Degradation and Range Straggling of High-Energy Radiations*

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A general analysis of the energy spectrum resulting from the degradation of ionizing radiations is presented in part A (Secs. 2–6) and of the range-energy straggling in part B (Secs. 7–13). A method for calculating the energy spectrum is developed, which requires that the differential cross sections of successive collisions are nearly equal. This method is applied to the slowing down of mesons and other heavy charged particles (Sec. 6). The same requirement on the cross section for successive collisions underlies the analysis of straggling into contributions from separate energy intervals traversed in the course of degradation. Successive approximation formulas for the straggling parameters (cumulants) are derived (Sec. 12) and applied to the case of heavy charged particles (Sec. 13). The connections among the theoretical approaches of several authors are discussed.

1. INTRODUCTION

IONIZING radiations which travel through a material experience a succession of inelastic collisions. Each collision results in a subdivision, or "degradation," of the radiation energy and in a deflection of its line of flight and may also change the nature of the incident radiation; for example, a photon may be replaced by a photoelectron. Therefore the propagation of high-energy radiation through a large mass of matter leads to a complex process of degradation and diffusion.

This process has been the object of much study in the case of *neutrons*.[1] Work on x-rays and γ rays up to energies of about 10 Mev has been in progress for several years in this laboratory. *Heavy charged particles*, including mesons, are scattered only to a minor extent and therefore do not experience true diffusion. Their multiple small-angle scattering is fairly complex, but the simpler problems of heavy particle degradation and range straggling have been solved long ago in a good approximation which will be discussed further below. The penetration, degradation, and diffusion of *electrons* up to about 10 Mev offers serious difficulties and is being approached systematically only now. The theory of high energy *electron-photon* showers has, for the most part, taken advantage of the smallness of deflections and of simplified cross sections for the elementary processes at very high energies. A recent semiempirical study by Wilson[2] has pointed up the errors due to unrealistic approximations. Here, too, a systematic study is still to come. The problems encountered in the study of different radiations have much in common.

A comprehensive formulation of a process of penetration, diffusion, and degradation in an infinite medium, with a point isotropic source or with any plane source, involves at least three independent variables: (1) the distance from the source, (2) the obliquity of propagation with respect to a fixed direction, and (3) the energy attained in the course of degradation. A more general type of source, namely point collimated, requires the introduction of two additional variables. Experience with neutrons[1] and x-rays emphasizes the convenience of proceeding by steps, taking into account *one variable at a time*. This is done by averaging out the distribution of radiation over the other variables.

The process of energy degradation may be isolated in this manner from the other aspects of penetration and diffusion, by averaging over all variables other than the energy of a particle or photon. The single-variable problem of simple energy degradation inquires about the distribution-in-energy of the particles or photons which result from a succession of collisions, irrespective of their positions in space. Once this problem has been solved, one may consider a two-variable problem. The second variable may be either the path length traveled by a particle from its source, or the obliquity of its direction. The study of the distribution-in-path length of radiation particles of various energies pertains to the problem of range-energy straggling. The study of the distribution-in-direction of particles of various energies, irrespective of their position in space, has received less attention than the other, but has proven instructive in the x-ray problem.[3] Either of the two-variable problems may be reduced to a set of independent single-variable, energy degradation problems (see Sec. 8, and Appendix I, respectively). Even the more complete three-variable problem, including the distribution-in-distance from the source, can be reduced to a set of energy degradation problems, at least in certain cases.[4]

As a preliminary to an effort to solve the electron problems it was desired to make a general study of the energy degradation problem and of the two-variable problem of range-energy straggling. These studies are reported in parts *A* and *B* of the present paper, respec-

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[1] See, e.g., R. E. Marshak, Revs. Modern Phys. **19**, 185 (1947), and in particular, the work of Placzek, Adler, and Wick quoted therein; also, I. Waller, Arkiv. Mat. Astron. Fysik **34A**, No. 3 (1948).

[2] R. R. Wilson, Phys. Rev. **86**, 261 (1952).

[3] See, e.g., L. V. Spencer and F. A. Jenkins, Phys. Rev. **76**, 1885 (1949).

[4] See, e.g., L. V. Spencer and U. Fano, J. Research Natl. Bur. Standards **46**, 446 (1951).

tively. Each problem is first surveyed qualitatively, mostly on the basis of ideas developed in the study of the neutron[1] and of the x-ray problems. Direct methods of solution, which work well in the case of neutrons or x-rays, are inadequate for charged-particle problems where the distribution of energy losses in individual collisions is extremely skew. A different line of approach is afforded by the fact that the energy degradation process exhibits a steady-state feature when successive energy losses obey *nearly equal* statistical laws, that is, when the differential cross section changes little from one collision to the next. This condition is fulfilled by "elastic" neutron collisions,[1] by all heavy charged particle collisions owing to the smallness of the maximum fractional energy loss and by most electron collisions. (The occasional hard knock-on collisions and the radiative collisions experienced by electrons do not fit into this class.) The steady-state feature is exploited in the development of an analytical method of successive approximations (Sec. 5) which assumes a slow variation of the cross sections for successive collisions. This method is not yet adequate to solve the electron problems, but it applies to heavy charged particles and serves to calculate corrections to the earlier elementary theories. These corrections are of little quantitative importance, however, because they correspond to an expansion in powers of m/M (electron mass divided by the mass of the incident particle). The steady-state feature is also shown to afford the necessary basis for resolving the range-energy straggling into a sum of contributions due to the various energy intervals traversed by particles in the course of degradation (Sec. 9).

In conclusion, this paper intends to clarify a number of concepts in the theories of energy degradation and of range-energy straggling as well as the relationships among earlier theories. As a by product it contributes new corrective terms on the slowing-down and straggling of heavy charged particles.

A. ENERGY DEGRADATION

2. Formulation of the Problem

Consider a uniform medium containing a source of high energy particles or photons. The initial energy of each particle or photon is reduced in discrete steps as a consequence of successive collisions. Other secondary radiations may arise from these collisions. What is the resulting energy spectrum of the primary radiation and of its secondaries?

This problem belongs to a broad class of one-dimensional random walk problems and much of the following treatment appears to have a correspondingly broad range of applications.

The problem is, perhaps, formulated most visually if one considers a uniformly distributed source of constant intensity which emits $S(E)dE$ particles or photons of energy between E and $E+dE$ per unit volume and per unit time. Spectral equilibrium of the radiation in the medium requires that the rate of destruction of radiation of each energy (number of particles absorbed per unit volume and per unit time) equal the rate of generation of the same radiation by the source or by degradation from higher energy radiation. If $n(E)$ indicates the spectral density of the *flux* of particles or photons of energy E at each point (number traversing a small spherical probe of unit cross-sectional area per unit time per unit spectral range), $\mu(E)$ the total probability of inelastic collision per unit path, $k(E, \epsilon)d\epsilon$ the differential probability per unit path of a particle of energy E to undergo a collision with an energy loss between ϵ and $\epsilon+d\epsilon$, the degradation obeys the equation

$$\mu(E)n(E)=\int_0^\infty k(E+\epsilon, \epsilon)n(E+\epsilon)d\epsilon+S(E). \quad (1)$$

The range of integration is indicated as extended to infinity, but in practice infinitely large losses cannot occur and $k(E, \epsilon)$ will vanish for ϵ larger than some maximum value ϵ_M. The probability distribution $k(E, \epsilon)$ may contain terms $k_i\delta(\epsilon-\epsilon_i)$ corresponding to the excitation of discrete levels of energy ϵ_i in the atoms or nuclei of the medium.

Actually, the source need not be considered as uniformly distributed in space and time. $S(E)dE$ may be regarded as the *total number* of particles or photons, emitted with energy between E and $E+dE$, no matter where or when. The spectral distribution $n(E)dE$ is then normalized as the total *track length* covered by particles or photons while their energy lies between E and $E+dE$. This normalization will be adhered to in the following. For convenience we shall also assume a unit source strength, $\int_0^\infty S(E)dE=1$.

If one considers secondary radiations, such as the delta rays ejected by protons, one may classify the different kinds of radiation by an index $\alpha=1, 2\cdots$. The degradation obeys then the system of equations

$$\mu(\alpha, E)n(\alpha, E)=\sum_{\alpha'}\int_0^\infty k(\alpha, \alpha'; E+\epsilon, \epsilon)n(\alpha', E+\epsilon)d\epsilon+S(\alpha, E), \quad (2)$$

where α takes all its values and $k(\alpha, \alpha'; E, \epsilon)$ indicates the differential probability that a particle of radiation α', with energy E collides and generates a particle of radiation α with energy $E-\epsilon$. The terms of the $\Sigma_{\alpha'}$ with $\alpha'\neq\alpha$ represent in effect additional sources of the radiation α. The whole system (2) is more complicated but in principle no more difficult to solve than the single equation (1), provided that there are only a few kinds of radiation. In this paper, we consider only the single equation (1).

Knowledge of the spectral distribution $n(E)$ serves to answer numerous questions such as the following one: "How much energy is radiated as *Bremsstrahlung* by electrons in the course of their degradation?" The basic data to be provided by *Bremsstrahlung* theory is the energy $W(E)$ radiated per unit distance by electrons of energy E. The total energy radiated is then obtained by multiplying $W(E)$ by the track length $n(E)$ of electrons of energy E and integrating over the energy, $\int_0^\infty n(E)W(E)dE$. Notice that the usual calculations of this quantity assume that $n(E)$ is given by the approximate formula (3), below, which is inadequate just when *Bremsstrahlung* is intense.

3. Qualitative Discussion

(a) *The Limiting Case of Continuous Slowing Down*

Heavy charged particles dissipate their energy in such small bits that their slowing down may be regarded for many purposes as a continuous process. According to this model the track length covered by particles of energy E is given by the reciprocal of the stopping power $\langle -dE/ds \rangle$. This is to say that for this model the degradation Eq. (1) has the solution

$$n(E) \sim \langle -dE/ds \rangle^{-1} = \left[\int_0^\infty k(E, \epsilon) \epsilon d\epsilon \right]^{-1}. \quad (3)$$

Much of the effort of this paper is directed to formulate the specific assumptions which underlie the approximate solution (3) and to develop a procedure for taking into account the fact that individual losses are actually finite.

The model of continuous slowing down applies to a considerable extent also to electrons. However, an electron occasionally loses a large fraction of its energy in a single collision especially when *Bremsstrahlung* is important. These rare events have a great influence on the range straggling of electrons. Hence there arises a need for considering radical departures from the model of continuous slowing down, but also for making the best use of this model in so far as it offers a valid approximation.

(b) *Direct Integration*

The degradation Eq. (1) determines $n(E)$ if a knowledge of $n(E+\epsilon)$ is assumed ($\epsilon > 0$). (Similarly, the system (2) determines $n(\alpha, E)$ if all $n(\alpha', E+\epsilon)$ are known.) Therefore, one is naturally led to stepwise methods of solution, beginning with the highest source energy E_0, such that $n(E_0+\epsilon)$ vanishes, and progressing from high to low energies along the course of energy degradation.

Placzek[1] approached the neutron slowing-down problem along this line, solving (1) stepwise in successively lower energy intervals by analytical means. The degradation of x-ray photons by successive Compton scatterings has been treated by Karr and Lamkin by numerical evaluation of the integral in (1) at successively lower values of the energy.[5]

The numerical procedure may well be the most expeditious one in many circumstances. One of its main limitations lies in the necessity of extrapolating the knowledge of $n(E+\epsilon)$ over the last finite interval of integration. This leads into difficulties unless $n(E+\epsilon)$ and $k(E+\epsilon, \epsilon)$ vary smoothly and not too rapidly with ϵ. In the slowing down of charged particles the energy loss distribution $k(E+\epsilon, \epsilon)$ is sharply peaked at low values of ϵ, so that this direct approach is not immediately applicable.

(c) *Transient and Steady State*

An additional element in the qualitative analysis of the degradation problem emerges from the following considerations. Both in neutron and in x-ray degradation the spectral distribution tends to a shape independent of the source energy, at all energies sufficiently lower than the source energy itself. This trend arises from the randomness of the energy loss in individual collisions, which erases the "memory" of the initial energy of each particle or photon after a few collisions. (As noted below, not every collision need be equally significant in this connection.) One may say that the process of energy degradation consists of an initial "transient" phase, which involves the first few collisions, followed by a simpler "steady state."

(d) *Expansion According to Orders of Scattering*

The first few collisions can be treated, if necessary, by an elementary iteration procedure, or expansion according to "orders of scattering." According to (1), the spectrum of undegraded (unscattered) particles is $n_0(E) = S(E)/\mu(E)$, that of "first scattered" particles is $n_1(E) = \int_0^\infty k(E+\epsilon, \epsilon) n_0(E+\epsilon) d\epsilon / \mu(E)$; the recurrence formula $n_r(E) = \int_0^\infty k(E+\epsilon, \epsilon) n_{r-1}(E+\epsilon) d\epsilon / \mu(E)$ gives the spectrum of particles that have experienced r collisions.[6] This iteration procedure is laborious and has been found to be of value only under special circumstances. A direct application to the energy loss of a charged particle, where a significant energy loss involves an extremely large number of collisions, appears unpromising.

(e) *Discussion of the Steady State*

The spectrum under steady-state conditions can be easily formulated in favorable circumstances, where the spectral density $n(E)$ equals the reciprocal of a suitable average of the energy loss per unit path. This situation was pointed out by Placzek for the neutron problem.[1] It would also obtain for low-energy x-rays if one could disregard the photoelectric absorption.[7] Notice finally that the solution (3) to the degradation equation, in the limiting case of continuous slowing down, is also equal to the reciprocal of the mean energy loss. We seek here to identify the common element in these examples.

The concept of a "steady state" implies, usually, more than independence of the initial conditions. In addition it tends to imply, in our case, a smooth variation of the spectral density $n(E)$. This implies, in turn, a smooth variation of the degradation law $k(E, \epsilon)$

[5] P. R. Karr and J. C. Lamkin, Phys. Rev. **76**, 1843 (1949). The same method has been applied in references 3 and 4 and in further extensive work still in progress, by means of desk and of automatic computers.

