Group these terms in the following manner:

$$E=[W+\sum_i(i|H(1)|i)+\sum_{\substack{i,j\\ i\neq j}}(ij|H(1,2)|ij)]$$
$$+[\sum_{i,j}S_{ij}S_{ji}(i|H(1)|i)$$
$$+\sum_{\substack{i,j\\ i\neq j}}(2\sum_k S_{ik}S_{ki}-S_{ij}S_{ji})(ij|H(1,2)|ij)$$
$$-\sum_{\substack{i,j\\ i\neq j}}S_{ji}(i|H(1)|j)-\sum_{\substack{i,j\\ i\neq j}}(ij|H(1,2)|ji)$$
$$-2\sum_{\substack{i,j,k\\ i\neq j\neq k}}S_{ji}(ik|H(1,2)|jk)]+\cdots. \quad (29)$$

The first group on the right in (29) is E_0 of (11). The second group in (29) is equal to $-\sum_{j>i}J_{ij}\langle P_{ji}{}^{ij}\rangle$ of (11), which one may prove by expanding J_{ij} as defined in (10). In the proof it must be remembered that $S_{ii}=0$ and S_{ij}, $(i|\ \ |j)$, etc. are also zero if i and j have different spins.

We state without further proof, that the next group of terms in (29), which would involve three integrals over pairs of wave functions with different i and j, is equal to the third-order terms of (11).

If the series (17) converges sufficiently, one may neglect the remaining terms in (29) and consequently R in (11).

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A Modified Theory of Production of Secondary Electrons in Solids*

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A few difficulties in the previous theories of energy loss and secondary electron production by primary electrons in metals can be removed by replacing the Coulomb interaction between a primary electron and a lattice electron by a screened Coulomb interaction, such as required by the plasma theory. In the new theory Wooldridge's mechanism of secondary electron production seems to have lost most of its importance. The energy loss distribution due to the interaction with the conduction electrons of the metal is found to have a maximum at an energy loss slightly less than $(\hbar^2k_m^2/2m)$, where k_m is the value of the wave vector of the conduction electrons at the Fermi level. The theory is also applied to the bound electrons, and it is shown that the screened Coulomb interaction does not change the previous results very strongly. It is also shown that the polarization of the medium has only a small influence upon the previous results for insulators.

I. INTRODUCTION

DEKKER and van der Ziel[1] recently unified the various theories of secondary electron production. This theory, which is essentially a modification of the existing theories of energy loss of fast particles in matter, is based upon a Coulomb interaction between the primary electrons and the lattice electrons. It has the same difficulties for the conduction electrons of the metal as previous theories:

(a) In a single collision between a primary electron and a lattice electron the probability $P(E_{kk'})dE_{kk'}$ of an energy loss between $E_{kk'}$ and $E_{kk'}+dE_{kk'}$ becomes infinite for $E_{kk'}\rightarrow 0$.

(b) The probability $P(k')dk'$ of a transition of a lattice electron to an energy state having an absolute value of the wave vector between k' and $k'+dk'$ becomes infinite at the Fermi level.

(c) The rate of energy loss (dE_p/dx) due to the lattice electrons for a primary electron of energy E_p varies as $E_p{}^{-1}\log(E_p/E_0')$, with a very small value of E_0'.

It is the aim of this paper to investigate whether a screened Coulomb interaction might remedy these defects. It is reasonable to expect so, for, since the difficulties are caused by the interaction of primary electrons and those lattice electrons that are passed by at larger distances, this interaction should be removed by a screened potential function.

Such a screened potential should be expected in metals according to the plasma theory of electron interaction. In that theory the interaction between a primary electron and the electrons of the metal is split into two parts.[2]

(1) An "organized" part, consisting of the interaction with the electron gas as a *whole*, resulting in relatively sharp to very sharp energy losses caused by the excitation of "plasma oscillations." This part of the interaction can explain the discrete energy losses of electrons in metal foils. It is not known how important this process

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[1] A. J. Dekker and A. van der Ziel, Phys. Rev. **86**, 755 (1952).

