigma}$:

$$-2\,{}_0\langle\boldsymbol{p}\boldsymbol{p}_y\beta(\boldsymbol{r}+m)+(\boldsymbol{r}+m)\beta\boldsymbol{p}_y\boldsymbol{p}+2m^2\boldsymbol{p}_y\rangle_0$$
$$=-2m\,{}_0\langle\boldsymbol{p}_y\beta\boldsymbol{r}+\boldsymbol{r}\beta\boldsymbol{p}_y\rangle_0-2m^2\,{}_0\langle\boldsymbol{p}_y\beta+\beta\boldsymbol{p}_y\rangle_0-4m^3$$
$$=-2m\,{}_0\langle 2m\boldsymbol{r}+2m\boldsymbol{p}_y-2\mathfrak{p}_y\cdot\mathfrak{r}\beta\rangle_0-8m^3$$
$$=-16m^3+4m(m^2-ys^2)=-4m(3m^2+ys^2)$$

The other coefficients are computed in the same way. The result is, for T_1:

$$2mT_1=-\frac{4}{\sigma^2}\left(\ln\frac{\lambda}{m}+\frac{1+\sigma^2}{\sigma^2}\ln(1+\sigma^2)-\ln\sigma^2\right)$$
$$+\int_0^1(1-y)dy\left\{\frac{4y^3}{[y+(1-y)u^2](y+u^2)}\right.$$
$$+\left[\frac{-6}{(1-u^2)^2}+\frac{1}{y}\frac{u^4-5u^2}{(1-u^2)^3}+\frac{2}{y^2}\frac{u^6+5u^4}{(1-u^2)^4}\right]\ln\frac{y+(1-y)u^2}{u^2}$$
$$+\left[y\frac{9-u^2}{1-u^2}+2\frac{u^2(3-u^2)}{(1-u^2)^2}+\frac{1}{y}\frac{u^6-5u^4}{(1-u^2)^3}\right]\frac{1}{y+(1-y)u^2}$$
$$\left.+\frac{3u^2-1}{(1-u^2)^2}-\frac{1}{y}\frac{3u^4+5u^2}{(1-u^2)^3}\right\}, \quad (79)$$

where σ is defined by (77) and $u=y\sigma$. Although powers of $(1-u^2)=\mathfrak{p}_y^2/m^2$ appear in the denominators, this is only an apparent pole and the whole expression stays finite for $u=1$.

T_3 is similar to T_1, only simpler. The denominators are combined according to

$$(\mathfrak{k}^2-2\mathfrak{r}\cdot\mathfrak{k})^{-2}(\mathfrak{k}^2-2\mathfrak{r}\cdot\mathfrak{k}-s^2)^{-1}(\mathfrak{k}^2-\lambda^2)^{-1}$$
$$=6\int_0^1(1-y)dy\int_0^1 x^2dx[\mathfrak{k}^2-2x\mathfrak{r}\cdot\mathfrak{k}-xys^2-(1-x)\lambda^2]^{-4}.$$

$$2mT_3=-\frac{4}{\sigma^2}\ln\frac{\lambda}{m}+\left[\frac{4}{\sigma^2}+\frac{7}{2}\frac{1}{1-\sigma^2}-\frac{1}{2}\frac{1}{(1-\sigma^2)^2}\right]\ln\sigma^2$$
$$+\frac{1}{\sigma^2}\left(\frac{6}{\sigma^2}+\frac{3}{2}\right)\ln|1-\sigma^2|+\frac{2}{\sigma^2}-\frac{1}{2}\frac{1}{1-\sigma^2}. \quad (80)$$

Here again, the powers of $(1-\sigma^2)=\mathfrak{r}^2/m^2$ in the denominators do not constitute a pole. Poles arise in the integrations over the auxiliary variables x and y, and the way of going around them is defined by giving m a small negative imaginary part. Only the real contribution from these integrations is written above. For each integration, the imaginary contribution can be dropped after we have made sure that no real term can arise in it from a pole in subsequent integrations.

For T_2, an additional auxiliary variable z has to be introduced:

$$[(\mathfrak{k}^2-2\mathfrak{p}\cdot\mathfrak{k})(\mathfrak{k}^2-2\mathfrak{r}\cdot\mathfrak{k})(\mathfrak{k}^2-2\mathfrak{r}\cdot\mathfrak{k}-s^2)(\mathfrak{k}^2-\lambda^2)]^{-1}$$
$$=6\int_0^1 ydy\int_0^1 x^2dx\int_0^1 dz$$
$$\times[\mathfrak{k}^2-2x\mathfrak{p}_y\cdot\mathfrak{k}-xyzs^2-(1-x)\lambda^2]^{-4}.$$