[6] See, e.g., G. H. Peebles and M. S. Plesset, Phys. Rev. **81**, 430 (1951).

[7] The actual steady-state shape of the x-ray degradation spectra can be qualitatively understood in this manner, as pointed out by M. H. Johnson in a private discussion.

as a function of the energy E.[8] After all, any convenient treatment of the problem must rely on some regularity of the degradation law.

In fact the examples mentioned above have this in common, that their degradation law can be expressed in a form independent of the energy of the incident particle or photon. Neutrons below 10 Mev are slowed down by elastic scattering against nuclei with a probability of deflection which is uniform in the center-of-mass system. Consequently the probability of a neutron losing various *fractions* of its energy depends only on the ratio of its mass to the mass of the nuclei in the medium and not on the energy of the incident neutron. The variations of the *logarithm* of the energy in successive collisions obey identical probability laws. The steady-state spectrum of neutrons in a *logarithmic energy scale* is the reciprocal of the *mean logarithmic energy* loss per unit path. Similarly the Compton scattering of x-rays approaches a constant angular distribution at low energies. Therefore the *wavelength changes* of the x-rays in successive scatterings obey identical probability laws and the x-ray spectrum in the *wavelength scale* tends to the reciprocal of the *mean wavelength increase* per unit path. In the slowing down of protons, on the other hand, the determining factor is not so much that the relative probability of various energy losses varies slowly with the energy of the incident particle. Rather the energy loss in each collision is so small that successive collisions obey practically equal probability laws. Therefore, the approximate solution (3) of the degradation equation for protons does not rely strictly on the model of continuous slowing down but on the fact that the individual energy losses are so small that the changes of cross section from one collision to the next can be disregarded.

The neutron slowing down with constant probability law has been treated analytically by Adler.[1] This treatment brings out clearly the emergence of a steady state after a transient stage of damped oscillations. It is presented in Sec. 4, in a form suitable for application to any radiation, because it constitutes the point of departure for further development.

(*f*) *Proposed Line of Advance*

The Adler treatment assumes that successive energy losses experienced by a particle are governed by identical probability laws. In order to permit a flexible exploitation of the regularities indicated above, we shall develop in Sec. 5 an analytical solution of the degradation equation, assuming only that successive energy losses obey "nearly equal" probability laws. More specifically we shall calculate corrective terms to the approximate solution (3) for the slowing down of protons. Since the maximum fraction of energy lost by a proton in a collision is of the order m/M (i.e., electron mass/proton mass), successive corrections to the results based on continuous slowing down will involve successively higher powers of m/M.

As indicated in the introduction, the justification for deriving these corrections does not lie primarily in their practical importance for proton problems (where $m/M\sim10^{-3}$) or even for mesons (where $m/M\sim10^{-2}$). Rather, it is desired to ascertain whether the qualitative argumentation of the preceding sections can be developed into a consistent formalism.

The solution of the problem of the degradation of electron energy will still have to wait for the development of a technique to deal with the rare collisions with large energy loss. The "order of scattering" procedure indicated above in Sec. 3 might perhaps serve for this purpose, when combined with the techniques appropriate to the smaller energy losses. Large energy losses are not so rare for high-energy electrons, when bremsstrahlung is important. Here too, however, one may see some hope in the fact that the spectrum of bremsstrahlung losses is nearly energy-independent when expressed in a logarithmic scale.†

In order to improve the initial approximation represented by the Adler treatment, one may want to express the energy variable in any scale suitable for minimizing the variations of the probability law. Furthermore, for simplicity, we may assume that the radiation source is monochromatic, with energy

$$E_0,\ S(E)=\delta(E-E_0). \tag{4}$$

(No generality is lost here, since the degradation equation is linear.) Therefore the energy will be indicated by a general variable x, such that $x=0$ when $E=E_0$ and $x>0$ when $E<E_0$. We also set

$$y(x)dx=n(E)dE,\quad K(x,\xi)d\xi=k(E,\epsilon)d\epsilon. \tag{5}$$

The degradation equation (1) becomes then

$$\mu(x)y(x)=\int_0^\infty K(x-\xi,\xi)y(x-\xi)\,d\xi+\delta(x). \tag{6}$$

The near equality of probability laws in successive collisions does not apply so much to the absolute as to the relative probability of different energy losses. Therefore it is not $K(x,\xi)$ which should vary slowly with x but rather $K(x,\xi)/\mu(x)$, and it is convenient to introduce the additional substitution:

$$\bar{y}(x)=\mu(x)y(x),\quad \bar{K}(x,\xi)=K(x,\xi)/\mu(x). \tag{7}$$

Equation (6) takes now the simpler form:

$$\bar{y}(x)=\int_0^\infty \bar{K}(x-\xi,\xi)\bar{y}(x-\xi)d\xi+\delta(x). \tag{8}$$

[8] Sharp, localized, irregularities of the degradation law can be treated separately from the smooth variations. For example, Placzek has treated the absorption of neutrons in a sharp spectral line as a "negative source" which gives rise to a new transient (see reference 1).

† *Note added in proof:*—A solution of the electron degradation problem, which combines the features of continuous slowing down and of direct integration, is presented in a forthcoming paper by L. V. Spencer and U. Fano.

However, the final results will be expressed just as easily in terms of $K(x, \xi)$ as of $\bar{K}(x, \xi)$.

Equation (8) is a general one-dimensional random walk equation, with the restriction that every step is forward, i.e., $K(x, \xi)$ vanishes for $\xi<0$. Therefore it corresponds to the games of chance of the parcheesi type. $K(x, \xi)$ is by its nature a non-negative function (except in the applications of Appendix I).

Since $k(E, \epsilon)$ represents the probability of collision with a finite energy loss ϵ, the $\int_0^\infty k(E, \epsilon)d\epsilon$ represents the probability of all collisions with a finite energy loss. This probability cannot exceed the total probability $\mu(E)$ of all collisions. On the other hand $\mu(E)$ may exceed $\int_0^\infty k(E, \epsilon)d\epsilon$ if there is some process of outright absorption of the particles. Thus we have, in general:

$$\int_0^\infty k(E, \epsilon)d\epsilon \leqslant \mu(E), \tag{9}$$

$$\int_0^\infty K(x, \xi)d\xi \leqslant \mu(x), \tag{9.1}$$

$$\int_0^\infty \bar{K}(x, \xi)d\xi \leqslant 1, \tag{9.2}$$

where the equality sign holds in the absence of absorption, the inequality in the presence of absorption.

4. Constant Degradation Law

If $\bar{K}(x, \xi)=\bar{K}(\xi)$ in (8),

$$\bar{y}(x)=\int_0^\infty \bar{K}(\xi)\bar{y}(x-\xi)d\xi+\delta(x). \tag{10}$$

The homogeneous portion of this equation is invariant with respect to translations along x and, therefore, must have exponential solutions. One takes advantage of this fact by a Laplace transformation. In this type of problem many formulas are expressed more conveniently by introducing a Laplace transform variable p with sign opposite to the usual one, i.e., by setting

$$\begin{aligned}\bar{v}(p)&=\int_0^\infty \exp(px)\bar{y}(x)dx,\\ \bar{F}(p)&=\int_0^\infty \exp(p\xi)\bar{K}(\xi)d\xi.\end{aligned} \tag{11}$$

Multiplying the degradation equation (10) by $\exp(px)$ and integrating over x from 0 to ∞, we obtain the transform equation

$$\bar{v}(p)=\bar{F}(p)\bar{v}(p)+1. \tag{12}$$

The solution of (12) is $\bar{v}(p)=[1-\bar{F}(p)]^{-1}$, and the solution of (10) is represented by the inverse transform

$$\bar{y}(x)=(2\pi i)^{-1}\int_{-a-i\infty}^{-a+i\infty} \exp(-px)[1-\bar{F}(p)]^{-1}dp. \tag{13}$$

The singularities of the transform $\bar{v}(p)$ are poles located at the points $p=p_j$ which are roots of the equation

$$\bar{F}(p)=\int_0^\infty \bar{K}(\xi)\exp(p\xi)d\xi=1. \tag{14}$$

Assuming that these roots are simple, which appears in the following to be true, the integral (13) can be evaluated by shifting the path of integration toward $p=\infty$ and reduces to the residues at the poles,

$$\begin{aligned}\bar{y}(x)&=\textstyle\sum_j \exp(-p_jx)/\bar{F}'(p_j)\\ &=\textstyle\sum_j \exp(-p_jx)\Big/\int_0^\infty \bar{K}(\xi)\exp(p_j\xi)\xi d\xi.\end{aligned} \tag{15}$$

Since Eq. (13) is real, its roots are real or complex conjugate in pairs; pairs of terms of (15) with complex conjugate p_j's represent real damped sinusoids.

The distribution of the roots of (14) may be discussed by setting $p=\alpha+i\beta$ so that (14) splits into two real equations:

$$\int_0^\infty \bar{K}(\xi)\exp(\alpha\xi)\cos(\beta\xi)d\xi=1, \tag{14.1}$$

$$\int_0^\infty \bar{K}(\xi)\exp(\alpha\xi)\sin(\beta\xi)d\xi=0. \tag{14.2}$$

Equation (14.2) has a solution $\beta=0$. When $\beta=0$, $\cos\beta\xi=1$ in (14.1); the resulting expression $\int_0^\infty \bar{K}(\xi)\times\exp(\alpha\xi)d\xi$ is an increasing function of α, since $\bar{K}(\xi)$ is non-negative. Therefore (14) has a single real root, which we indicate as $p=p_0=\alpha_0$, $\beta=0$. In the absence of absorption $\int_0^\infty \bar{K}(\xi)d\xi=1$ and $p_0=0$; otherwise $\int_0^\infty \bar{K}(\xi)d\xi<1$ and $p_0>0$.

The other roots of (14), if any, must have values of β of such magnitude that the positive and negative values of the integrand of (14.2) average out. This requires that $\beta \gtrsim \pi/\xi_M$, where ξ_M is the largest value of ξ, beyond which $\bar{K}(\xi)$ vanishes. The $\cos(\beta\xi)$ in the integrand of (14.1) oscillates also, and therefore α will have to be larger than α_0 if (14.1) has to be fulfilled despite the partial cancellation of positive and negative values of the integrand. Two examples may be quoted. If $\bar{K}(\xi)=b\exp(-a\xi)$, there is no root other than $p_0=a-b$. If $\bar{K}(\xi)=1/a$ for $0\leqslant\xi\leqslant a$, $\bar{K}(\xi)=0$ for $\xi>a$, and we set $\beta_n=\pm(2n\pi+\epsilon_n)$, then $\epsilon_0=0$, $\epsilon_1\sim\pi/4$, $\lim_{n\to\infty}\epsilon_n=\pi/2$ and $\alpha_n=(1/a)\log[(2n\pi+\epsilon_n)/\sin\epsilon_n]$.

This discussion indicates that all terms of the sum (15) other than $j=0$ are damped out more rapidly than the $j=0$ term, as x increases, because $\alpha_j>\alpha_0$ for $j\neq 0$. The rate of damping is inversely related to the magnitude of the maximum energy loss ξ_M. All these terms together represent the "transient."

The $j=0$ term is a constant in the absence of absorption, where $p_0=0$. Its value, which represents the *steady-*

state value of $\bar{y}(x)$, is

$$\bar{y}(x)=1\Big/\int_0^\infty \bar{K}(\xi)\xi d\xi=1/\bar{M}_1 \text{ (steady state, no absorption),} \quad (16)$$

where

$$\bar{M}_1=\int_0^\infty \bar{K}(\xi)\xi d\xi \quad (17)$$

represents the first moment of the probability distribution $\bar{K}(\xi)$. According to (7) we have also

$$y(x)=1\Big/\int_0^\infty K(x,\xi)\xi d\xi=1/M_1(x) \text{ (steady state, no absorption),} \quad (18)$$

with

$$M_1(x)=\int_0^\infty K(x,\xi)\xi d\xi, \quad (19)$$

whenever $K(x,\xi)=\mu(x)\bar{K}(\xi)$ is the product of $\mu(x)$ and of a function of ξ, that is, whenever the relative probability of different energy losses is independent of x. Equation (18) includes (3) and the results for neutrons and x-rays mentioned in Sec. 3*e*.

In the presence of absorption, the formulas (16), (17), (18), and (19) become:

$$\bar{y}(x)=\exp(-p_0x)\Big/\int_0^\infty \bar{K}(\xi)\exp(p_0\xi)\xi d\xi = \exp(-p_0x)/\bar{L}_1 \text{ (steady state),} \quad (20)$$

$$\bar{L}_1=\int_0^\infty \bar{K}(\xi)\exp(p_0\xi)\xi d\xi, \quad (21)$$

$$y(x)=\exp(-p_0x)\Big/\int_0^\infty K(x,\xi)\exp(p_0\xi)\xi d\xi = \exp(-p_0x)/L_1(x) \text{ (steady state),} \quad (22)$$

and

$$L_1(x)=\int_0^\infty K(x,\xi)\exp(p_0\xi)\xi d\xi. \quad (23)$$

The parameters $\bar{L}_1$ and L_1 are the first moments of the statistical distributions $\bar{K}(\xi)\exp(p_0\xi)$ and $K(x,\xi)\times\exp(p_0\xi)$, respectively. These are not the probability distributions of the "energy losses" ξ which can be experienced by the particles with "energy" x, but rather the distributions of the "energy losses" experienced in the preceding collision which had led to x. The factor $\exp(p_0\xi)$ takes into account the fact that there were $\exp(p_0\xi)$ as many particles at $x-\xi$ than at x, the balance, $\exp(p_0\xi)-1$, having been absorbed.