[2] R. Kronig and J. Korringa, Physica **10**, 406, 800 (1943); H. A. Kramers, Physica **13**, 401 (1947); D. Bohm and E. P. Gross, Phys. Rev. **75**, 1851, 1864 (1949); D. Bohm and D. Pines, Phys. Rev. **80**, 903 (1950); **82**, 625 (1951); D. Pines and D. Bohm, Phys. Rev. **85**, 338 (1952); D. Pines, Phys. Rev. **85**, 931 (1952).

is for the production of secondary electrons; this question deserves further study.

(2) An "unorganized" part consisting of the residual interaction with the *individual* lattice electrons. This part can be described by a screened Coulomb potential,

$$V(\mathbf{R},\mathbf{r})=\frac{e^2}{|\mathbf{R}-\mathbf{r}|}\exp[-\lambda|\mathbf{R}-\mathbf{r}|], \tag{1}$$

where $\mathbf{R}$ and $\mathbf{r}$ are the radius vectors of the primary electron and the lattice electron, respectively, whereas λ is determined by the properties of the electron gas; it is estimated that $\lambda\simeq 10^8$ cm^{-1} for metals. This part of the interaction is investigated here.

We assume in our discussion that the metal is in the form of a cube of 1 cm^3 volume and that the primary beam has an intensity of 1 electron per cm^2 per second. Let $\mathbf{K}$ and $\mathbf{K}'$ be the wave vectors of the primary electron and $\mathbf{k}$ and $\mathbf{k}'$ the wave vectors of the lattice electron before and after the impact, respectively. Energy is conserved in the collision process; that is, if E and E' are the energies of the lattice electron before and after the impact, then

$$\hbar^2K^2/2m+E=\hbar^2K'^2/2m+E'. \tag{2}$$

For given values of $\mathbf{K}$, $\mathbf{k}$ and $\mathbf{k}'$ the magnitude of $\mathbf{K}'$ is thus fixed but its direction is not.

We apply the method employed in Dekker and van der Ziel's paper and rewrite Eq. (11) of that paper. According to that equation the rate of collision processes, in which the wave vector $\mathbf{k}$ is changed into $\mathbf{k}'$ and the wave vector of the primary electron after the collision is found within a solid angle $d\Omega'$ around $\mathbf{K}'$, is

$$P(\mathbf{K},\mathbf{k}\rightarrow\mathbf{K}',\mathbf{k}')d\Omega'=\frac{m^2(K'/K)}{4\pi^2\hbar^4}J^2|I|^2d\Omega', \tag{3}$$

where

$$J=\int V(\mathbf{R},\mathbf{r})\exp[i\mathbf{q}\cdot(\mathbf{R}-\mathbf{r})]d\mathbf{R}, \tag{3a}$$

$$I=\int\psi_{\mathbf{k}}(\mathbf{r})\psi_{\mathbf{k}'}{}^*(\mathbf{r})\exp[i(\mathbf{q}\cdot\mathbf{r})]d\mathbf{r}. \tag{3b}$$

The integrations are to be extended over the crystal and $\mathbf{q}=\mathbf{K}-\mathbf{K}'$. The potential energy $V(\mathbf{R},\mathbf{r})$ describes the interaction between a primary electron and a lattice electron, and $\psi_{\mathbf{k}}(\mathbf{r})$ and $\psi_{\mathbf{k}'}(\mathbf{r})$ are the wave functions of the lattice electrons before and after the impact. It follows directly from Dekker and van der Ziel's calculations that (3) holds for an arbitrary potential function.

Substituting (1), we obtain

$$J=4\pi e^2/(q^2+\lambda^2), \tag{3c}$$

and

$$P(\mathbf{K},\mathbf{k}\rightarrow\mathbf{K}',\mathbf{k}')d\Omega'=\frac{4m^2e^4(K'/K)}{\hbar^4(q^2+\lambda^2)^2}|I|^2d\Omega'. \tag{4}$$

For the case $\lambda\rightarrow 0$, the potential energy (1) becomes a Coulomb potential function and (4) reduces to Dekker and van der Ziel's formula (11), as should be expected.