If the integration over z is performed immediately after integrating on $\mathfrak{k}$, we get expressions very similar

to T_1:

$$2mT_2=-\frac{4}{\sigma^3}\int_0^1\frac{dx}{[x^2+(1-x)\lambda^2/m^2]^{\frac{1}{2}}}\ln\frac{x\sigma+[x^2+(1-x)\lambda^2/m^2]^{\frac{1}{2}}}{|x\sigma-[x^2+(1-x)\lambda^2/m^2]^{\frac{1}{2}}|}$$
$$+\int_0^1 dy\left\{\left[\frac{8y^2}{u^2(1-u^2)}+\frac{12y}{(1-u^2)^2}-\frac{4u^2}{(1-u^2)^2}-\frac{2u^2(1+u^2)}{y(1-u^2)^3}\right]\ln\frac{y+(1-y)u^2}{u^2}\right.$$
$$\left.+4u^2(1-u^2)^{-2}+2\sigma^{-2}\ln[(1+y(1-y)\sigma^2)/|1-y^2\sigma^2|]\right\}. \quad (81)$$

This is only the real part. The imaginary part has been dropped, because it cannot give any real term in subsequent integrations. There is a real pole, this time, for $u=y\sigma=1$. But if one subtracts from $2mT_2$ the following expression:

$$\int_0^1 dyu^2(12y^2-2y-2)(1-u^2)^{-1}, \quad (82)$$

one cancels the principal part of the pole. Moreover the contribution of the subtracted part (82) in later integrations is 0.

The last step is to compute

$$\int_0^\infty \sigma^{-2}d\sigma 2m(T_1+T_2+T_3) \quad (83)$$

according to (73). But, since this integral is divergent for small σ, its "small-σ limit" will be subtracted from each term before integrating. By small-σ limit is meant the terms in σ^{-4}, σ^{-2}, $\sigma^{-4}\ln\sigma^2$, $\sigma^{-2}\ln\sigma^2$, in the expansion for small σ, after (79), (80), (81) have been divided by σ^2. The integral is then convergent. It will be seen at the end that the small-σ limit cancels with terms B and C of (65).

We find for instance, for $2mT_3$, from (80), in the small-σ limit:

$$-4\sigma^{-4}\ln(\lambda/m)+4\sigma^{-4}\ln\sigma^2+3\sigma^{-2}\ln\sigma^2-4\sigma^{-4}-5\sigma^{-2} \quad (84)$$

and the contribution to (83) is then

$$\int_0^\infty \sigma^{-2}d\sigma(2mT_3-\text{small-}\sigma\text{ limit})=-(11/8)\pi^2, \quad (85)$$

by making use of the integrals:

$$\int_0^\infty d\sigma[\sigma^{-6}\ln|1-\sigma^2|+\sigma^{-4}+\tfrac{1}{2}\sigma^{-2}]=0,$$
$$\int_0^\infty d\sigma[\sigma^{-4}\ln|1-\sigma^2|+\sigma^{-2}]=0,$$
$$\text{p.v.}\int_0^\infty d\sigma(1-\sigma^2)^{-1}=0$$
$$\int_0^\infty d\sigma(1-\sigma^2)^{-1}\ln\sigma^2=-\pi^2/2$$
$$\text{p.v.}\int_0^\infty d\sigma(1-\sigma^2)^{-2}\ln\sigma^2=-\pi^2/4$$

(p.v. means principal value). In the same way the first line of (79) gives, in the small-σ limit:

$$-4\sigma^{-4}\ln(\lambda/m)+4\sigma^{-4}\ln\sigma^2-4\sigma^{-4}-2\sigma^{-2}; \quad (86)$$

and the contribution to (83) is then

$$8\pi/15. \quad (87)$$

The first term of (81) gives for the small-σ limit:

$$-\frac{4}{\sigma^5}\int_0^1\frac{dx}{[x^2+(1-x)\lambda^2/m^2]^{\frac{1}{2}}}\left[\frac{2x\sigma}{[x^2+(1-x)\lambda^2/m^2]^{\frac{1}{2}}}+\frac{2}{3}\frac{x^3\sigma^3}{[x^2+(1-x)\lambda^2/m^2]^{\frac{3}{2}}}\right]$$
$$=8(\sigma^{-4}+\tfrac{1}{3}\sigma^{-2})\ln(\lambda/m)+(4/3)\sigma^{-2}\ (\text{assuming }\lambda/m\ll 1) \quad (88)$$

and no contribution to (83), by virtue of

$$\int_0^\infty\frac{d\sigma}{\sigma^5}\left[\ln\frac{1+k\sigma}{|1-k\sigma|}-2k\sigma-\tfrac{2}{3}k^3\sigma^3\right]=0.$$