The procedure of this section is still applicable if the source term in the initial equation (10) is a function $s(x)$ instead of $\delta(x)$. The unit term in the transform equation (12) must be replaced by the transform of the source function, $g(p)=\int_0^\infty s(x)\exp(px)dx$. When one evaluates the inverse transform

$$\bar{y}(x)=(2\pi i)^{-1}\int_{-a-i\infty}^{-a+i\infty}\exp(-px)[1-\bar{F}(p)]^{-1}g(p)dp = (2\pi i)^{-1}\int_{-a-i\infty}^{-a+i\infty}\exp(-px)[1-\bar{F}(p)]^{-1}dp\int_0^\infty s(x')\times\exp(px')dx', \quad (24)$$

the source strength at $x'>x$ does not contribute anything to $\bar{y}(x)$, as one would expect, that is the integral over x' is effectively limited at $0\leqslant x'\leqslant x$. This is verified by shifting the path of integration over p toward $p=-\infty$, since $\exp[p(x'-x)]$ vanishes there when $x'>x$. The separation of the steady state from the transient according to (15) is also applicable, but the transient does not become negligible unless the source strength $s(x')$ is confined to values of x' sufficiently smaller than x.

The procedure of this section is also applicable if the kernel $\bar{K}(x,\xi)$ of (8) is not constant but equal to different constant functions $\bar{K}_1(\xi)$, $\bar{K}_2(\xi)$, etc. in successive intervals $0\leqslant x\leqslant x_1$, $x_1\leqslant x\leqslant x_2$, etc. The constant kernel problem can be solved for each interval taking as a source the input of particles which are degraded from higher energy intervals. A new transient will occur at the beginning of each interval, but it will have a low amplitude if the difference of the kernels in successive intervals is small. Thus one might approach the problem of a continuous kernel variation $K(x,\xi)$ by regarding it as the limit of a succession of variations by finite steps.

5. Slowly Variable Degradation Law

The preceding results suggest that, when the degradation kernel $\bar{K}(x,\xi)$ is a slowly variable function of x, Eq. (8) should have solutions similar to individual terms of (15), $\exp(-p_jx)/\bar{L}_1$, modified by a slow variation of the parameter p_j. A form of this type is:

$$\bar{y}(j,x)\sim\exp\Big[-\int^x p_j(x')dx'\Big]\Big/\bar{L}_1(j,x), \quad (25)$$

$$y(j,x)\sim\exp\Big[-\int_0^x p_j(x')dx'\Big]\Big/L_1(j,x). \quad (25.1)$$

Here $\bar{L}_1$ and L_1 are given by (21) and (23), with p_j in place of p_0 and $p_j(x)$ is defined, for each value of x, as a root of the equation.

$$\int_0^\infty \bar{K}(x,\xi)\exp[p(x)\xi]d\xi=1, \quad (26)$$

or

$$\int_0^\infty K(x,\xi)\exp[p(x)\xi]d\xi=\mu(x), \quad (26.1)$$

which is analogous to (14).

This surmise is supported by the following developments. In particular, it will be shown that:

(a) For all values of $x>0$, where there is no source and the degradation equation is homogeneous, there is a set of solutions $y(j,x)$, whose form is approximately (25.1).

(b) The accuracy of each solution can be improved progressively by a procedure of successive approximations.

(c) The solutions of the homogeneous equation can be superposed, with suitable coefficients, so that

$$y(x)=\sum_j A_jy(j,x) \quad (27)$$

obeys the degradation equation (6) over the whole range of x, inclusive of the point $x=0$ where there is the source $\delta(x)$.

(d) As x increases, all terms of the $\sum_j$ in (27) with $j>0$ are rapidly damped out and the term with $j=0$ emerges as the steady-state solution.

The homogeneous equations corresponding to (8) and (6) are

$$\bar{y}(x)=\int_0^\infty \bar{K}(x-\xi,\xi)\bar{y}(x-\xi)d\xi \tag{28}$$

and

$$\mu(x)y(x)=\int_0^\infty K(x-\xi,\xi)y(x-\xi)d\xi, \tag{28.1}$$

respectively.

For purpose of orientation we consider first the case of no absorption, where $\int_0^\infty \bar{K}(x,\xi)d\xi=1$ and (26) has the solution $p_0(x)=0$. The corresponding spectral distribution of the form (26) reduces then to $\bar{y}(0,x)\sim 1/\bar{M}_1(x)$ and is, therefore, a slowly variable function of x. To determine this spectral distribution in a deductive manner, rather than by surmise, one may treat (28) by expanding $\bar{K}(x-\xi,\xi)\bar{y}(x-\xi)$ in the form $\bar{K}(x,\xi)\bar{y}(x)-\xi\partial[\bar{K}(x,\xi)\bar{y}(x)]/\partial x+\cdots$. This procedure is carried out in Appendix II. The solution $y(0,x)$ of the equation (28.1) in terms of the moments of the "degradation step size" ξ, averaged over the step-size distribution $K(x,\xi)$, i.e., in terms of

$$M_n(x)=\int_0^\infty K(x,\xi)\xi^n d\xi, \tag{29}$$

is

$$y(0,x)=(C/M_1)[1+\tfrac{1}{2}(d/dx)(M_2/M_1)+\tfrac{1}{4}(d/dx)(M_2/M_1)(d/dx)(M_2/M_1)-\tfrac{1}{6}(d^2/dx^2)(M_3/M_1)+\cdots], \tag{30}$$

where C is an undetermined integration constant.

The factor $1/M_1$ in front of the bracket coincides with (25.1), with $p_0=0$, as expected. To visualize the expression (30) as an expansion into successive orders of approximation, one may analyze the dimensions of its terms as follows. A ratio M_{n+1}/M_1 represents the mean of the nth power of the "step size" ξ averaged over the distribution $K(x,\xi)\xi$, and has the dimensions of ξ^n. The percent-variations of the ratios M_r/M_1 depend on the variations of the kernel $K(x,\xi)$, which are assumed to be small and are represented by the logarithmic derivatives of the ratios. A term of the expansion (30) which contains a total of n logarithmic differentiations-in-x, together with ratios M_r/M_1 with total dimensions ξ^n, constitutes a quantity small of the nth order. Thus, e.g., the second term within the bracket of (30) may be written in the form $\frac{1}{2}\{d[\log(M_2/M_1)]/dx\}(M_2/M_1)$ and clearly represents a first-order correction. Similarly the last two terms of (30) may be expressed by means of terms like $\{d^2[\log(M_3/M_1)]/dx^2\}(M_3/M_1)$ and represent second-order corrections.

As an illustration, consider the example: $K(x,\xi)=\mu(x)/a(x)$ for $0\leqslant\xi\leqslant a(x)$, $K(x,\xi)=0$ for $\xi>a(x)$. According to (29), $M_n(x)=\mu(x)a^n(x)/(n+1)$, and, according to (30), $y(0,x)=[2C/\mu(x)a(x)]\cdot[1+\frac{1}{3}da/dx-(1/36)d^2(a^2)/dx^2+\cdots]$.

Consider now the general case where one does not assume $p=0$ in the tentative solution (25). This case is of importance for various reasons, even though the steady-state distribution of particles corresponds ordinarily to $p=0$. In the first place the treatment of transients requires one to work with $p_j\neq 0$. Even with regard to steady states only, one may want to consider some outright absorption of particles by nuclear collisions. Moreover, in the degradation of electrons it might prove convenient to treat unusually large energy losses as separate processes which deplete the number of electrons that have experienced only small losses. Finally, the applications to two-variable problems, considered in part B and in Appendix I, introduce in effect an additional, positive or negative, absorption.

The variations of $\bar{y}(j,x)$ as a function of x need not be "slow" according to (25), when $p_j\neq 0$, unless p_j itself happens to be so small that $p_j\langle\xi\rangle\ll 1$. On the other hand, p_j itself, as a solution of (26), varies only on account of variations of $\bar{K}(x,\xi)$ and can therefore be assumed to vary slowly. This argument suggests the replacement of y with a slowly varying dependent variable. This is achieved by the substitution

$$y(j,x)=\exp\left[-\int_0^x q(j,x')dx'\right], \tag{31}$$

where $q(j,x)$ need not be small and is expected to be approximately equal to the root $p_j(x)$ of (26) and to vary slowly as a function of x. The lower limit of the integral in (31) is fixed arbitrarily in such a way that $y(j,0)=1$. The treatment of the homogeneous equation (28) by means of the substitution (31) is developed in Appendix III and we give here only the results.

Whereas the solution (30) for $p=0$ is expressed in terms of the moments M_n, the general solution for $p\neq 0$ is expressed in terms of the moments

$$L_n(j,x)=\int_0^\infty K(x,\xi)\exp[p_j(x)\xi]\xi^n d\xi \tag{32}$$

of the modified distribution $K(x,\xi)\exp[p_j(x)\xi]$. L_n corresponds to M_n in the same way as the first moment L_1 of (23) corresponds to the M_1 of (19). The index j will be dispensed with whenever it is not necessary for clarity.

The calculations of Appendix III show that $q(j,x)$

is given, up to second-order corrections inclusive, by

$$q(j,x)=p_j(x)+\left\{\frac{d\log L_1}{dx}-\frac{1}{2}\frac{dp_j}{dx}\frac{L_2}{L_1}\right\}$$
$$-\frac{1}{2}\left\{\frac{d^2}{dx^2}\frac{L_2}{L_1}-\frac{dp_j}{dx}\frac{d}{dx}\left[\frac{L_3}{L_1}-\frac{3}{4}\left(\frac{L_2}{L_1}\right)^2\right]\right.$$
$$\left.+\left(\frac{dp_j}{dx}\right)^2\left[\frac{1}{4}\frac{L_4}{L_1}-\frac{L_3}{L_1}\frac{L_2}{L_1}+\frac{1}{2}\left(\frac{L_2}{L_1}\right)^3\right]\right.$$
$$\left.-\frac{2}{3}\frac{d^2p_j}{dx^2}\left[\frac{L_3}{L_1}-\frac{3}{4}\left(\frac{L_2}{L_1}\right)^2\right]\right\}+\cdots. \quad (33)$$

The first term, $p_j(x)$, indicates a solution of (26) or (26.1). The discussion of these solutions is the same as for the equivalent equation (14). The solution with $j=0$ is the only one which is real; all other solutions have a real part, $\alpha_j(x)$, larger than $p_0(x)$. The solution of (26) for each particular kernel $K(x,\xi)$ constitutes a special problem, which may have to be solved numerically. The terms in the two curly brackets of (33) represent the first- and second-order corrections to the zero-order value p_j, respectively. The classification of these terms according to order of approximation is given by the number of differentiations which they contain.

To obtain $y(j,x)$, one enters the expression (33) for $q(j,x)$ into the substitution (31). The terms of q which do not contain p_j explicitly can be integrated directly. The first-order term $d\log L_1/dx$ contributes to y a factor $\exp(-\log L_1)=1/L_1$. This yields just the denominator of the expression (25.1) which had been surmised to indicate the approximate form of $y(j,x)$. The second-order term $-\frac{1}{2}d^2(L_2/L_1)/dx^2$ contributes to y a factor $\exp[\frac{1}{2}d(L_2/L_1)/dx]\sim 1+\frac{1}{2}d(L_2/L_1)/dx$ which is analogous to the first-order correction term in (30). Thus, first- and second-order terms in the expansion of $q(j,x)$, yield zeroth and first-order effects on $y(j,x)$, after integration over x. In general, the integration of the "small" terms of (33) over x implies an accumulation of corrective effects from all along the course of energy degradation. Accordingly, care must be exercized in determining what approximation is required in the calculation of $q(j,x)$.

There remains now to construct a solution to the full inhomogeneous degradation equation (6) by superposing the various solutions $y(j,x)$ of the homogeneous equation with suitable coefficients A_j, as indicated in (27). To determine the A_j's we construct the solution $\bar{y}(x)=\mu(x)y(x)$ of the inhomogeneous equation (8) by an entirely different method valid only for $x\sim 0$. The degradation kernel $\bar{K}(x-\xi,\xi)$ is expanded into powers of $x-\xi$, that is, beginning with its value at $x-\xi=0$ rather than at $x-\xi=x$. The calculation, whose details are given in Appendix IV, proceeds by the method of the Laplace transform, as in Sec. 4. The result is expressed in terms of the transforms of the derivatives $[\partial^n\bar{K}(x,\xi)/\partial x^n]_{x=0}$, namely,

$$\bar{F}_n(p)=\int_0^\infty [\partial^n\bar{K}(x,\xi)/\partial x^n]_{x=0}\exp(p\xi)d\xi, \quad (34)$$

which are treated as small quantities of order n. [If $\bar{K}$ is independent of x, as in Sec. 4, $\bar{F}_n$ vanishes for $n>0$, and $\bar{F}_0(p)$ coincides with $\bar{F}(p)$ of (11).] The solution, carried to first-order corrections only, is

$$\bar{y}(x)=(2\pi i)^{-1}\int_{-a-i\infty}^{-a+i\infty}\exp(-px)[1-\bar{F}_0(p)]^{-1}$$
$$\times[1+\bar{F}_1\bar{F}_0'/(1-\bar{F}_0)^2]dp$$
$$=\sum_j\{[\exp(-px)/\bar{F}_0']\left[1+\tfrac{1}{2}(\bar{F}_1''/\bar{F}_0'\right.$$
$$\left.-\bar{F}_1'\bar{F}_0''/\bar{F}_0'^2-\tfrac{3}{2}\bar{F}_1\bar{F}_0''^2/\bar{F}_0'^3)\right.$$
$$\left.-(\bar{F}_1'/\bar{F}_0'-\tfrac{1}{2}\bar{F}_1\bar{F}_0''/\bar{F}_0'^2)x\right.$$
$$\left.+\tfrac{1}{2}(F_1/\bar{F}_0')x^2+\cdots\right]\}_{p=p_j}, \quad (35)$$

where the primes indicate differentiation with respect to p.