To calculate the rate of production of secondaries the integral I has to be evaluated. The cases of weakly bound electrons (conduction electrons) and strongly bound electrons have to be treated separately.

II. THE CASE OF WEAKLY BOUND LATTICE ELECTRONS

For weakly bound lattice electrons $\psi_{\mathbf{k}}(\mathbf{r})$ is of the form $u_{\mathbf{k}}(\mathbf{r})\exp[i(\mathbf{k}\cdot\mathbf{r})]$, where $u_{\mathbf{k}}(\mathbf{r})$ is periodic in $\mathbf{r}$ with the period of the lattice. A Fourier expansion of the periodic part of the wave function yields.

$$\psi_{\mathbf{k}}(\mathbf{r})=\sum_{\mathbf{n}=0}^{\infty}c_{\mathbf{n}}(\mathbf{k})\exp[i(\mathbf{n}\cdot\mathbf{r})+i(\mathbf{k}\cdot\mathbf{r})],$$

$$\psi_{\mathbf{k}'}(\mathbf{r})=\sum_{\mathbf{m}=0}^{\infty}c_{\mathbf{m}}(\mathbf{k}')\exp[i(\mathbf{m}\cdot\mathbf{r})+i(\mathbf{k}'\cdot\mathbf{r})].$$

Substituting this into (3b) yields that

$$I=\sum_{\mathbf{n}}\sum_{\mathbf{m}}c_{\mathbf{n}}(\mathbf{k})c_{\mathbf{m}}{}^*(\mathbf{k}'), \tag{5}$$

if

$$\mathbf{C}\equiv\mathbf{K}+\mathbf{k}+\mathbf{n}-\mathbf{m}=\mathbf{K}'+\mathbf{k}' \text{ (momentum law)} \tag{5a}$$

and zero otherwise; $-h(\mathbf{n}-\mathbf{m})=-h\mathbf{p}$ is the momentum taken up by the lattice.

For a given momentum $-h\mathbf{p}$ taken up by the lattice the transition probability becomes

$$P_{\mathbf{p}}(\mathbf{K},\mathbf{k}\rightarrow\mathbf{K}',\mathbf{k}')d\Omega'$$

$$=\left|\sum_{\mathbf{m}}c_{\mathbf{m}+\mathbf{p}}(\mathbf{k})c_{\mathbf{m}}{}^*(\mathbf{k}')\right|^2\frac{4m^2e^4(K'/K)}{\hbar^4(q^2+\lambda^2)^2}d\Omega', \tag{6}$$

where

$$q=|\mathbf{K}-\mathbf{K}'|=|\mathbf{k}'-\mathbf{k}-\mathbf{p}|. \tag{6a}$$

Now introduce the vector $\mathbf{C}_p=(\mathbf{K}+\mathbf{k}+\mathbf{p})$, and use it as the Z axis of a polar coordinate system and introduce the polar angle ϑ between $\mathbf{K}'$ and $\mathbf{C}_p$ and the azimuthal angle ϕ of $\mathbf{K}'$ as new variables. Since $\mathbf{C}_p\simeq\mathbf{K}$ if $\mathbf{K}\gg|\mathbf{k}+\mathbf{p}|$,

$$d\Omega'\simeq\frac{k'dk'}{C_pK'}d\phi=\frac{k'dk'}{KK'}d\varphi, \tag{6b}$$

we find for the rate of transitions whereby a lattice electron of wave vector $\mathbf{k}$ is scattered into a state between k' and $k'+dk'$

$$P_{\mathbf{p}}(k')dk'=\frac{4m^2e^4}{\hbar^4K^2}k'dk'$$

$$\times\int_0^{2\pi}\frac{d\phi}{(|\mathbf{k}'-\mathbf{k}-\mathbf{p}|^2+\lambda^2)^2}\left|\sum_{\mathbf{m}}c_{\mathbf{m}+\mathbf{p}}(\mathbf{k})c_{\mathbf{m}}{}^*(\mathbf{k}')\right|^2. \tag{7}$$

We now have to discuss this for the cases (a) momentum $-\hbar\mathbf{p}$ taken up by the lattice ($p\neq 0$), (b) no momentum taken up by the lattice ($p=0$).