We note that λ appears only in the small-σ limit, which is to be canceled against B and C, and this supports the consistency of our calculation. For the last term in the curly bracket of (81), we find, in the small-σ limit:

$$\sigma^{-2}; \quad (89)$$

and the contribution to (83) is then

$$-\pi^2/2^6. \quad (90)$$

For the remainder of (79) and (81), the small-σ limit is

$$-8\sigma^{-4}\ln\sigma^2-(3+8/3)\sigma^{-2}\ln\sigma^2$$
$$+8\sigma^{-4}+(5+7/18)\sigma^{-2}. \quad (91)$$

The contribution to (83) of this last part is easily evaluated by using y and $u=y\sigma$ as new variables of integration, so that

$$\int_0^\infty \sigma^{-2}d\sigma\int_0^1 dy\cdots$$

becomes

$$\int_0^\infty u^{-2}du\int_0^1 ydy\cdots.$$

This change of variables presents no difficulty since there is no pole in the integrand for either $\sigma=0$ or $y\sigma=1$, after subtraction of the small-σ limit and of (82). One can first transform the term containing $\ln[(y+(1-y)u^2)/u^2]$ by an integration by parts, such as

$$\int_0^1 y^n dy \ln\frac{y+(1-y)u^2}{u^2}=\frac{(1-u^2)}{n+1}\int_0^1\frac{1-y^{n+1}}{y+(1-y)u^2}dy \quad (\text{for } n\geq 0).$$

Then the integration over u is carried on first, by taking a contour running along the real axis and closed by an infinite semicircle above. The only singularities are poles inside the contour, for $u=i[y/(1-y)]^{\frac{1}{2}}$ and $u=iy^{\frac{1}{2}}$. Then the residues are integrated over y. We find

$$-8\pi/15+\pi^2(5/2-3/128-\tfrac{1}{2}\ln 2) \qquad (92)$$

as the contribution of that part to (83).

The reader can now check that the sum of the various parts, (84), (86), (88), (89), (91), of the small-σ limit is equal to $B+C$. Adding the contributions (85), (87), (90), (92), we obtain the corrections to the many-potential Lamb sift:

$$D'=4\alpha(Z\alpha)^2\pi m^{-2}|\psi(0)|^2(1+11/128-\tfrac{1}{2}\ln 2) \qquad (93)$$

which is the result previously reported.[24] For the $2S$ state, $|\psi(0)|^2=(Z\alpha m)^3/8\pi$. For the $2P$ state, it is 0, in this approximation.

VI. VACUUM POLARIZATION TERM

To the desired approximation, the electrons forming the loop in diagram 1(c) can be considered free. This is because, due to Furry's theorem, the next correction to this is a diagram which contains a loop with 4 vertices, which is expected to be of order $\alpha(Z\alpha)^6$. Diagram 1(c) with free electrons in the loop is known[25] to be equivalent to the potential

$$\frac{e^2}{4\pi m^2}V(q)\int_0^1 du\frac{u^2(1-u^2/3)q^2}{1+(1-u^2)q^2/4m^2},$$

where q is the magnitude of the 3-momentum transferred. This is after charge has been renormalized. To get the shift due to vacuum polarization effects, we just have to take the expectation value of this potential. Moreover, we can use the Schrödinger approximation to the wavefunction, which is, for the $2S$ state,

$$\frac{(Z\alpha m)^{5/2}}{\pi}\frac{p^2-\frac{1}{4}(Z\alpha m)^2}{[p^2+\frac{1}{4}(Z\alpha m)^2]^3}.$$

That this is correct can be easily shown by the methods developed in the appendix. The integration involved in taking the expectation value is straightforward, if one does not integrate over u until the end, and yields the result:

$$-\frac{\alpha(Z\alpha)^4}{8\pi}m\int_0^1 u^2\left(1-\frac{u^2}{3}\right)\frac{1+\frac{1}{8}(Z\alpha)^2(1-u^2)}{[1+\frac{1}{2}Z\alpha(1-u^2)^{\frac{1}{2}}]^4}du.$$

Expanding in powers of $Z\alpha$, we first find

$$-(1/30\pi)\alpha(Z\alpha)^4m,$$

which is the well-known contribution of about -27 megacycles. The next term is

$$(5/384)\alpha(Z\alpha)^5m. \qquad (94)$$

The same calculation for the $2P$ state gives a result of order $\alpha(Z\alpha)^6$, which we do not consider.