We shall now compare this expression of $\bar{y}(x)$ with the expression derived from (27) and (31), namely,

$$\bar{y}(x)=\mu(x)y(x)$$
$$=\mu(x)\textstyle\sum_j A_j\exp\left[-\int_0^x q(j,x')dx'\right]. \quad (36)$$

At $x=0$ this expression reduces to $\mu(0)\sum_j A_j$ and it coincides with the corresponding value of (35) provided we set

$$A_j=[1/\mu(0)]\{(1/\bar{F}_0')[1+\tfrac{1}{2}(\bar{F}_1''/\bar{F}_0'$$
$$-\bar{F}_1'\bar{F}_0''/\bar{F}_0'^2-\tfrac{3}{2}\bar{F}_1\bar{F}_0''^2/\bar{F}_0'^3)-\cdots]\}_{p=p_j}. \quad (37)$$

The $\bar{F}$'s and their derivatives with respect to p can be expressed in terms of the $\bar{L}$'s or L's and of their derivatives with respect to x. Thereby (37) may be rewritten in the form

$$A_j=\left\{\frac{1}{L_1}\left[1+\frac{1}{2}\frac{d}{dx}\frac{L_2}{L_1}+\frac{1}{2}\frac{dp_j}{dx}\left(\frac{L_3}{L_1}+\frac{1}{2}\frac{L_2^2}{L_1^2}\right)+\cdots\right]\right\}_{x=0}. \quad (38)$$

The comparison of (36) with (35) has been extended to values of $x\neq 0$ by expanding (36) into powers of x up to x^2. The expansion is found to coincide with (35) provided A_j has the value (37). This comparison verifies the consistency of the independent calculations which lead to (33) and to (35).

6. Application to Heavy Charged Particles

As mentioned before, the formulas of the preceding section cannot be applied immediately to the slowing down of electrons but rather to protons and other heavy charged particles for which the maximum value

of ξ is small, so that the successive approximations [the successive terms of (30)] are bound to converge rapidly. This application serves to calculate corrective terms to the elementary equation (3).

Protons and other heavy charged particles are not subject to outright absorption except at high energies where the chance of head-on nuclear collisions becomes appreciable. If the absorption is disregarded, i.e. if p_0 is assumed to vanish, the steady-state term of the spectral density (27) reduces to

$$A_0 y(0, x)=\frac{1}{M_1}\left\{1+\frac{1}{2}\frac{d}{dx}\frac{M_2}{M_1}+\cdots\right\}, \qquad (39)$$

taking into account (31), (33) and (38). [Notice that this expression (39) coincides with the homogeneous solution (28), as it should, provided the integration constant C of (28) is set equal to 1.]

We take $x=E_0-E$, where E is the proton energy and E_0 its initial value. The moment

$$M_1=\int_0^\infty K(x, \xi)\xi d\xi=\int_0^\infty k(E, \epsilon)\epsilon d\epsilon$$

represents the stopping power (average energy loss per unit path). Its value may be indicated as

$$M_1=(k/2E)B, \quad B=\log(\gamma E/I)^2, \qquad (40)$$

where $k=2\pi NZz^2e^4M/m$; M, E, and ze are the mass, energy, and charge of the incident particle; NZ is the number of electrons per unit volume; e, m, and I are the charge, mass, and effective binding energy of the electrons in the medium; and $\gamma=4(m/M)(1+m/M)^{-2}\sim 4m/M$ is the maximum energy fraction lost in a single collision. B represents the stopping number.

The moment $M_2=\int_0^\infty K(x, \xi)\xi^2 d\xi=\int_0^\infty k(E, \epsilon)\epsilon^2 d\epsilon$ represents the mean square energy loss per unit path. From the theory of stopping power it follows that M_2 is given to a good approximation by[9]

$$M_2=(k/2E)[\gamma E+(4/3)TB], \qquad (41)$$

where T indicates the mean kinetic energy of the electrons in the medium. Therefore we have

$$M_2/M_1=\gamma E/B+(4/3)T. \qquad (42)$$

Taking into account the relationship between x and E, the spectral distribution (39) becomes in the energy scale,

$$n(E)=\frac{1}{M_1}\left[1-\frac{1}{2}\frac{d}{dE}\frac{M_2}{M_1}+\cdots\right]$$
$$=\frac{2E}{kB}\left[1-\frac{1}{2}\frac{\gamma}{B}+\frac{\gamma}{B^2}+\cdots\right]. \qquad (43)$$

The factor in front of the brackets coincides with the elementary solution (3). The first-order corrective terms in the brackets are proportional to γ, as expected. The term γ/B^2 arises from the derivative of B, and is considerably smaller than $\frac{1}{2}\gamma/B$, since B is of the order of 10.

The corrective terms are, of course, small owing to the smallness of γ. It may be worth noticing, however, that the accuracy attained in the zero-order approximation depends on the relationship between x and E which one assumes. For example, one might take into account from the outset that the maximum *fraction* of energy lost in a collision, γ, is independent of the energy. Accordingly, the degradation kernel $\bar{K}(x, \xi)$ is more nearly energy independent if represented in a logarithmic energy scale, that is, if one takes $x=\log(E_0/E)$. This choice of x leads to a result equivalent to (43), but with the corrective term $\frac{1}{2}\gamma/B$ incorporated in the zero-order approximation $1/M_1$ (see also Appendix VII).

The corrected value (43) of $n(E)$ is a little lower than the value $2E/kB$ derived from the model of continuous slowing down. This shows that the finite size of the degradation steps effectively shortens the average track length traveled by protons of each energy. A related effect of the fluctuations of energy loss is familiar from the theory of neutron slowing down, where the mean number of collisions required for a certain degradation depends on the mean logarithmic energy loss per collision rather than on the simple mean energy loss. However, it should not be concluded here that the mean range of protons is a little shorter than expected from the model of continuous slowing down. The systematic treatment of the range distribution in Sec. 13 will show that the range shortening effect considered above is overcompensated by the chance of unusually low energy losses during the transient phase of slowing down, at energies just below the initial energy.

B. RANGE-ENERGY STRAGGLING

7. Background

The energy loss of radiations along their path in a material is subject to statistical fluctuations. This effect of "straggling" is rather small for heavy charged particles and rather large for electrons; for neutrons and x-rays the fluctuations are so large that the mean energy loss per unit path length is hardly a significant parameter.

Certain basic features of the phenomenon of straggling were established in Bohr's early theory of stopping power.[10] Among these is the fact that successive intervals of energy degradation contribute additive amounts to the mean square fluctuation of energy loss. The Bohr theory applies primarily to heavy charged particles since it regards the fluctuations as sufficiently small to be well represented by a Gaussian distribution.

[9] The inelastic collisions may be classified according to the energy Q that an atomic electron would receive if it could recoil freely. The probability of collision can then be written as $k(E, \epsilon, Q)d\epsilon dQ=(k/2E)(dQ/Q^2)|F(\epsilon, Q)|^2 d\epsilon$ where $|F(\epsilon, Q)|^2$ is the "generalized form factor." One finds from sum rules that $\int_0^\infty |F(Q, \epsilon)|^2\epsilon d\epsilon=Q$ and $\int_0^\infty |F(Q, \epsilon)|^2\epsilon^2 d\epsilon=Q^2+4QT/3$. The first of these formulas serves to calculate the stopping power M_1, the second serves to calculate M_2.

[10] N. Bohr, Phil. Mag. **30**, Series 6, 581 (1915).

In the following sections we propose to formulate a general treatment of the energy distribution of particles which have traveled a certain path length from their source and of the path length distribution of particles which have been degraded to a certain energy. Attention will be directed to the conditions under which successive intervals of energy degradation contribute additively to the straggling. The problem will be treated as a two-variable problem from the point of view outlined in Sec. 1. The treatment will rely heavily on the concepts and techniques of the energy degradation problem. Since these techniques are primarily applicable to the slowing down of heavy charged particles, the immediate application of our methods will be to verify, analyze and extend the recent results of Lewis.[11]

It is hoped that these methods will also prove useful for analyzing and developing further the work of Wilson,[12] of Blunck,[13] and of Landau and Symon,[14] on the straggling of electrons. These authors, as well as Lewis, rely on simplified representations of the collision cross sections and do not make it quite obvious how critically their results depend on the initial assumptions.

Finally, it should be noted that the study of the distribution in energy and path length finds a direct application to certain penetration problems. Whenever the penetration of radiation in a material is accompanied by only small deflections, the depth of penetration attained is approximately equal to the path length traveled. Increments of path length and increments of depth of penetration are in a ratio equal to the cosine of the angle φ between the instantaneous direction of travel and the initial direction. Approximate theories are often based on the assumption $\cos\varphi\sim 1$, which is called the "straight ahead" approximation. These theories may be regarded as theories dealing with the distribution in path length rather than with the distribution in depth.

8. The Energy-Path Length Equation and Its Laplace Transform

Consider the energy-path length distribution function $f(E, s)$ of particles or photons which have an energy E and which have traveled a path length s from their source. The distance s is to be measured along the zigzag path actually followed. If one deals with a source distributed uniformly in space and time, with a strength measured in particles emitted per unit volume and per unit time, $f(E, s)dEds$ represents the flux of particles traversing a small spherical probe of unit cross-sectional area per unit time, with energy between E and $E+dE$ and pathlength between s and $s+ds$ (see Sec. 2). If, on the contrary, one considers only the total number of particles emitted in the medium, no matter where and when, then $f(E, s)dEds$ represents the track length covered by the particles while their energy and path length lie within the specified limits, again as in Sec. 2. We shall follow the latter point of view, so that $f(E, s)dEds$ has the dimensions of a length and $f(E, s)$ the dimensions of a reciprocal energy. Accordingly, $\int_0^\infty f(E, s)dE$ is a number, the number of particles which travel at least a distance s in the course of their degradation. On the other hand, $\int_0^\infty f(E, s)ds$ coincides with the spectral density $n(E)$ of part A.

The distribution function obeys a familiar transport equation, which is formulated on the same basis as the degradation equation (1) and is

$$\partial f(E, s)/\partial s = -\mu(E)f(E, s) + \int_0^\infty k(E+\epsilon, \epsilon)f(E+\epsilon, s)d\epsilon + S(E)\delta(s), \quad (44)$$

with the boundary condition $f(E, s)=0$ for $s<0$. The Dirac function $\delta(s)$ indicates that the source constitutes the point of zero path length.

It will prove advantageous to treat the distribution function $f(E, s)$ in terms of its Laplace transform

$$\phi(E, \sigma) = \int_0^\infty \exp(\sigma s)f(E, s)ds. \quad (45)$$

(Here, as in part A, it is convenient to define the transform variable σ with a sign opposite to that of common practice.) The transform of the transport equation (44), obtained by multiplication by $\exp(\sigma s)$ and integration over s, is

$$[\mu(E)-\sigma]\phi(E, \sigma) = \int_0^\infty k(E+\epsilon, \epsilon)\phi(E+\epsilon, \sigma)d\epsilon + S(E). \quad (46)$$

This is a degradation equation of the type (1), with $\mu(E)-\sigma$ in the place of $\mu(E)$. The variable σ may be regarded as a parameter and the equation may be solved separately for different values of σ, that is, as though it were an equation with the single variable E.

By inverting the Laplace transform we represent the distribution function as

$$f(E, s) = (2\pi i)^{-1}\int_{-a-i\infty}^{-a+i\infty} \exp(-\sigma s)\phi(E, \sigma)d\sigma. \quad (47)$$

The integral

$$\int_0^\infty f(E, s)ds = \phi(E, 0) \quad (48)$$

represents the spectral distribution without reference to the path length traveled and is the solution of (46) for $\sigma=0$, i.e., of the degradation equation (1), as men-

[11] H. W. Lewis, Phys. Rev. **85**, 20 (1952).
[12] R. R. Wilson, Phys. Rev. **84**, 100 (1951).
[13] O. Blunck, Z. Physik **131**, 354 (1952).
[14] L. D. Landau, J. Phys. (U. S. S. R.) **8**, 201 (1944); K. R. Symon, Harvard thesis, 1948 (unpublished).

tioned above. Accordingly the ratio

$$f(E, s)/\phi(E, 0) \tag{49}$$

represents the *path length distribution* of all particles which happen to have the energy E.

Often one may want to consider the related distribution of the path length at which the particle *energy drops below* the value E. This distribution is

$$g(E, s) = -(d/ds)\int_E^\infty f(E', s)dE', \tag{50}$$

and its transform is

$$\Gamma(E, \sigma) = \int_0^\infty \exp(\sigma s) g(E, s) ds = \sigma \int_E^\infty \phi(E', \sigma) dE'. \tag{51}$$

The distributions (49) and (50) must tend to coincide in the limiting case of continuous slowing down (Sec. 3a). Their relationship will be discussed in the following sections.

9. Laplace Transform and the Additivity of Straggling

The representation of the distribution function by a Laplace transform serves not only to "separate" the energy and path length variables in the transport equation (46) but also to combine conveniently the effects of independent statistical fluctuations. This is a well-known method of statistics which can be illustrated by the following schematic example.

Suppose that a quantity x is the sum of two quantities x_1 and x_2, which are subject to independent statistical fluctuations. Call $f_1(x_1)$ the statistical distribution of x_1 and $f_2(x_2)$ that of x_2; call $\phi_1(\sigma)$ and $\phi_2(\sigma)$ the Laplace transforms of f_1 and f_2. The statistical distribution of $x = x_1 + x_2$ is $f(x) = \int_0^\infty f_1(x_1) f_2(x - x_1) dx_1$. According to the well-known "folding" theorem, which can be easily verified, the Laplace transform of $f(x)$, $\phi(\sigma)$, is simply the product of $\phi_1(\sigma)$ and of $\phi_2(\sigma)$. Thus the combination law of the independent statistical fluctuations of x_1 and x_2 takes a simple form when expressed in terms of the distribution transforms ϕ_1 and ϕ_2. The combination law can be cast in the form of an addition, corresponding to the addition of x_1 and x_2 by writing

$$\log\phi(\sigma) = \log\phi_1(\sigma) + \log\phi_2(\sigma). \tag{52}$$

Let us see how this combination law applies to the straggling of charged particles. Bohr's theory of straggling relies on the model of "continuous slowing down" (Sec. 3a). According to this model each particle traverses every small energy interval dE in the course of its progressive degradation. The statistical fluctuations of pathlength within different intervals dE are, of course, independent and should accordingly contribute additive amounts to $\log\phi(E, \sigma)$. This argument is formulated by writing

$$\log\phi(E, \sigma) = \int_E \varphi(E', \sigma) dE'; \tag{53}$$

$\varphi(E', \sigma)dE'$ represents the logarithm of the transform of the distribution of path lengths traveled during the degradation through dE'.