For $\lambda=0$ these cases lead to Wooldridge's theory and Baroody's theory, respectively.[3,4]

Comparing the cases $p\neq 0$ and $p=0$ and observing that

$$\left|\sum_{\mathbf{m}} c_{\mathbf{m}+\mathbf{p}}(\mathbf{k})c_{\mathbf{m}}^*(\mathbf{k}')\right| \ll 1 \text{ for } p\neq 0 \text{ and } \simeq 1 \text{ for } p=0,$$

we see that the case $p\neq 0$ has a much smaller probability than the case $p=0$. For $\lambda=0$ this would be offset by the fact that the factor $|\mathbf{k}'-\mathbf{k}-\mathbf{p}|^{-4}$ has a very sharp maximum around $\mathbf{k}'\simeq(\mathbf{k}+\mathbf{p})$, but for $\lambda\neq 0$ this is no longer the case. The unimportance of the case $p\neq 0$ is further strengthened by the fact that

$$\left|\sum_{\mathbf{m}} c_{\mathbf{m}+\mathbf{p}}(\mathbf{k})c_{\mathbf{m}}^*(\mathbf{k}')\right| = 0 \text{ for } \mathbf{k}'=\mathbf{k}+\mathbf{p},$$

as was first shown by Marshall.[5] It seems therefore that Wooldridge's mechanism has lost all its importance in the case of a screened potential and that it is sufficient to consider the case $p=0$ (Baroody's mechanism).

We first make the simplifying assumption that $k'\gg k$; Eq. (7) then becomes

$$P(k')dk' = \frac{8\pi m^2e^4}{\hbar^4K^2}\frac{k'dk'}{(k'^2+\lambda^2)^2}, \tag{7a}$$

and the rate at which an energy loss between $E_{kk'}$ and $E_{kk'}+dE_{kk'}$ occurs is

$$P(E_{kk'})dE_{kk'} = \frac{\pi Ne^4}{E_p}\frac{dE_{kk'}}{(E_{kk'}+E_0)^2}, \tag{8}$$

where N is the number of conduction electrons/cm^3, $E_0=(\hbar^2/2m)\lambda^2$, $E_p=(\hbar^2/2m)K^2$, and $E_{kk'}=E'=(\hbar^2/2m)k'^2$ is the energy loss. The total rate of energy loss is, therefore,

$$-\frac{dE_p}{dx} = \int_0^{E_p} E_{kk'}P(E_{kk'})dE_{kk'} = \frac{\pi Ne^4}{E_p}\left[\log\left(\frac{E_p+E_0}{E_0}\right)-\left(\frac{E_p}{E_p+E_0}\right)\right]. \tag{9}$$

For $E_p\gg E_0$ this may be written

$$-\frac{dE_p}{dx} = \frac{\pi Ne^4}{E_p}\log\left(\frac{E_p}{\epsilon E_0}\right), \tag{9a}$$

where ϵ is the base of natural logarithms.

[3] D. E. Wooldridge, Phys. Rev. **56**, 562 (1939).
[4] E. M. Baroody, Phys. Rev. **78**, 780 (1950).
[5] J. F. Marshall, Phys. Rev. **88**, 416 (1952); E. M. Baroody, Phys. Rev. **89**, 910 (1953).

We now investigate how these results are modified if the velocity distribution of the lattice electrons is taken into account. We observe that (7) expresses the number of transitions from a single state of wave vector $\mathbf{k}$ to a new wave vector having an absolute value between k' and $k'+dk'$.