VII. RESULT AND DISCUSSION

Adding (93) and (94), we find all $\alpha(Z\alpha)^5$ corrections to the Lamb shift:[26]

$$\tfrac{1}{2}\alpha(Z\alpha)^5m(1+11/128-\tfrac{1}{2}\ln 2+5/192).$$

There is a question of determining whether we should use the mass of the free electron m_0, or the reduced mass in the hydrogen atom, $m_r=m_0(1+m_0/M)^{-1}$. This question is of no consequence with the present accuracy (theoretical and experimental). However, it could be settled after the very small corrections of order $Z\alpha m/M$ with respect to the main Lamb shift, coming from the 2-body aspect of the problem, have been calculated.[27] At the present time, the most logical course to take is to use m_r in $|\psi(0)|^2$ and m_0 everywhere else. This has also proved to be the best way for the main term of the Lamb shift. Therefore, we write the $\alpha(Z\alpha)^5$ corrections in the form

$$\begin{aligned}&\tfrac{1}{2}\alpha(Z\alpha)^5(m_r^3/m_0^2)(1+11/128-\tfrac{1}{2}\ln 2+5/192)\\&\quad=\alpha(Z\alpha)^3Ry_H[1-m_0/(M+m_0)]^2\\&\qquad\qquad\times(1+11/128-\tfrac{1}{2}\ln 2+5/192)\\&\quad=7.130\text{ Mc/sec}.\end{aligned}$$

The result for the whole Lamb shift, and the comparison with experiment, has been given in a recent paper by Salpeter.[28] The discrepancy between theory and experiment has now been reduced to 0.6 Mc/sec, with the experimental value still higher than the theoretical. With the discrepancy reduced to such a small amount, there can no longer be any doubt that the general principles of quantum electrodynamics, especially the idea of renormalization, are correct. In pr-a ticular, it can now be considered as experimentally established that the vacuum polarization term (-27 Mc/sec) must be included.

It is also interesting to compare the remaining discrepancy of 0.6 Mc/sec with the total energy of the $n=2$ state, 8×10^8 Mc/sec. This means that there

[24] M. Baranger, Phys. Rev. **84**, 866 (1951).

[25] For instance, see E. A. Uehling, Phys. Rev. **48**, 55 (1935), Eq. (14), and integrate by parts over u.

[26] The same result has been obtained by Karplus, Klein, and Schwinger, Phys. Rev. **86**, 288 (1952).

[27] See E. E. Salpeter, Phys. Rev. **89**, 92 (1953). We are indebted to Professor Salpeter for the remarks of this paragraph.

[28] See reference in footnote 27.

cannot be an over-all deviation from the Coulomb field between proton and electron of as much as 1 part in a billion.

It is very unlikely that the small remaining discrepancy of 0.6 Mc/sec can be attributed to experimental error since the given experimental accuracy is 0.1 Mc/sec. In principle, the discrepancy could be attributed to a small deviation from the Coulomb field at small distances, due to the mesonic structure of the proton; a repulsion of 80 kev over a sphere of radius e^2/mc^2 would give the desired effect. However, it is known that the neutron-electron interaction amounts to only 4 kev (attractive) over this radius, and it is reasonable to assume that the proton-electron interaction (apart from the Coulomb force) has about the same magnitude. Moreover, if the discrepancy was of nuclear origin, one would expect it to have different values for hydrogen and deuterium. But the experimental value of the difference between the hydrogen and deuterium Lamb shifts is in good agreement with the present theoretical value.[28]

On the other hand, it does not seem to us unreasonable that the discrepancy could be explained by the relativistic correction of next order. We have shown in Sec. IV that the ratio of this correction to the correction calculated in this paper, is of order $\alpha Z \ln(\alpha Z)$, i.e., 1/28. Since our result is 7 Mc/sec, the next order is expected to be about 0.25 Mc/sec. Our result in this paper has also shown that higher order effects are apt to have a larger numerical coefficient, and only twice the coefficient is needed to give 0.5 Mc/sec.

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APPENDIX

In this appendix, we give a method for computing the corrections to the one-potential Lamb shift, in order $\alpha(Z\alpha)^6$. Because these corrections contain many non-gauge-invariant terms, and an infrared catastrophy, which are supposed to cancel against contributions from the many-potential Lamb shift, they are rather uninteresting as long as similar corrections for the many-potential part are not available. For this reason, we shall not carry out the calculations, but limit ourselves to showing that terms of order $\alpha(Z\alpha)^5$ do not arise.

We can consider two kinds of corrections: (1) Corrections arising when (29) is evaluated exactly, but with the approximate nonrelativistic Schrödinger wave-function in place of $\varphi_0(\mathbf{p})$, (2) Corrections arising from the use of the exact Dirac wave function.