The relationship between ϕ and φ in (53) is essentially the same as that between y and $-q$ in (31), which suggests a close connection between the additivity of pathlength fluctuations and the solution of the degradation problem by the method of Sec. 5. True enough, one can always introduce the relationship (53) in a formal manner, by defining φ as the logarithmic derivative of ϕ. However, the substitution of ϕ with φ appears physically meaningful only when the value of φ for each energy interval can be calculated independently of its values in other energy intervals. Independent calculation is, in fact, achieved in Sec. 5, where $q(x)$ is given in terms of the values at x of the "moments" L_n, i.e., of the kernel $K(x, \xi)$, and of their derivatives. Therefore the path length fluctuations will be determined by adding contributions from successive energy intervals provided the transform equation can be solved by the method of Sec. 5 and the solution represented adequately by its steady-state component. In other words, the additivity of straggling hinges on the near equality of the probability laws in successive collisions and on the attainment of a steady state rather than on the model of continuous slowing down.

10. Inversion of the Laplace Transform

In general one may expect to have to solve the transform equation (46) numerically for particular values of σ. Even so, the subsequent problem of inverting the transform by means of (47) need not be very difficult. This was shown, for example, by Spencer in an analogous study of x-ray penetration.[15]

The topography of the Laplace transform on the complex plane has an important common feature in the penetration problem and in the present one, namely, that the singularities are confined to the portion of the positive real axis where $\sigma = \mu(E)$. Inspection of the transform equation (46) shows that a singularity arises when $\sigma = \mu(E)$. If the right side of (46) is finite and $\mu(E) - \sigma$ vanishes, $\phi(E, \sigma)$ becomes infinite. Furthermore, if $\phi(E, \sigma)$ is infinite for a certain value of E, this infinity will appear in the right side, under the integral, when the equation is solved for lower values of E and thereby will propagate through the rest of the degradation process. This is to say that the transform $\phi(E, \sigma)$ is singular whenever σ equals any value which μ takes in the whole energy range between E and the source energy. The smallest value of σ which leads to a singularity equals the smallest value, μ_s, of $\mu(E)$ in the range of integration. To avoid singularities σ must remain $< \mu_s$, if it is real. Owing to this topography, the steepest descent path of integration in the inverse transform (47) passes through a single saddle point to the left of $\sigma = \mu_s$.[15,16]

[15] L. V. Spencer, Phys. Rev. **88**, 793 (1952).

[16] See also U. Fano, Phys. Rev. **76**, 739 (1949) which deals with the equivalent problem of x-ray penetration in the straight-ahead approximation and a more general discussion by U. Fano, J. Research Natl. Bur. Standards **51**, 95 (1953).

As discussed by Spencer,[15] it is possible under these conditions to evaluate the inverse transform integral (47) on the basis of the values of $\phi(E, \sigma)$ for a few values of σ on both sides of the saddle point. One method of evaluation consists of approximating $\phi(E, \sigma)$ over an interval of the real axis by a suitable analytical function of σ, whose inverse transform is known. The resulting analytical function of s constitutes a good approximation to $f(E, s)$ for values of s such that the saddle point of $\exp(-\sigma s)\phi(E, \sigma)$ lies within the interval where the approximation holds.

11. Survey of the Path Length Distribution

In order to apply the technique indicated above, or any similar procedure, one must have good understanding of the topography of $\phi(E, \sigma)$, particularly on the real axis of σ. The calculation by Landau[14] of the energy straggling of charged particles after a short travel serves as an illustration of the questions which arise in this connection. This calculation is discussed in Appendix V. Examination of the transform equation (46) shows that $\phi(E, \sigma)$ diverges when $\sigma \geqslant \mu_s$, and is finite when $\mu(E)-\sigma$ remains positive for all values of E. Any decrease of σ, toward the left on the real axis, makes $\mu(E)-\sigma$ larger and therefore causes $\phi(E, \sigma)$ to decrease. In fact, not only $\phi(E, \sigma)$ but also its slope and all the higher derivatives are increasing functions of σ. This is seen by taking successive derivatives of (46) with respect to σ, which yields

$$[\mu(E)-\sigma]\phi^{(n)}(E, \sigma)=\int_0^\infty k(E+\epsilon, \epsilon)\phi^{(n)}(E+\epsilon, \sigma) + S(E)\delta_{n0}+n\phi^{(n-1)}(E, \sigma), \quad (54)$$

where $\phi^{(n)}=\partial^n\phi/\partial\sigma^n$. This equation shows that each derivative obeys the same Eq. (46) as ϕ itself but with a "source" proportional to the previous derivative. Therefore, since $\phi(E, \sigma)$ is itself positive, all its derivatives are in turn positive. Furthermore this chainwise relationship among the derivatives makes it so that successive derivatives increase more and more steeply as σ increases toward μ_s and also as the energy E decreases.

It follows that the saddle point, whose position is defined by the condition[17]

$$-\sigma s+\log\phi(E, \sigma)=\min, \quad (55)$$

or

$$\phi^{(1)}(E, \sigma)/\phi(E, \sigma)=s, \quad (56)$$

moves steadily to the right, toward $\sigma=\mu_s$, as s increases. Accordingly, the trend of the transform $\phi(E, \sigma)$ for positive σ, approaching μ_s, determines the trend of the path length distribution $f(E, s)/\phi(E, 0)$ over the largest values of s, i.e., it determines the long path length tail of the distribution. Similarly, the trend of $\phi(E, \sigma)$ for negative σ, approaching $-\infty$, determines the short path length side of the distribution. The trend of $\phi(E, \sigma)$ for intermediate values of σ, near $\sigma=0$, determines the features of the middle and main part of the distribution.

[17] H. and B. S. Jeffreys, *Mathematical Physics* (Cambridge University Press, Cambridge, 1946), Chap. 17.

(a) *The Middle Portion of the Path Length Distribution*

It is widely known, and can be verified easily from (47), that the moments of a statistical distribution are equal to the derivatives of its transform, at $\sigma=0$. Thus the nth moment of the path length distribution (49), of the particles with energy E is

$$\langle s^n\rangle_E=\int_0^\infty ds s^n f(E, s)/\phi(E, 0)=\phi^{(n)}(E, 0)/\phi(E, 0). \quad (57)$$

Similarly the nth moment of the distribution (50) pertaining to particles whose energy drops below E is

$$\langle s^n\rangle_{[E]}=\int_0^\infty ds s^n g(E, s)=\Gamma^{(n)}(E, 0)=n\int_E^\infty \phi^{(n-1)}(E', 0)dE'.^{18} \quad (58)$$

Notice that the mean path length $\langle s\rangle_E$ equals $\phi^{(1)}(E, 0)/\phi(E, 0)$. Hence, according to (56), the saddle point lies exactly at $\sigma=0$ when s equals its mean value $\langle s\rangle_E$.

In order to take advantage of the additivity of straggling, discussed in Sec. 9, we consider parameters of the statistical distribution which relate to the derivatives of $\log\phi(E, \sigma)$ rather than to the derivatives of ϕ itself. These parameters are familiar to statisticians and are called *cumulants* because of their additivity property or also "semi-invariants of Thiele."[19] They have been used by Lewis[11] to characterize the range distribution of heavy charged particles.

According to (53) the derivatives of $\log\phi$ are straggling parameters which are represented as the sum of contributions from different energy intervals,

$$\partial^n \log\phi(E, \sigma)/\partial\sigma^n=\int_E \varphi^{(n)}(E', \sigma)dE'. \quad (59)$$

The cumulants are just the values of these parameters

[18] The relationship between the moments (57) and (58) is as follows. The expression (57) is the ratio of the solutions $\phi^{(n)}$ and ϕ of the equations (54) and (46). These differ only in their source terms which are $n\phi^{(n-1)}(E, \sigma)$ and $S(E)$, respectively. On the other hand (58) may be regarded as the ratio of the integrals of the sources, since the total source strength $\int_E^\infty S(E')dE'$ is understood to equal 1. Therefore (57) relates to (58) as the ratio of two solutions of the transform equations relates to the ratio of the corresponding source strengths. This relationship is obviously very nearly an identity under steady-state conditions, as we shall verify later on.

[19] See, e.g., H. Cramér, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, 1946), p. 185 ff. The relationships between the first four cumulants κ_n and the moments $\langle s^n\rangle_E$ are listed here for convenience.

$$\kappa_1=\langle s\rangle_E, \quad \kappa_2=\langle s^2\rangle-\langle s\rangle^2=\langle[s-\langle s\rangle]^2\rangle,$$
$$\kappa_3=\langle s^3\rangle-3\langle s^2\rangle\langle s\rangle+2\langle s\rangle^3=\langle[s-\langle s\rangle]^3\rangle,$$
$$\kappa_4=\langle s^4\rangle-3\langle s^2\rangle^2-4\langle s^3\rangle\langle s\rangle+12\langle s^2\rangle\langle s\rangle^2-6\langle s\rangle^4 = \langle[s-\langle s\rangle]^4\rangle-3\langle[s-\langle s\rangle]^2\rangle^2,$$
$$\langle s\rangle=\kappa_1, \quad \langle s^2\rangle=\kappa_2+\kappa_1^2, \quad \langle s^3\rangle=\kappa_3+3\kappa_2\kappa_1+\kappa_1^3,$$
$$\langle s^4\rangle=\kappa_4+3\kappa_2^2+4\kappa_3\kappa_1+6\kappa_2\kappa_1^2+\kappa_1^4.$$

for $\sigma=0$ and are usually indicated by

$$\kappa_n=[\partial^n \log\phi(E,\sigma)/\partial\sigma^n]_{\sigma=0}. \quad (60)$$

They represent, of course, the coefficients of the power-series expansion of $\log\phi(E,\sigma)$, that is of the following expansion of ϕ,

$$\phi(E,\sigma)=\phi(E,0)\exp\left(\sum_n \kappa_n\sigma^n/n!\right).$$

In Sec. 12 we shall calculate $\phi(E,\sigma)$ by the method of Sec. 5, i.e., by means of (27), (31) and (33), for the slowing down of heavy charged particles, to which the method is applicable. The cumulants will then be given, in the main, by an integral over the derivatives of the parameter q, which coincide with the $\varphi^{(n)}$ of (59).

The corresponding calculation for electrons will have to wait until the method of Sec. 5 is suitably adapted (see Sec. 3f). The additivity property of cumulants is not easily exploited in the slowing down of x rays and neutrons. However in these cases the equation (54) for successively higher values of n can be solved in succession, by direct numerical approach (see Sec. 3*b*).

Information on the transform $\phi(E,\sigma)$ for $\sigma\sim 0$, as represented by a knowledge of a few terms of its power expansion, i.e., of a few $\phi^{(n)}(E,0)$, can be utilized in various manners to construct an approximation to the path length distribution $f(E,s)/\phi(E,0)$ (see Appendix VI).

(*b*) *The Long Path Length Tail of the Distribution*

The behavior of $\phi(E,\sigma)$ when σ approaches μ_s has been studied particularly for the equivalent problem of x-ray penetration in the straight-ahead approximation.[16] For example, in the usual case where $\mu(E)$ increases monotonically as E decreases, ϕ turns out to vary in proportion to $\mu_s-\sigma$ raised to a power of the order of -1 or -2, which depends on the logarithmic derivative of $\mu(E)$ at the source energy. The range straggling of neutrons is rather analogous to that of x-rays.

The occurrence of an unusually long path length is favored by a succession of numerous collisions with a particularly low-energy loss. The calculation of $\phi(E,\sigma)$ for σ near μ_s reflects this relationship and can take advantage of it. A positive value of σ in the transform equation (46) is equivalent to a decrease of the absorption coefficient μ. Such a decrease may be visualized as indicating an effective multiplication of the particles at every collision. Therefore the particle population of low energies grows larger as σ begins to approach μ_s. It also consists to an increasing extent of particles that have experienced a long succession of small energy losses and thus have been boosted by repeated "effective multiplication." The preference, in the region of σ near μ_s, of low energy losses as compared to large energy losses tends to improve the accuracy of calculations in this region by the method of Sec. 5 which assumes a small change of probability law from one collision to the next.

To verify this surmise, we begin the application of Sec. 5 and specifically of Eq. (26.1), which reads now:

$$\mu(E)-\sigma=\int_0^\infty k(E,\epsilon)\exp(p_0\epsilon)d\epsilon. \quad (61)$$

The parameter p_0 pertains to the steady-state solution of the transform equation (46). As σ grows positive and $\mu-\sigma$ grows small, p_0 is forced to take negative values of increasingly large magnitude. Now, the solution of the transform equation by means of (31) and (33) depends on the moments L_n of the modified distribution of energy losses $k(E,\epsilon)\exp(p_0\epsilon)$. As p_0 grows large and negative, this distribution becomes increasingly confined to small energy losses ϵ. The convergence of the approximation method of Sec. 5 depends on the smallness of moment ratios such as L_2/L_1 and will obviously improve rapidly when p_0 grows large and negative.

The solution of the problem of straight-ahead x-ray penetration[16] was in effect a resort to the zero-order approximation of Sec. 5, justified by the large negative values of p_0.

In addition, that solution took advantage of the circumstance that the differential cross section $k(E,\epsilon)$ for x-rays and neutrons varies slowly for $\epsilon\sim 0$. As a result the $\int_0^\infty k(E,\epsilon)\exp(p_0\epsilon)d\epsilon$ in (61) converges rapidly toward a limit

$$\int_0^\infty k(E,\epsilon)\exp(p_0\epsilon)d\epsilon \sim k(E,0)\int_0^\infty \exp(p_0\epsilon)d\epsilon=-k(E,0)/p_0, \quad (62)$$

and the solution of (61) becomes

$$p_0\sim -k(E,0)/[\mu(E)-\sigma]. \quad (63)$$

This is just the result of reference 16. Its new and more general derivation points to possible improvements in the approximation.