We therefore multiply (7) by the number of electrons occupying a given region in $\mathbf{k}$ space. The number of states in $\mathbf{k}$ space from which an electron may be scattered into a new state k' by an increase in momentum equal to $\hbar|\mathbf{k}'-\mathbf{k}|$ is then equal to

$$2.2\pi k^2dk\,\sin\theta d\theta/8\pi^3,$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{k}'$. The number of transitions per unit time from an initial state with a wave vector of magnitude between k and $k+dk$ to a final state with a wave vector of magnitude between k' and $k'+dk'$ is found approximately by integrating with respect to θ and ϕ, this yields

$$P(k,k')dkdk' = \frac{8m^2e^4k'dk'k^2dk}{\pi\hbar^4K^2[(k^2+k'^2+\lambda^2)^2-(2kk')^2]}. \tag{10}$$

Integrating this expression with respect to k between the limits 0 and k_m, where k_m corresponds to the absolute value of the wave vector at the Fermi level, gives the total transition probability to a state between k' and $k'+dk'$,

$$P_{\text{tot}}(k')dk' = \frac{4m^2e^4k'dk'}{\pi\hbar^4K^2}\left\{-\frac{1}{4k'}\log\left[\frac{(k'+k_m)^2+\lambda^2}{(k'-k_m)^2+\lambda^2}\right] + \frac{1}{2\lambda}\tan^{-1}\left(\frac{2k_m\lambda}{k'^2+\lambda^2-k_m^2}\right)\right\}. \tag{11}$$

Considering the logarithmic term as a function of $2k'k_m/(k_m^2+k'^2+\lambda^2)$ and making a series expansion of (11) gives in first approximation

$$P(E')dE' \simeq \frac{\pi Ne^4}{E_p}\frac{dE'}{(E'+E_0)^2}, \tag{12}$$

where $E'=(\hbar^2/2m)k'^2$, which is identical with (8). This result is obtained by using only the first terms in the series expansion, rearranging some terms, bearing in mind that usually $k'^2+\lambda^2\gg k_m^2$, and substituting $k_m^3=3\pi^2N$, where N is the number of conduction electrons/cm^3. This expression remains finite at the Fermi level and extends to relatively large energies if $E_0\gg E_m$. The infinite transition probability to unoccupied states at the Fermi level has thus been eliminated.

Introducing into (10) the energy loss $E_{kk'}=(\hbar^2/2m)\times(k'^2-k^2)$ as a new variable and $E_0=(\hbar^2/2m)\lambda^2$ as a constant and integrating with respect to k, we obtain the rate $P(E_{kk'})dE_{kk'}$ at which an energy loss between $E_{kk'}$ and $E_{kk'}+dE_{kk'}$ occurs. In carrying out this integration we have to observe that all conduction elec-

trons can contribute to $E_{kk'}$ if $E_{kk'} \geqq E_m$. For $E_{kk'} \leqq E_m$ only those electrons can contribute for which $E(k) + E_{kk'} \geqq E_m$; here $E_m = (\hbar^2/2m)k_m^2$ and k_m is the absolute value of the wave vector at the Fermi level. For $E_{kk'} \geqq E_m$ the limits of integration are therefore 0 and k_m, whereas $k_m(1-E_{kk'}/E_m)^{\frac{1}{2}}$ and k_m are the limits of integration for $E_{kk'} \leqq E_m$. Carrying out the integration we obtain

$$P(E_{kk'})dE_{kk'} = \frac{\pi N e^4 dE_{kk'}}{E_p} \cdot \frac{3}{4E_0E_m} \times \left[1 - \left(\frac{E_{kk'}+E_0}{2(E_0E_m)^{\frac{1}{2}}}\right) \tan^{-1}\left(\frac{2(E_0E_m)^{\frac{1}{2}}}{E_{kk'}+E_0}\right)\right], \quad (13)$$