The first kind can be obtained in closed form, by using integrals such as

$$K_{ab}=\Re\int\frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2}{(\mathbf{p}_1^2+a^2)(\mathbf{p}_2^2+b^2)\mathbf{q}^2(E^2-\mathbf{p}_x^2)}$$
$$=\frac{2\pi^4}{E^2+(1-x)a^2+xb^2}\ln\frac{(E^2+a^2)(E^2+b^2)}{(a+b)^2[E^2+((1-x)a+xb)^2]},$$

$$J_{a^2b}=\Re\int\frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2}{(\mathbf{p}_1^2+a^2)^2(\mathbf{p}_2^2+b^2)(E^2-\mathbf{p}_x^2)}$$
$$=-\frac{2\pi^4}{ax}\frac{(1-x)a+xb}{E^2+[(1-x)a+xb]^2},$$

where $\Re$ means the real part, and

$$\mathbf{q}=\mathbf{p}_2-\mathbf{p}_1,$$
$$\mathbf{p}_x=x\mathbf{p}_2+(1-x)\mathbf{p}_1.$$

From these two integrals and a similar one, J_{ab^2}, one can, by differentiating with respect to a and b and taking various combinations, obtain the values of all other integrals of this sort. In particular, we shall need

$$K_{21}=\Re\int\frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2}{(\mathbf{p}_1^2+a^2)^2(\mathbf{p}_2^2+a^2)\mathbf{q}^2(E^2-\mathbf{p}_x^2)}$$
$$=\frac{2\pi^4(1-x)}{(E^2+a^2)^2}\ln\frac{E^2+a^2}{4a^2}+\frac{2\pi^4}{E^2+a^2}\left(\frac{1}{2a^2}-\frac{x}{E^2+a^2}\right),$$

$$K_{12}=\Re\int\frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2}{(\mathbf{p}_1^2+a^2)(\mathbf{p}_2^2+a^2)^2\mathbf{q}^2(E^2-\mathbf{p}_x^2)}$$
$$=\frac{2\pi^4 x}{(E^2+a^2)^2}\ln\frac{E^2+a^2}{4a^2}+\frac{2\pi^4}{E^2+a^2}\left(\frac{1}{2a^2}-\frac{1-x}{E^2+a^2}\right),$$

$$K_{22}=\Re\int\frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2}{(\mathbf{p}_1^2+a^2)^2(\mathbf{p}_2^2+a^2)^2\mathbf{q}^2(E^2-\mathbf{p}_x^2)}$$
$$=\pi^4\left[\frac{2x(1-x)}{(E^2+a^2)^3}\left(2\ln\frac{E^2+a^2}{4a^2}+3\right)\right.$$
$$\left.-\frac{2}{(E^2+a^2)^3}+\frac{1-x(1-x)}{a^2(E^2+a^2)^2}+\frac{1}{4a^4(E^2+a^2)}\right],$$

$$K_{20}=\Re\int\frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2}{(\mathbf{p}_1^2+a^2)^2\mathbf{q}^2(E^2-\mathbf{p}_x^2)}=-\frac{2\pi^4}{x}\frac{1}{E^2+a^2},$$

$$J_{21}=\Re\int\frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2}{(\mathbf{p}_1^2+a^2)^2(\mathbf{p}_2^2+a^2)(E^2-\mathbf{p}_x^2)}=-\frac{2\pi^4}{x}\frac{1}{E^2+a^2},$$

$$J_{12}=\Re\int\frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2}{(\mathbf{p}_1^2+a^2)(\mathbf{p}_2^2+a^2)^2(E^2-\mathbf{p}_x^2)}=-\frac{2\pi^4}{1-x}\frac{1}{E^2+a^2},$$

$$J_{22}=\Re\int\frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2}{(\mathbf{p}_1^2+a^2)^2(\mathbf{p}_2^2+a^2)^2(E^2-\mathbf{p}_x^2)}=\frac{\pi^4}{a^2}\frac{E^2-a^2}{(E^2+a^2)^2}.$$

For the sake of simplicity, let us consider the shift of the $1S$ level. The treatment of the $2S$ level is quite similar. For the $1S$ level, we have the normalized Schrödinger wave function

$$\pi^{-1}(8Z^5\alpha^5m^5)^{\frac{1}{2}}(\mathbf{p}^2+Z^2\alpha^2m^2)^{-2}.$$

Taking for instance I_b, Eq. (31), and using (8b),

$$I_b=-\frac{Z\alpha\beta}{4\pi^2}\int_0^1 dx\left[\frac{1-3x(1-x)}{\mathfrak{p}_x^2}-\frac{(1-2x)(\mathbf{p}_2^2-\mathbf{p}_1^2)}{\mathfrak{p}_x^2\mathbf{q}^2}\right],$$

we obtain for the corresponding energy-shift:

$$\Delta E_{1b}=(\alpha/\pi)\int\bar{\varphi}_0(\mathbf{p}_2)I_b\varphi_0(\mathbf{p}_1)d^3\mathbf{p}_2d^3\mathbf{p}_1$$