The corresponding study for charged particles requires a different approach since $k(E,\epsilon)$ is peaked sharply at $\epsilon\sim 0$. For particles that have lost only a small fraction of their total energy, the answer is provided by the Landau theory[14] (see Appendix V). For larger energy losses, this study may be of minor importance in so far as excessively long path lengths of charged particles, and especially of heavy particles, are quite unlikely and their distribution might be determined rather well by the behavior of $\phi(E,\sigma)$ near $\sigma=0$.

(*c*) *The Short Path Length Tail of the Distribution*

The trend of $\phi(E,\sigma)$ for σ negative and large corresponds to the probability distribution of unusually short path lengths. It depends on the occurrence of an

unusually large proportion of collisions with large energy losses. No general analysis of this problem has been given. For x-rays and neutrons this problem is not of great interest. For heavy or light charged particles that have lost a small fraction of their energy, it is solved by the Landau theory.[14] For heavy charged particles that have lost much energy, extreme fluctuations of this kind are unlikely and the tail of the distribution can be predicted by the behavior of $\phi(E, \sigma)$ near $\sigma=0$ as was done by Symon[14] (see Appendix V).

12. Calculation of the Cumulants

We propose here to calculate the cumulants of the path length distribution, i.e., the parameters

$$\kappa_n=[\partial^n \log\phi/\partial\sigma^n]_{\sigma=0} \tag{64}$$

of Sec. 11*a*, under the assumption of a slowly variable degradation law, i.e., by the approximation method of Sec. 5. This method, in its present form, yields results which are applicable primarily to the slowing down of heavy charged particles.

In order to maintain flexibility in the choice of the energy scale and to utilize the formulas of part *A*, we proceed as in *A*, Sec. 3*f*, and represent the energy by a variable x, such that $x=0$ at a fixed source energy, $E=E_0$. The transform equation (46) will be written here in a form analogous to (6), namely

$$[\mu(x)-\sigma]y(x, \sigma)=\int_0^\infty K(x-\xi, \xi)y(x-\xi, \sigma)+\delta(x), \tag{65}$$

where $y(x, \sigma)dx=\phi(E, \sigma)dE$. Since dE/dx is independent of σ, the cumulants are given by

$$\kappa_n=[\partial^n \log y(x, \sigma)/\partial\sigma^n]_{\sigma=0}. \tag{66}$$

The solution of (65), for any particular value of σ, is given by (31) and (27) in the form

$$y(x, \sigma)=\sum_j A_j(\sigma)\exp\left[-\int_0^x q(j, x', \sigma)dx'\right]. \tag{67}$$

Confining ourselves to the steady-state term, $j=0$, which is permissible whenever x is much larger than its maximum change in a single collision, we take

$$y(x, \sigma)=A_0(\sigma)\exp\left[-\int_0^x q(0, x', \sigma)dx'\right] \quad \text{(steady state)}. \tag{68}$$

Dropping from now on the index $j=0$, we write

$$\log y(x, \sigma)=\log A(\sigma)-\int_0^x q(x', \sigma)dx'$$

and

$$\kappa_n=[\partial^n \log A(\sigma)/\partial\sigma^n]_{\sigma=0}-\int_0^x q^{(n)}(x')dx \quad \text{(steady state)}, \tag{69}$$

where $q^{(n)}=[\partial^n q/\partial\sigma^n]_{\sigma=0}$.

The parameters q and A are given, in turn, by the successive approximation formulas (33) and (38) in terms of still other parameters, p and L_n. Like q and A, p and the L_n's should be calculated, in principle, for each value of σ. However, the cumulants κ_n which we want to calculate are coefficients of the expansion of $\log y(x, \sigma)$ into powers of σ. Therefore we seek here the expansion of each parameter (p, L_n, q and A) into powers of σ. This problem is straightforward but its solution is algebraically complicated since it involves a double expansion, into powers of σ and into orders of approximation of the procedure of Sec. 5.

The expansion of $p(x, \sigma)$ may be written as

$$p(x, \sigma)=\sum_n p^{(n)}(x)\sigma^n/n!. \tag{70}$$

Since $p(x, \sigma)$ itself—the zero-order approximation value of $q(x, \sigma)$—has to be determined by solving Eq. (26.1),

$$\mu(x)-\sigma=\int_0^\infty K(x, \xi)\exp[p(x, \sigma)\xi]d\xi=L_0(\xi), \tag{71}$$

the coefficients $p^{(n)}(x)$ of (70) are to be found through the expansion of (71) into powers of σ. To begin with, the expansion of the factor $\exp[p(x, \sigma)\xi]$ yields[20]

$$\begin{aligned}&\exp[p(x, \sigma)\xi]\\&=\exp[p^{(0)}(x)\xi]\exp\Big[\sum_{n>0} p^{(n)}(x)\xi\sigma^n/n!\Big]\\&=\exp[p^{(0)}(x)\xi]\{1+\sigma p^{(1)}\xi+\tfrac{1}{2}\sigma^2[p^{(2)}\xi+p^{(1)2}\xi^2]\\&\qquad+\tfrac{1}{6}\sigma^3[p^{(3)}\xi+3p^{(2)}p^{(1)}\xi^2+p^{(1)3}\xi^3]+\cdots\}.\end{aligned} \tag{72}$$

From this expansion we derive the expansion of the L_n's by multiplying each side of (72) by $K(x, \xi)\xi^n$ and integrating over ξ. There result integrals of the type $\int_0^\infty K(x, \xi)\exp[p^{(0)}(x)\xi]\xi^n d\xi$. These integrals have the same form as the L_n's, but with $p^{(0)}$ in the place of p. They are the L_n values for $\sigma=0$, i.e., the L_n values pertaining to the simple degradation process, irrespective of path length distribution. Accordingly they will be indicated as $L_n{}^{(0)}(x)$.

$$\begin{aligned}L_n(x)=L_n{}^{(0)}&+\sigma p^{(1)}L_{n+1}{}^{(0)}\\&+\tfrac{1}{2}\sigma^2[p^{(2)}L_{n+1}{}^{(0)}+p^{(1)2}L_{n+2}{}^{(0)}]\\&+\tfrac{1}{6}\sigma^3[p^{(3)}L_{n+1}{}^{(0)}+3p^{(2)}p^{(1)}L_{n+2}{}^{(0)}\\&\qquad+p^{(1)3}L_{n+3}{}^{(0)}]+\cdots.\end{aligned} \tag{73}$$

The expansion of Eq. (71), which determines the $p^{(n)}$'s, results when the expansion (73) of L_0 is entered

[20] Each of the expressions in the brackets that multiply σ^2, $\sigma^3\cdots$ has the same structure as the expression in the footnote (17) which gives $\langle s^2\rangle$, $\langle s^3\rangle\cdots$ in terms of the cumulants $\kappa_1, \kappa_2\cdots$, with $p^{(n)}\xi$ in the place of κ^n, If $\exp[\Sigma_{n>0} p^{(n)}\xi\sigma^n/n!]=\Sigma_{\nu\geq 1}B_\nu\sigma^\nu/\nu!$, the coefficients B_ν are determined in the following manner. Consider all the partitions of the number ν, which are represented by a set of numbers ν_r such that $\Sigma_{r=1}^{\nu} r\nu_r=\nu_1+2\nu_2+\cdots=\nu$. For example, for $\nu=4$ the sets of ν_r's are (4000), (2100), (0200), (1010), (0001). We find that

$$B_\nu=\sum_{\text{all partitions}} \nu!\prod_r \frac{1}{\nu_r!}\left(\frac{p^{(r)}}{r!}\xi\right)^{\nu_r}.$$

into (71):

$$\mu(x)-\sigma=L_0^{(0)}+\sigma p^{(1)}L_1^{(0)}+\tfrac{1}{2}\sigma^2[p^{(2)}L_1^{(0)}+p^{(1)2}L_2^{(0)}]+\tfrac{1}{6}\sigma^3[p^{(3)}L_1^{(0)}+3p^{(2)}p^{(1)}L_2^{(0)}+p^{(1)3}L_3^{(0)}]+\cdots. \quad (74)$$

Separating out the groups of terms with the same power of σ, we obtain the set of equations:

$$L_0^{(0)}=\mu(x),\quad L_1^{(0)}p^{(1)}=-1,$$
$$L_1^{(0)}p^{(2)}+L_2^{(0)}p^{(1)2}=0, \quad (75)$$
$$L_1^{(0)}p^{(3)}+3L_2^{(0)}p^{(2)}p^{(1)}+L_3^{(0)}p^{(1)3}=0.$$

The first of these equations,

$$L_0^{(0)}(x)=\int_0^\infty K(x,\xi)\exp[p^{(0)}(x)\xi]d\xi=\mu(x),$$

is the equation (2.61) pertaining to the simple degradation process. From the standpoint of the path length distribution problem we regard this equation as solved in advance. (In the absence of absorption, $p_0=0$, of course.) If $p^{(0)}(x)$ is regarded as known, all the $L_n^{(0)}$ are known as well.

From this point on, the successive equations (75) yield the successive parameters $p^{(n)}(x)$ in terms of the moments $L_n^{(0)}(x)$, and one finds

$$\begin{aligned}
&p^{(1)}(x)=-1/L_1^{(0)},\\
&p^{(2)}(x)=-L_2^{(0)}/(L_1^{(0)})^2,\\
&p^{(3)}(x)=L_3^{(0)}/(L_1^{(0)})^4-3(L_2^{(0)})^2/(L_1^{(0)})^5,\\
&p^{(4)}(x)=-L_4^{(0)}/(L_1^{(0)})^5+10L_3^{(0)}L_2^{(0)}/(L_1^{(0)})^6-15(L_2^{(0)})^3/(L_1^{(0)})^7,\\
&p^{(5)}(x)=L_5^{(0)}/(L_1^{(0)})^6-[15L_4^{(0)}L_2^{(0)}+10L_3^{(0)}]^2/(L_1^{(0)})^7+105L_3^{(0)}(L_2^{(0)})^2/(L_1^{(0)})^8-105(L_2^{(0)})^4/(L_1^{(0)})^9.
\end{aligned} \quad (76)$$

The expression (69) of the cumulants κ_n is obtained in zero-order approximation, by replacing $q^{(n)}$ with its zero-order value $p^{(n)}$ given by (76). Here one may disregard $[\partial^n \log A/\partial\sigma^n]_{\sigma=0}$, since this term yields only a first-order correction as shown further below. Thus we write

$$\kappa_n(x)\sim-\int_0^x p^{(n)}(x')dx'. \quad (77)$$

The values thus obtained are those which measure the straggling according to the model of "continuous slowing down."

To obtain the higher order approximations of κ_n one must now take Eqs. (33) for $q(x)$ and (38) for A and expand them into powers of σ, utilizing the expansion (73) of the L_n's. This procedure is tedious but straightforward. We give here only the values of $q^{(1)}(x)$ and of $[\partial \log A/\partial\sigma]_{\sigma=0}$, calculated to first-order corrections inclusive:

$$\begin{aligned}
q^{(1)}(x)&=-\frac{1}{L_1^{(0)}}\left\{1+\left[\frac{d}{dx}\frac{L_2^{(0)}}{L_1^{(0)}}-\frac{1}{2}\frac{d\log L_1^{(0)}}{dx}\frac{L_2^{(0)}}{L_1^{(0)}}-\frac{dp^{(0)}}{dx}\left(\frac{L_3^{(0)}}{L_1^{(0)}}-\frac{(L_2^{(0)})^2}{(L_1^{(0)})^2}\right)\right]+\cdots\right\}\\
&=-\frac{1}{L_1^{(0)}}\left\{1+\left[\frac{1}{2}\frac{d}{dx}\frac{L_2^{(0)}}{L_1^{(0)}}-\frac{dp^{(0)}}{dx}\left(\frac{L_3^{(0)}}{L_1^{(0)}}-\frac{(L_2^{(0)})^2}{(L_1^{(0)})^2}\right)\right]+\cdots\right\}-\frac{1}{2}\frac{d}{dx}\frac{L_2^{(0)}}{(L_1^{(0)})^2}+\cdots;
\end{aligned} \quad (78)$$

$$\left[\frac{\partial\log A}{\partial\sigma}\right]_{\sigma=0}=\left[\frac{L_2^{(0)}}{(L_1^{(0)})^2}-\frac{1}{2}\frac{d}{dx}\left(\frac{L_3^{(0)}}{(L_1^{(0)})^2}-\frac{(L_2^{(0)})^2}{(L_1^{(0)})^3}\right)+\cdots\right]_{x=0}. \quad (79)$$

The value of the first cumulant κ_1 is given by (69), which combines the two formulas (78) and (79)[21] into

$$\begin{aligned}
\kappa_1(x)=\langle s\rangle_x&=\left(\frac{L_2^{(0)}}{(L_1^{(0)})^2}\right)_{x=0}+\int_0^x\frac{dx'}{L_1^{(0)}(x')}\left\{1+\left[\frac{1}{2}\frac{d}{dx'}\frac{L_2^{(0)}}{L_1^{(0)}}-\frac{dp^{(0)}}{dx'}\left(\frac{L_3^{(0)}}{L_1^{(0)}}-\frac{(L_2^{(0)})^2}{(L_1^{(0)})^2}\right)\right]+\cdots\right\}\\
&\quad+\frac{1}{2}\left(\frac{L_2^{(0)}}{(L_1^{(0)})^2}\right)_x-\frac{1}{2}\left(\frac{L_2^{(0)}}{(L_1^{(0)})^2}\right)_{x=0}+\cdots.
\end{aligned} \quad (80)$$

The significance of the various terms of this expression may be illustrated by comparing it with the expression for the related mean path length,

$$\langle s\rangle_{[E]}=\int_E^\infty\phi(E',0)dE'=\int_0^x y(x',0)dx', \quad (81)$$

of Eq. (58). We assume here absence of absorption, i.e., $p^{(0)}=0$ and $L_n^{(0)}=M_n$. Under these conditions, the integrand of (80), $(1/M_1)\{1+\frac{1}{2}d(M_2/M_1)dx+\cdots\}$ represents the steady-state value of $y(x',0)$ in (81), according to (30). The sum of the first and the last term of (80), namely, $\frac{1}{2}(M_2/M_1^2)_{x=0}$ can be shown to equal the integral of the transient terms of $y(x',0)$, which are appreciable only for $x\sim0$.[22] The remaining term of (80), $\frac{1}{2}(L_2^{(0)}/(L_1^{(0)})^2)_x$

[21] Only the zero-order term of $[\partial\log A/\partial\sigma]_{\sigma=0}$ has been carried, because the next term has an effect comparable to that of the second-order terms of $q^{(1)}$.