for $E_{kk'} \geqq E_{m'}$ whereas for $E_{kk'} \leqq E_m$,

$$P(E_{kk'})dE_{kk'} = \frac{\pi N e^4 dE_{kk'}}{E_p} \cdot \frac{3}{4E_0E_m}\left\{1 - \left(\frac{E_{kk'}+E_0}{2(E_0E_m)^{\frac{1}{2}}}\right) \tan^{-1}\left(\frac{2(E_0E_m)^{\frac{1}{2}}}{E_{kk'}+E_0}\right) - \left(1-\frac{E_{kk'}}{E_m}\right)^{\frac{1}{2}}\left[1 - \left(\frac{E_{kk'}+E_0}{2[E_0(E_m-E_{kk'})]^{\frac{1}{2}}}\right) \tan^{-1}\left(\frac{2[E_0(E_m-E_{kk'})]^{\frac{1}{2}}}{E_{kk'}+E_0}\right)\right]\right\}. \quad (14)$$

We thus see that $P(E_{kk'}) \to 0$ if $E_{kk'} \to 0$ so that the previous infinity at $E_{kk'}=0$ has been removed. Moreover, (14) has a maximum value for a value of $E_{kk'}$ that is slightly smaller than E_m. Previously such a maximum had been found experimentally by Rudberg and Slater;[6] they could only explain these maxima theoretically by treating the free electrons as bound. The screened potential thus leads to the same result in a less artificial manner.

Calculating the energy loss distribution we obtain a complicated expression which in first approximation is identical with (9). In the case $\lambda=0$ it was found that

$$-\frac{dE_p}{dx} = \frac{\pi N e^4}{E_p} \log\left(\frac{E_p}{E_0'}\right), \quad (15)$$

where $E_0' = 0.277E_m$ and $E_m = (\hbar^2/2m)k_m^2$ as before. Comparing this expression with (9), we see that $-(dE_p/dx)$ decreases much more slowly with increasing E_p for the case $\lambda \neq 0$ than for the case $\lambda=0$ so that the screened potential also removes the third objection of the introduction. This result is important for the relative shape of the curve representing the secondary emission coefficient δ as a function of E_p.

[6] E. Rudberg and J. C. Slater, Phys. Rev. **50**, 150 (1936).

III. THE CASE OF STRONGLY BOUND ELECTRONS

Though the application of a screened potential function to the interaction between primary electrons and bound electrons does perhaps not have as firm a foundation as for the conduction electrons, it is at least interesting to investigate the result. The wave function $\psi_k(\mathbf{r})$ for an electron bound to an atom in a lattice point of radius vector $\mathbf{r}_i$ closely resembles an atomic wave function. In calculating the integral I of (3b) we may use the approximation

$$\exp[i(\mathbf{q}\cdot\mathbf{r})] = [1+i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}_i)]\exp[i(\mathbf{q}\cdot\mathbf{r}_i)]. \quad (16)$$

Making use of the orthogonality of $\psi_k(\mathbf{r})$ and $\psi_{k'}(\mathbf{r})$ leads to

$$|I|^2 = q^2\left|\int \frac{1}{q}[\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}_i)]\psi_k(\mathbf{r})\psi_{k'}^*(\mathbf{r})d\mathbf{r}\right|^2 = |L_{kk'}|^2 q^2, \quad (17)$$

where $|L_{kk'}|^2$ is the optical transition probability.[7] Substituting into (4) yields

$$P(\mathbf{K}, \mathbf{k}\to\mathbf{K}', \mathbf{k}')d\Omega' = \frac{4m^2e^4K'/K}{\hbar^4}|L_{kk'}|^2\frac{q^2}{(q^2+\lambda^2)^2}d\Omega'. \quad (18)$$