$$=-2\pi^{-5}\alpha(Z\alpha)^6m^5\int_0^1 dx[(1-3x(1-x))J_{22}(E_0,Z\alpha m)-(1-2x)(K_{21}-K_{12})_{(E_0,Z\alpha m)}],$$

where $J_{22}(E_0, Z\alpha m)$ means that, in the expression for J_{22}, we replace E by E_0 and a by $Z\alpha m$. The first term of the expansion of ΔE_{1b} in powers of $Z\alpha$ is of order $\alpha(Z\alpha)^4$, and the next one is of order $\alpha(Z\alpha)^6\ln(Z\alpha)$. No term of order $\alpha(Z\alpha)^5$ occurs, due to the fact that J_{22}, K_{21}, K_{12}, etc.$\cdots$ are functions of a^2 only, not of a. I_a can be dealt with in the same way, while I_c can be written

$$I_c=\tfrac{3}{8}V\int_0^1 dx\int_{E_0^2}^\infty d(E^2)[2(E^2-\mathfrak{p}_x^2)^{-1}-(E^2-\mathbf{p}_1^2)^{-1}-(E^2-\mathbf{p}_2^2)^{-1}],$$

giving an energy shift

$$\Delta E_{1c}=(\alpha/\pi)\int\varphi_0(\mathbf{p}_2)I_c\varphi_0(\mathbf{p}_1)d^3\mathbf{p}_1d^3\mathbf{p}_2$$

$$=-\tfrac{3}{2}\pi^{-5}\alpha(Z\alpha)^6m^5\int_0^1 dx\int_{E_0^2}^\infty d(E^2)$$

$$\times[2K_{22}(E,Z\alpha m)-K_{22}^0(E,Z\alpha m)-K_{22}^1(E,Z\alpha m)],$$

where x has been put equal to 0 and 1, respectively, in K_{22}^0 and K_{22}^1. The bracket is equal to

$$2\pi^4x(1-x)\left[\frac{2}{(E^2+a^2)^3}\left(2\ln\frac{E^2+a^2}{4a^2}+3\right)-\frac{1}{a^2(E^2+a^2)^2}\right]_{(a=Z\alpha m)},$$

and there again no term of order $\alpha(Z\alpha)^5$ is found. For the next term I_d, we write

$$I_d=\tfrac{1}{2}V\int_0^1 dx[(\mathfrak{p}_1\cdot\mathfrak{p}_2/\mathfrak{p}_x^2)-1]\int_{E_0^2}^\infty[(E^2-\mathfrak{p}_x^2)^{-1}-E^{-2}]d(E^2)$$

$$=-(Z\alpha\beta/8\pi^2\mathbf{q}^2)\int_0^1 dx\int_{E_0^2}^\infty d(E^2)\{[1-2x(1-x)]\mathbf{q}^2-(1-2x)(\mathbf{p}_2^2-\mathbf{p}_1^2)\}\{(E^2-E_0^2)^{-1}[(E_0^2-\mathfrak{p}_x^2)^{-1}-(E^2-\mathfrak{p}_x^2)^{-1}]-E^{-2}(E_0^2-\mathfrak{p}_x^2)^{-1}\},$$

$$\Delta E_{1d}=-\pi^{-5}\alpha(Z\alpha)^6m^5\int_0^1 dx\int_{E_0^2}^\infty d(E^2)\cdot\left\{[1-2x(1-x)]\left[\frac{1}{E^2-E_0^2}(J_{22}(E_0)-J_{22}(E))-\frac{1}{E^2}J_{22}(E_0)\right]-(1-2x)\left[\frac{1}{E^2-E_0^2}(K_{21}(E_0)-K_{12}(E_0)-K_{21}(E)+K_{12}(E))-\frac{1}{E^2}(K_{21}(E_0)-K_{12}(E_0))\right]\right\}_{(a=Z\alpha m)}$$

$K_{21}-K_{12}$ is of order $\ln a$, while the $1/a^2$ factor in J_{22} actually disappears, so that ΔE_{1d} is of order $\alpha(Z\alpha)^6\times\ln(Z\alpha)$, in the Schrödinger approximation.

Before investigating I_e, we consider the second type of corrections, coming from the use of the Dirac wave function. For the $1S_{\frac{1}{2}}$ state, and $m=+\frac{1}{2}$, the four components of the Dirac wave function are

$$\varphi_{01}=-Y_{00}(\theta,\varphi)G(p),$$
$$\varphi_{02}=0,$$
$$\varphi_{03}=(1/\sqrt{3})Y_{10}(\theta,\varphi)f(p),$$
$$\varphi_{04}=-(\sqrt{2}/\sqrt{3})Y_{11}(\theta,\varphi)f(p),$$