[22] To evaluate $\int_0^x y(x',0)dx'$ in (81) correctly in the range $x\sim0$ one can use the solution (35) valid in this range and carry out the integration over x before the integration over p. The error that would have been made by integrating over p first and then discarding the transient terms is seen to equal $\frac{1}{2}[L_2^{(0)}/(L_1^{(0)})^2]_{x=0}$. See also the treatment at the end of Appendix VII.

does not appear in the evaluation of (81) and therefore represents the true difference between the mean path lengths $\langle s\rangle_E$ and $\langle s\rangle_{[E]}$. Notice that the corrective terms,

$$[L_2^{(0)}/(L_1^{(0)})^2]_{x=0}-\tfrac{1}{2}[L_2^{(0)}/(L_1^{(0)})^2]_{x=0}=\tfrac{1}{2}[L_2^{(0)}/(L_1^{(0)})^2]_{x=0}$$

and $\frac{1}{2}[L_2^{(0)}/(L_1^{(0)})^2]_x$, are approximately equal to $\int dx'/L_1^{(0)}$ extended respectively, over one half-interval of $L_2^{(0)}/L_1^{(0)}$ at $x=0$ and over a similar interval at x. The ratio $L_2^{(0)}/L_1^{(0)}$ represents one "step of degradation." Therefore one may visualize the value (80) of the cumulant as equal to the integral of the steady-state value of $y(x', 0)$ from 0 to x, plus one half-step of degradation at the beginning of the degradation and one half-step at the end. These corrections represent effects of the discontinuity of the process of degradation.

13. Application to Heavy Charged Particles. Comparison with the Lewis Results

The formulas of the preceding section can be applied directly to the calculation of the mean path length and of the straggling of heavy charged particles. As in the application of the formulas on spectral distribution, we assume that there is no true absorption, so that $p^{(0)}(x)=0$ and $L_n^{(0)}(x)=M_n(x)$. We also choose $x=E_0-E$.

The first moment M_1, that is, the stopping power, will be indicated by $(k/2E)B$, as in (40). The higher moments can be shown to be, approximately

$$M_n\sim\tfrac{1}{2}k\gamma^{n-1}E^{n-2}/(n-1). \tag{82}$$

This approximation disregards corrective terms of the order of the atomic binding energy divided by the maximum energy loss in a collision, γE. [The second term in the bracket of (41), which constitutes a correction $4T/3B\gamma E$, is an example of the terms disregarded in (82).] These assumptions about the M's rely on the ordinary, nonrelativistic, theory of stopping power. Improved formulas may be entered in the theory without any additional difficulty.

The Bohr theory[10] of the mean range and of the mean square straggling may be described here as a calculation of the first and second cumulant in zero-order approximation. It utilizes (77), that is,

$$\kappa_n(E=0)\sim-\int_0^{E_0}p^{(n)}(E_0-E)dE, \tag{83}$$

with the values of $p^{(1)}$ and $p^{(2)}$ from (74). Blunck[12] gives also the formulas for κ_3 and κ_4.

An explicit calculation of the cumulants up to κ_5 in this approximation has been carried out by Lewis.[11] The Lewis results can be derived by entering the values (76) of $p^{(n)}$ into (83), substituting the M_n values (81) in place of the $L_n^{(0)}$ and performing the integration over the energy. If the stopping number B is regarded as constant (as in the Lewis "simplified problem") one obtains the Lewis formula (L23),[23] with B and γ in the place of Lewis' A_1 and ϵ_1. If the dependence of B on the energy is taken into account, one obtains (L23.1).

Lewis also introduced a new notion in the theory of the mean range, namely the need for a correction of the order of $\gamma\sim m/M$, which amounts to a fraction of 1 percent for μ mesons. The Lewis correction should appear in the present treatment as a first-order correction. According to (80) the "mean range," i.e., the mean path length of particles with no remaining energy, is

$$\begin{aligned}\kappa_1(0)=\langle s\rangle_0&=\frac{1}{2}\left(\frac{M_2}{M_1^2}\right)_{E=E_0}\\&+\int_0^{E_0}\frac{dE}{M_1}\left\{1-\frac{d}{dE}\frac{M_2}{M_1}+\cdots\right\}+\frac{1}{2}\left(\frac{M_2}{M_1^2}\right)_{E=0}\\&=\frac{\gamma E_0^2}{kB^2}+\int_0^{E_0}\frac{2EdE}{kB}\left\{1-\frac{1}{2}\frac{\gamma}{B}+\frac{\gamma}{B^2}+\cdots\right\}.\end{aligned} \tag{84}$$

The last term in the middle expression vanishes at $E=0$. Otherwise it would have to be discounted in the comparison with the Lewis theory, since Lewis calculates the mean $\langle s\rangle_{[0]}$, according to (58), instead of $\langle s\rangle_0$. The comparison must be made at $E=0$ because the Lewis theory deals only with the path length for zero energy.

The Lewis formula for the mean range (L20) is $\langle R\rangle=\int_0^{E_0}2EdE/kA_1$, where $A_1=B-\frac{1}{2}\gamma$.[24] If A_1 is expanded into powers of γ, $\langle R\rangle=\int_0^{E_0}(2EdE/kB)\times(1+\frac{1}{2}\gamma/B+\cdots)$. The discrepancy in sign between the correction term in this formula and the term γ/B in the curly bracket of (84) is compensated by the term in front of the integral of (84), whose value is $\int_0^{E_0}(2EdE/kB)(\gamma/B-2\gamma/B^2)$. However, the term γ/B^2 of (84) has no counterpart in the Lewis treatment. This discrepancy is small, since $(\gamma/B^2)/(\gamma/B)=1/B\sim0.1$, but real. The corrective terms γ/B^2 arise from the derivative of B which is disregarded by Lewis.[25]

A further discussion of the Lewis theory is given in Appendix VII.

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[23] The Lewis equation numbers will be accompanied by the letter L. The coefficient 230 in β_5 of (L23) should apparently be replaced with 210. Notice also that B does not quite coincide with Lewis' $A_1=B-\frac{1}{2}\gamma$ nor does γ coincide with $\epsilon_1=\gamma(1-\gamma)$. However this difference is immaterial in the zero-order approximation. It does not seem consistent for Lewis to regard A_1 as different from B in this calculation, since terms of the order of ϵ have been disregarded in his derivation of (L23).

[24] In Lewis's notation γ and B are called ϵ and $\log(\epsilon/\delta)$. The value of A_1 given by Lewis is equal to $\log(\epsilon/\delta)+\epsilon/2$ and appears to be incorrect, possibly because of confusion between his symbols ϵ and ϵ_1. The final statement that the range is lengthened by a factor $1+3\epsilon_1/2\log(\epsilon/\delta)+\cdots$ seems to be incorrect for the same reason, since the stopping number in the Bohr formula is $B=2\log(\epsilon E/I)$ as shown in (L2) and not $2\log(\epsilon_1 E/I)$.

[25] Lewis begins by treating a "simplified problem" in which B is regarded as constant and then allows for the variation of B only *after* having obtained the formula $\langle R\rangle=\int_0^{E_0}2EdE/k(B-\frac{1}{2}\gamma)$.

APPENDIX I. DISTRIBUTION IN DIRECTION AND ENERGY

Suppose that particles or photons diffuse in a medium under the conditions of Sec. 2 and that the source is nonisotropic, though with cylindrical symmetry. The angle between a direction ω and the axis of symmetry will be indicated by ϑ. The number of particles emitted with energy between E and $E+dE$ and with obliquity between ϑ and $\vartheta+d\vartheta$ will be indicated by $S(E, \vartheta)dE2\pi \times\sin\vartheta d\vartheta$. The track length of particles in the same energy range and in the same range of obliquity will be called $N(E, \vartheta)dE2\pi \sin\vartheta d\vartheta$. The probability per unit track of a collision with energy loss from ϵ to $\epsilon+d\epsilon$ and with a deflection of Θ to $\Theta+d\Theta$ will be called $k(E, \epsilon, \cos\Theta)d\epsilon 2\pi \sin\Theta d\Theta$. (This probability includes elastic scattering, for which $\epsilon=0$.)

The degradation and deflection equation is:

$$\mu(E)N(E, \vartheta)=\int_0^\infty d\epsilon\int_{4\pi} d\omega' k(E+\epsilon, \epsilon, \omega\cdot\omega') \times N(E+\epsilon, \vartheta')+S(E, \vartheta). \quad (85)$$

The source distribution, the scattering probability, and the particle distribution will be represented by Legendre polynomial expansions:

$$S(E, \vartheta)=\sum_l(2l+1)S_l(E)P_l(\cos\vartheta)/4\pi, \quad (86)$$

$$k(E, \epsilon, \cos\Theta)=\sum_l(2l+1)k_l(E, \epsilon)P_l(\cos\Theta)/4\pi, \quad (87)$$

$$N(E, \vartheta)=\sum_l(2l+1)N_l(E)P_l(\cos\vartheta)/4\pi. \quad (88)$$

It is well known that this expansion effectively separates the variables, as long as the kernel k depends only on $\cos\Theta=\omega\cdot\omega'$ and not on ω and ω' separately. For example if the source is isotropic the distribution will remain isotropic; if the source were distributed like $P_1(\cos\vartheta)=\cos\vartheta$, the particles would also be distributed in this manner. Multiplication of Eq. (85) by $P_l(\cos\vartheta)$ and integration over all directions ω yields the set of equations

$$\mu(E)N_l(E)=\int_0^\infty d\epsilon\, k_l(E+\epsilon, \epsilon)N_l(E+\epsilon)+S_l(E). \quad (89)$$

The equations with different values of l can be solved independently of one another. Each of the equations has the same form as (1) and can be treated accordingly. The equation with $l=0$ is identical with (1) and $N_0(E)$ with $n(E)$ of Sec. 2.

One difference, which may be important at times, is the fact that neither $N_l(E)$ nor $k_l(E, \epsilon)$ need be positive for $l\neq 0$. Each distribution $N_l(E)$ has no physical reality *per se*, since it corresponds to a source distribution P_l which is negative in some directions and therefore unrealistic. The discussion of the roots of (14) given in Sec. 4 hinges on the fact that $K(\xi)$ is positive. No study of the more general problem with a nonpositive K seems to have been made yet.

Spencer and Jenkins[3] solved the set (89) for x-ray degradation up to $l=5$, by the direct method of Sec. 3*b*. There was formerly some feeling that the breakdown of the problem in terms of Legendre coefficients, as in (89), is convenient only in the absence of peaked angular distributions, i.e., when only a few N_l's are significant. However it has become apparent that one can also handle long sequences of Legendre coefficients provided their values vary smoothly with l. For example, the Molière theory of multiple electron scattering[26] takes advantage, in essence, of the independence of the different Legendre coefficients and replaces the Σ_l in (88) with an integration. A procedure to carry out the Σ_l when many terms are important has recently been developed by Spencer.[27]

When the directional distribution of particles is peaked sharply at the source, it becomes progressively smeared out by multiple scattering. The smearing out is described, in the Legendre polynomial representation, by a damping of the coefficients N_l, for $l>0$, in the course of degradation. The larger l, the faster is the damping. This picture has been developed particularly in treatments where the directional distribution is considered as a function of the pathlength traversed.[28] Since the cross sections are functions of the energy, rather than of the path length, the formulation given here is a little more appropriate. A more complete treatment would combine the procedure of this appendix and of part *B* to give a combined distribution in energy, direction and path length.

The damping of the Legendre coefficients $N_l(E)$ comes about as follows. According to (9), $\int_0^\infty k_0(E, \epsilon)d\epsilon \leqslant\mu(E)$. Moreover, since $k(E, \epsilon, \cos\Theta)$ is non-negative, $k_l(E, \epsilon)=2\pi\int_0^\infty k(E, \epsilon, \cos\Theta)P_l(\cos\Theta)\sin\Theta d\Theta\leqslant k_0(E, \epsilon)$, for $l\neq 0$, owing to the oscillations of $P_l(\cos\Theta)$ which become increasingly rapid as l increases. Therefore the integral $\int_0^\infty k_l(E, \epsilon)d\epsilon$ over the kernel of (89) amounts to a smaller and smaller fraction of $\mu(E)$ as l increases and the solution p of Eq. (26.1), $\mu(E)=\int_0^\infty k_l(E, \epsilon) \times\exp(p\epsilon)d\epsilon$, tends to become large and positive. The value of p determines the damping of the solution of (89), $N_l(E)$, in the course of degradation.

APPENDIX II. CALCULATION OF THE STEADY STATE IN ABSENCE OF ABSORPTION

We wish to solve the equation

$$\bar{y}(x)=\int_0^\infty \bar{K}(x-\xi, \xi)\bar{y}(x-\xi)d\xi \quad (28)$$

with $\int_0^\infty \bar{K}(x, \xi)d\xi=1$, assuming that both $\bar{K}$ and $\bar{y}$

[26] G. Molière, Z. Naturforsch **3a**, 78 (1948). See particularly the derivation by H. W. Lewis, Phys. Rev. **78**, 526 (1950) and by H. A. Bethe, Phys. Rev. **89**, 1256 (1953).

[27] L. V. Spencer, Phys. Rev. **90**, 146 (1953); also L. V. Spencer and C. H. Blanchard, Phys. Rev. (to be published).