Choosing $\mathbf{K}$ as the Z axis of a polar coordinate system and specifying the direction of $\mathbf{K}'$ by its polar angle θ and its azimuthal angle ϕ gives

$$d\Omega' \simeq (qdq/KK')d\phi. \quad (18a)$$

Integrating with respect to ϕ (limits 0 and 2π) and with respect to q[limits $q_{\min} = mE_{kk'}/\hbar^2K$ and $q_{\max} = K$ (see reference 1)], we obtain for the rate at which transitions $\mathbf{k}\to\mathbf{k}'$ occur

$$P(\mathbf{k}, \mathbf{k}') = \frac{4\pi me^4}{\hbar^2E_p}|L_{kk'}|^2 \cdot \left[\frac{1}{2}\log\frac{4E_p(E_p+E_0)}{E_{kk'}^2+4E_pE_0} - \frac{1}{2}\left(\frac{4E_0E_p}{E_{kk'}^2+4E_0E_p} - \frac{E_0}{E_0+E_p}\right)\right]. \quad (19)$$

For $E_p \gg E_0$ and $E_{kk'}^2 \gg 4E_0E_p$, this expression reduces to the value found for ordinary Coulomb interaction ($E_0=0$),

$$P(\mathbf{k}, \mathbf{k}') = \frac{4\pi me^4}{\hbar^2E_p}|L_{kk'}|^2\log\left(\frac{2E_p}{E_{kk'}}\right). \quad (19a)$$

This should occur for the deeper-lying bound electron levels. If, however, $E_p \gg E_0$ and $E_{kk'}^2 \ll 4E_0E_p$, Eq. (19)

[7] It should be noticed that in general the value of the integral depends upon the direction of $\mathbf{q}$, unless the wave functions have spherical symmetry; we ignore this dependence here.

reduces to

$$P(\mathbf{k}, \mathbf{k}')=\frac{4\pi me^4}{\hbar^2 E_p}|L_{kk'}|^2 \cdot \frac{1}{2}\log\left(\frac{E_p}{\epsilon E_0}\right), \quad (19b)$$

where ϵ is the base of natural logarithms. This might occur for bound energy levels closest to the conduction band.

We conclude therefore that the screened potential does not modify Dekker and van der Ziel's results for the bound electrons very much; it only decreases the transition probability for those bound electrons that have energies closest to those in the conduction band. This is a very reasonable result.

IV. APPLICATION TO INSULATORS

In the case of the interaction between primary electrons and bound electrons in insulators, one has to take into account the polarization of the medium. The potential function is then equal to $e^2/|\mathbf{R}-\mathbf{r}|$ for small distances and $e^2/\epsilon_0|\mathbf{R}-\mathbf{r}|$ for larger distances, where ϵ_0 is the effective dielectric constant. It seems therefore that the potential function

$$V(\mathbf{R}, \mathbf{r})=\frac{e^2}{\epsilon_0|\mathbf{R}-\mathbf{r}|}+\frac{e^2(\epsilon_0-1)}{\epsilon_0|\mathbf{R}-\mathbf{r}|}\exp[-\lambda|\mathbf{R}-\mathbf{r}|], \quad (20)$$

where $1/\lambda$ is of the order of magnitude of atomic dimensions, should give a rough estimate of the influence of polarization. The integral J of (3a) then becomes

$$J=\frac{4\pi e^2}{\epsilon_0 q^2}+\frac{4\pi e^2(\epsilon_0-1)}{\epsilon_0(q^2+\lambda^2)}. \quad (21)$$

Since the electron of wave vector $\mathbf{k}$ is bound, the value of the integral I is given by (17). Introducing (21) and (17) into (3), using Eq. (18a) for $d\Omega$; and integrating with respect to ϕ and q yields the rate $P(\mathbf{k}, \mathbf{k}')$ at which transitions $\mathbf{k}\rightarrow\mathbf{k}'$ occur. The result is similar to the one obtained for bound electrons in a metal. For $E_p \gg E_0$ and $E_{kk'}^2 \gg 4E_0E_p$ the result reduces to the value found for zero polarization. For $E_p \gg E_0$ and $E_{kk'}^2 \ll 4E_0E_p$ we find that $P(\mathbf{k}, \mathbf{k}')$ somewhat smaller than for zero polarization. The polarization should thus only affect the transition probability for the more weakly bound electrons, which is a very reasonable result.

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