where the Y's are normalized spherical harmonics in momentum space,

$$Y_{00}=(\tfrac{1}{4}\pi)^{\frac{1}{2}},$$
$$Y_{10}=(\tfrac{3}{4}\pi)^{\frac{1}{2}}\cos\theta,$$
$$Y_{11}=-(\tfrac{3}{8}\pi)^{\frac{1}{2}}\sin\theta e^{i\varphi},$$

and $G(p)$ and $f(p)$ are obtained from the corresponding functions in ordinary space by a Fourier-Bessel transformation. $G(p)$, the large component, is proportional to

$$p^{-1}[1+(p^2/Z^2\alpha^2m^2)]^{-(1+\epsilon)/2}\sin[(1+\epsilon)\,\mathrm{tg}^{-1}(p/Z\alpha m)],$$

where ϵ is the energy of the level (including rest energy) divided by m [$\epsilon=(1-\alpha^2Z^2)^{\frac{1}{2}}$]. $G(p)$ can be expanded in powers of $Z\alpha$ and one finds that, if one neglects terms of order $(Z\alpha)^2$ with respect to the main term, a good approximation for almost all p's is given by

$$G(p)\simeq[32(Z\alpha m)^5/\pi]^{\frac{1}{2}}(p^2+Z^2\alpha^2m^2)^{-2}(1+\pi Z\alpha p/8m);$$

while, for the small component $f(p)$, it is enough to keep only the main term

$$f(p)\simeq-[8(Z\alpha)^5m^3/\pi]^{\frac{1}{2}}p(p^2+Z^2\alpha^2m^2)^{-2}.$$

ΔE_1 involves expressions of the form

$$\int\bar{\varphi}_0(\mathbf{p}_2)\beta\Lambda(\mathfrak{p}_1^2,\mathfrak{p}_2^2,\mathfrak{q}^2)\varphi_0(\mathbf{p}_1)d^3\mathbf{p}_1d^3\mathbf{p}_2,\qquad\text{(A1)}$$

where Λ is a function containing no Dirac operator. (A1) is equal to

$$(4\pi)^{-1}\int d^3\mathbf{p}_1d^3\mathbf{p}_2\Lambda(\mathfrak{p}_1^2,\mathfrak{p}_2^2,\mathfrak{q}^2)[G(p_2)G(p_1)+(\mathbf{p}_1\cdot\mathbf{p}_2/p_1p_2)f(p_2)f(p_1)],\qquad\text{(A2)}$$

the difference between the integrands in (A1) and (A2) giving 0 in the integration. Since $\mathbf{p}_1\cdot\mathbf{p}_2$ can be expressed in terms of $\mathbf{p}_1^2$, $\mathbf{p}_2^2$, $\mathbf{q}^2$ by

$$2\mathbf{p}_1\cdot\mathbf{p}_2=\mathbf{p}_1^2+\mathbf{p}_2^2-\mathbf{q}^2,$$

the corrections involving the small component, $f(p)$, in I_a, I_b, I_c, can be immediately evaluated as combinations of the integrals K, J, etc.$\cdots$, again, and it is seen that, due to the fact that these integrals depend on a^2 only, and not on a, there is no term of order $\alpha(Z\alpha)^5$.[29] As for the correction involving a first power of p in $G(p)$, we can transform it by use of the formula

$$(\pi/2)p=\int_0^\infty (p^2/p^2+b^2)db,$$

so that

$G(p)\simeq$Schrödinger wave function

$$+[2\pi^{-1}(Z\alpha)^7m^3]^{\frac{1}{2}}\int_0^\infty p^2(p^2+b^2)^{-1}(p^2+Z^2\alpha^2m^2)^{-2}db,$$

and we obtain combinations of the more general integrals, K_{ab}, J_{a^2b}, etc.$\cdots$, followed by an integration over b. In all cases, Dirac corrections prove to be of order $\alpha(Z\alpha)^6\ln(Z\alpha)$ or higher.

Going back to I_e now, we can first perform the integral over $\mathbf{p}_i$ and $\mathbf{p}_f$ in (33). Define 4 new functions by

$$\varphi_{0s}'(\mathbf{p})=\int V(\mathbf{p}-\mathbf{p}')\varphi_{0s}(\mathbf{p}')d^3\mathbf{p}' \quad (s=1,2,3,4).$$

φ_{0s}' has the same angular dependence as φ_{0s}. This is a consequence of Dirac's equation itself, which can be written (for an S state):

$$\int V(\mathbf{p}-\mathbf{p}')G(p')Y_{00}(\theta',\varphi')d^3\mathbf{p}'$$
$$=Y_{00}(\theta,\varphi)[(E_0-m)G(p)+pf(p)]\equiv Y_{00}(\theta,\varphi)G'(p),$$

$$\int V(\mathbf{p}-\mathbf{p}')f(p')Y_{1m}(\theta',\varphi')d^3\mathbf{p}'$$
$$=Y_{1m}(\theta,\varphi)[(E_0+m)f(p)+pG(p)]\equiv Y_{1m}(\theta,\varphi)f'(p),$$

defining two new functions $G'(p)$ and $f'(p)$. Approximations for $G'(p)$ and $f'(p)$ are given by

$$G'(p)\simeq-\frac{Z\alpha}{2\pi^2}\left(\frac{32(Z\alpha m)^5}{\pi}\right)^{\frac{1}{2}}\int\frac{d^3\mathbf{p}'}{(\mathbf{p}-\mathbf{p}')^2[p'^2+(Z\alpha m)^2]^2}$$
$$=-\left(\frac{8(Z\alpha)^5m^3}{\pi}\right)^{\frac{1}{2}}\frac{1}{p^2+(Z\alpha m)^2},$$

$$Y_{1m}f'(p)\simeq\frac{Z\alpha}{2\pi^2}\left(\frac{8(Z\alpha)^5m^3}{\pi}\right)^{\frac{1}{2}}\int\frac{p'Y_{1m}(\theta',\varphi')d^3\mathbf{p}'}{(\mathbf{p}-\mathbf{p}')^2[p'^2+(Z\alpha m)^2]^2}$$
$$=\left(\frac{8(Z\alpha)^7m^3}{\pi}\right)^{\frac{1}{2}}Y_{1m}(\theta,\varphi)\int_{Z\alpha m}^\infty\frac{pdb}{(p^2+b^2)^2}.$$

We can now evaluate the term with $3V$'s in (33), and find a result of order $\alpha(Z\alpha)^6\ln(Z\alpha)$. The other terms of (33) have already been evaluated with p_x^2 replaced by m^2; we now want the difference, involving

$$(1/\mathfrak{p}_x^2)-(1/m^2)=(m^2-E_0^2+\mathbf{p}_x^2)/m^2(E_0^2-\mathbf{p}_x^2).$$

Always by the same methods, this can be shown to yield linear combinations of integrals K, J, etc.$\cdots$ that depend only on a^2, and therefore there cannot be any term of order $\alpha(Z\alpha)^5$.

We complete the investigation of the one-potential Lamb shift by looking at I_f, given in (35), from which we subtract

$$\tfrac{3}{4}m\delta^3(\mathbf{p}_2-\mathbf{p}_1)\int_0^1(4x-1)(\not{p}-m)^2m^{-2}dx,$$

already evaluated in (36). The difference gives for the energy shift:

$$\frac{3}{4}\frac{e^2}{\pi}\int\bar{\varphi}_0(\mathbf{p})\int_0^1 dx\left[x(4x-1)\frac{(\not{p}-m)^2(m^2-\not{\mathfrak{p}}^2)}{m[\mathfrak{p}^2x+m^2(1-x)]}+2x\frac{(\not{p}-m)^3}{\mathfrak{p}^2x+m^2(1-x)}\right]\varphi_0(\mathbf{p})d^3\mathbf{p}$$
$$=\frac{3}{4}\frac{e^2}{\pi}\int\bar{\varphi}_0(\mathbf{p}_f)V(\mathbf{p}_f-\mathbf{p})\int_0^1 xdx\left[(4x-1)\frac{m^2-\not{\mathfrak{p}}^2}{m[\mathfrak{p}^2x+m^2(1-x)]}+2\frac{\not{p}-m}{\mathfrak{p}^2x+m^2(1-x)}\right]V(\mathbf{p}-\mathbf{p}_i)\varphi_0(\mathbf{p}_i)d^3\mathbf{p}d^3\mathbf{p}_fd^3\mathbf{p}_i$$
$$=\frac{3}{4}\frac{e^2}{\pi}\int_0^\infty p^2dp\int_0^1 xdx\left[(4x-1)\frac{m^2-E_0^2+\mathbf{p}^2}{m[\mathfrak{p}^2x+m^2(1-x)]}\{[G'(p)]^2-[f'(p)]^2\}\right.$$
$$\left.+\frac{2}{\mathfrak{p}^2x+m^2(1-x)}\{(E_0-m)[G'(p)]^2+(E_0+m)[f'(p)]^2-2pG'(p)f'(p)\}\right].$$

The integration can be performed with the help of

$$\Re\int_0^\infty\frac{p^2dp}{(\mathbf{p}^2+a^2)[\mathfrak{p}^2x+m^2(1-x)]}=-\frac{\pi}{2}\frac{a}{(E_0^2+a^2)x+m^2(1-x)},$$

and other formulas obtained by taking derivatives with respect to a. No term of order $\alpha(Z\alpha)^5$ is found.

[29] A similar procedure shows that there is no $\alpha(Z\alpha)^5$ term in the one-potential shift for the $2P$ state.