[28] See, e.g., S. Goudsmit and J. L. Saunderson, Phys. Rev. **57**, 24 (1940); **58**, 36 (1940), H. W. Lewis, reference 26, and also C. H. Blanchard and U. Fano, Phys. Rev. **82**, 767 (1951).

are slowly variable functions of x. In view of this assumption we utilize the expansion

$$\bar{K}(x-\xi,\xi)\bar{y}(x-\xi)=\sum_{n=0}^{\infty}\frac{(-\xi)^n}{n!}\frac{\partial^n}{\partial x^n}\bar{K}(x,\xi)\bar{y}(x). \quad (90)$$

This expansion reduces (28) to the form

$$\bar{y}(x)=\textstyle\sum_n(-d/dx)^n[\bar{M}_n(x)\bar{y}(x)]/n!, \quad (91)$$

where $\bar{M}_n=\int_0^\infty \bar{K}(x,\xi)\xi^n d\xi$, according to (29). The $\bar{y}(x)$ on the left side cancels with the $n=0$ term on the right, since $\bar{M}_0=1$. Integration of (91) yields

$$\sum_{n=0}(-d/dx)^n[\bar{M}_{n+1}(x)\bar{y}(x)]/(n+1)!=C, \quad (92)$$

where C is an integration constant.

In a zero-order approximation one would disregard all derivatives and consider only the term $n=0$ in the sum. A procedure of successive approximations may be established by writing

$$\bar{y}(x)=\bar{y}_0(x)+\bar{y}_1(x)+\bar{y}_2(x)+\cdots, \quad (93)$$

with the understanding that the rth term of the expansion, $\bar{y}_r$, is small of the rth order, i.e., that it contains r differential factors. This expansion brings (92) to the form

$$\textstyle\sum_n\sum_r(-d/dx)^n[\bar{M}_n(x)\bar{y}_r(x)]/(n+1)!=C. \quad (94)$$

Since $\bar{y}_r$ contains r differentiations, the total number of differentiations in each term of (94) is $n+r$. The procedure of successive approximations requires that (94) be fulfilled separately by each group of terms with the same value of $n+r$. Thereby (94) separates out into the set of equations:

$$\begin{aligned}
&\bar{M}_1\bar{y}_0=C, \quad (n+r=1),\\
&\bar{M}_1\bar{y}_1-\tfrac{1}{2}(d/dx)(\bar{M}_2\bar{y}_0)=0, \quad (n+r=2),\\
&\bar{M}_1\bar{y}_2-\tfrac{1}{2}(d/dx)(\bar{M}_2\bar{y}_1)+\tfrac{1}{6}(d/dx)^2(\bar{M}_3\bar{y}_0)=0,\\
&\qquad\qquad (n+r=3).\\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned} \quad (95)$$

The first equation yields $\bar{y}_0$, the second $\bar{y}_1$, etc., and the resulting expressions, when entered in (93), give

$$\bar{y}(x)=\frac{C}{\bar{M}_1}+\frac{1}{2}\frac{1}{\bar{M}_1}\frac{d}{dx}\bar{M}_2\frac{C}{\bar{M}_1}+\frac{1}{4}\frac{1}{\bar{M}_1}\frac{d}{dx}\frac{\bar{M}_2}{\bar{M}_1}\frac{d}{dx}\bar{M}_2\frac{C}{\bar{M}_1}-\frac{1}{6}\frac{1}{\bar{M}_1}\frac{d^2}{dx^2}\bar{M}_3\frac{C}{\bar{M}_1}+\cdots. \quad (96)$$

Since $\bar{M}_n/\bar{M}_{n'}=M_n/M_{n'}$, Eq. (96) reduces to the expression (30) for $y(x)$, when divided by $\mu(x)$.

APPENDIX III. CALCULATION OF SOLUTIONS OF THE HOMOGENEOUS EQUATION

We wish to solve Eq. (28),

$$\bar{y}(x)=\int_0^\infty \bar{K}(x-\xi,\xi)\bar{y}(x-\xi)d\xi,$$

without assuming that $\bar{y}(x)$ varies slowly. The equation may be written by means of operator symbols in the form

$$\bar{y}(x)=\int_0^\infty \exp[-\xi d/dx]\bar{K}(x,\xi)\bar{y}(x)d\xi. \quad (97)$$

The substitution (31), $\bar{y}(x)=\exp[-\int_0^x \bar{q}(x')dx']$, is such that $(d/dx)\bar{y}(x)=\bar{y}(x)[(d/dx)-\bar{q}(x)]$ and, therefore,

$$\exp[-\xi d/dx]\bar{y}(x)=\bar{y}(x)\exp\{\xi[\bar{q}(x)-d/dx]\}. \quad (98)$$

In this manner $\bar{y}(x)$ can be factored out of (97), which reduces to

$$\int_0^\infty \exp\{\xi[\bar{q}(x)-d/dx]\}\bar{K}(x,\xi)d\xi=1. \quad (99)$$

We wish to set up a procedure of approximation in which $\bar{q}(x)$ and $\bar{K}(x,\xi)$ are regarded as varying slowly with x. To this end we must expand the exponential operator of (99) into powers of the differential operator d/dx. The Feynman operator calculus[29] yields the expansion formula

$$\begin{aligned}
\exp(\alpha+\beta)=\exp\alpha&+\int_0^1 \exp[(1-s)\alpha]\beta\exp(s\alpha)ds\\
&+\int_0^1 ds\int_0^s ds'\exp[(1-s)\alpha]\beta\\
&\times\exp[(s-s')\alpha]\beta\exp(s'\alpha)+\cdots.
\end{aligned} \quad (100)$$

In our problem, $\alpha=\xi\bar{q}(x)$ and $\beta=-\xi d/dx$ and the commutator of β and $\exp[(1-s)\alpha]$ is independent of β. Therefore β may be shifted everywhere to the left of α, after which the integrals in (100) reduce to ordinary integrals. This procedure gives

$$\begin{aligned}
\exp\left\{\xi\left[\bar{q}(x)-\frac{d}{dx}\right]\right\}&=\left\{1-\xi\left[\frac{d}{dx}-\xi\frac{d\bar{q}}{dx}\right]+\frac{1}{2}\xi^2\left[\frac{d^2}{dx^2}-\xi\frac{d\bar{q}}{dx}-\frac{2}{3}\xi\frac{d^2\bar{q}}{dx^2}\right.\right.\\
&\left.\left.+\frac{1}{4}\xi^2\left(\frac{d\bar{q}}{dx}\right)^2\right]+\cdots\right\}\exp[\bar{q}(x)\xi].
\end{aligned} \quad (101)$$

If we enter this expansion in Eq. (99), the integrals over ξ can be represented as the moments

$$N_t(x)=\int_0^\infty \bar{K}(x,\xi)\exp[\bar{q}(x)\xi]\xi^t d\xi \quad (102)$$

of a distribution $\bar{K}(x,\xi)\exp[\bar{q}(x)\xi]$, and (99) takes the

[29] R. P. Feynman, Phys. Rev. **84**, 108 (1951), Sec. 1.

form:

$$N_0-\left[\frac{d\bar{N}_1}{dx}-\frac{1}{2}\frac{d\bar{q}}{dx}\bar{N}_2\right]+\left[\frac{1}{2}\frac{d^2\bar{N}_2}{dx^2}-\frac{1}{2}\frac{d\bar{q}}{dx}\frac{d\bar{N}_3}{dx}-\frac{1}{3}\frac{d^2\bar{q}}{dx^2}\bar{N}_3+\frac{1}{8}\left(\frac{d\bar{q}}{dx}\right)^2\bar{N}_4\right]+\cdots=1. \quad (103)$$

The terms in successive brackets contain an increasing number of differentiations.

At this point we proceed as in Appendix II, i.e., we represent $\bar{q}(x)$ as a sum of terms of successively smaller order, with the understanding that the term of rth order contains an r-fold differential. In the zero-order approximation, (103) reduces to $\bar{N}_0=1$, which coincides with (26) and shows that the zero-order value of $\bar{q}(x)$ is in fact $p(x)$. Hence we write:

$$\bar{q}(x)=p(x)+\bar{q}_1(x)+\bar{q}_2(x)+\cdots. \quad (104)$$

The exponential in the integral of (101) can be expanded into powers of $\sum_{r>0}\bar{q}_r(x)\xi$. The value of $\bar{N}_n$ with $\bar{q}=p$ is the moment $\bar{L}_n$ analogous to that of (32), so that

$$\bar{N}_n(x)=\sum_h[\sum_r \bar{q}_r(x)]^h\bar{L}_{n+h}(x)/h!. \quad (105)$$

We must now enter this expansion into (103) and separate out the groups of terms of the same order, for which the number of differential factors plus the sum of the indices of the factors $\bar{q}_r$ equals a given total. The groups of terms of the same order must fulfil the equation separately. In this manner we obtain a system of equations of which the first three are written out explicitly:

$$\bar{L}_0=1,$$

$$\bar{q}_1\bar{L}_1-\frac{d\bar{L}_1}{dx}+\frac{1}{2}\frac{dp}{dx}\bar{L}_2=0,$$

$$\bar{q}_2\bar{L}_1+\frac{1}{2}\bar{q}_1^2\bar{L}_2-\frac{d\bar{q}_1\bar{L}_2}{dx}+\frac{1}{2}\frac{d\bar{q}_1}{dx}\bar{L}_2+\frac{1}{2}\frac{dp}{dx}\bar{q}_1\bar{L}_3+\frac{1}{2}\frac{d^2\bar{L}_2}{dx^2}-\frac{1}{2}\frac{dp}{dx}\frac{d\bar{L}_3}{dx}-\frac{1}{3}\frac{d^2p}{dx^2}\bar{L}_3-\frac{1}{8}\frac{d^2p}{dx^2}\bar{L}_4=0. \quad (106)$$

The first equation coincides with (26) and yields $p(x)$. The second yields $\bar{q}_1(x)$ in terms of $p(x)$, the third yields $\bar{q}_2(x)$ in terms of $p(x)$ and $\bar{q}_1(x)$, etc. When the $\bar{q}_r$'s are expressed entirely in terms of $p(x)$ and of the $\bar{L}_n$'s, it turns out that the $\bar{L}_n$'s are combined in ratios $\bar{L}_n/\bar{L}_{n'}=L_n/L_{n'}$ with the exception of one term, $d\log\bar{L}_1/dx$, in $\bar{q}_1$. On the other hand the relationship $\bar{y}(x)=\mu(x)y(x)$ implies $\bar{q}(x)=q(x)-d\log\mu/dx$, so that one can pass from the $\bar{q}_r$'s to the q_r's simply by replacing the $\bar{L}_n$'s with L_n's. In this manner one arrives at (33).

APPENDIX IV. CALCULATION OF THE SOLUTION FOR ENERGIES NEAR THE SOURCE

We wish to solve the equation

$$\bar{y}(x)=\int_0^\infty \bar{K}(x-\xi,\xi)\bar{y}(x-\xi)d\xi+\delta(x) \quad (8)$$

by means of the expansion

$$\bar{K}(x-\xi,\xi)=\sum_n[\partial^n\bar{K}(x,\xi)/\partial x^n]_{x=0}(x-\xi)^n/n!. \quad (107)$$

The Laplace transform of (8) is

$$\begin{aligned}\bar{v}(p)&=\int_0^\infty \exp(px)\bar{y}(x)dx\\&=\int_0^\infty dx\exp(px)\int_0^x \bar{K}(x-\xi,\xi)\bar{y}(x-\xi)d\xi+1\\&=\int_0^\infty d\xi\int_0^\infty d\eta\exp[-p(\xi+\eta)]\bar{K}(\eta,\xi)\bar{y}(\eta)+1\\&=\sum_n n!^{-1}\int_0^\infty[\partial^n\bar{K}(x,\xi)/\partial x^n]_{x=0}\\&\qquad\times\exp(p\xi)d\xi\int_0^\infty \eta^n\bar{y}(\eta)\exp(p\eta)d\eta+1\\&=\sum_n\frac{\bar{F}_n(p)}{n!}\left(\frac{d}{dp}\right)^n\bar{v}(p)+1, \quad (108)\end{aligned}$$

where $\bar{F}_n(p)$ is the transform of $[\partial^n\bar{K}/\partial x^n]_{x=0}$, according to (34).

The $\bar{F}_n$'s are treated as small quantities of order u and $\bar{v}(p)$ is represented as a sum of terms of successively smaller order,

$$\bar{v}(p)=\bar{v}_0(p)+\bar{v}_1(p)+\cdots+\bar{v}_r(p)+\cdots. \quad (109)$$

This expansion, entered into (108), yields

$$\sum_r\bar{v}_r(p)=\sum_n\sum_r\bar{F}_n(p)(d/dp)^n\bar{v}_r(p)/n!+1. \quad (110)$$

The groups of terms of equal order, i.e., with equal $n+r$, when separated out yield the system of successive equations:

$$\begin{aligned}\bar{v}_0(p)&=\bar{F}_0(p)\bar{v}_0(p)+1,\\\bar{v}_1(p)&=\bar{F}_0(p)\bar{v}_1(p)+\bar{F}_1(p)d\bar{v}_0/dp, \quad (111)\\\bar{v}_2(p)&=\bar{F}_0(p)\bar{v}_2(p)+\bar{F}_1(p)d\bar{v}_1/dp+\tfrac{1}{2}\bar{F}_2(p)d^2\bar{v}_0/dp^2,\\&\cdots\cdots\cdots\cdots\cdots\end{aligned}$$

the first of which coincides with (12). The system, solved chainwise, has the solution

$$\begin{aligned}\bar{v}_0(p)&=[1-\bar{F}_0(p)]^{-1},\\\bar{v}_1(p)&=\bar{F}_1(1-\bar{F}_0)^{-1}d\bar{v}_0/dp=\bar{F}_1(d\bar{F}_0/dp)(1-\bar{F}_0)^{-3}, \quad (112)\\\bar{v}_2(p)&=\bar{F}_1(1-\bar{F}_0)^{-1}d\bar{v}_1/dp\\&\qquad+\tfrac{1}{2}\bar{F}_2(1-\bar{F}_0)^{-1}d^2\bar{v}_0/dp^2,\\&\cdots\cdots\cdots\cdots\cdots\end{aligned}$$

All the terms $v_r(p)$ have poles only at values of p which are roots of Eq. (14), $\bar{F}_0(p)=1$. The Laplace transform (109) can be inverted in the same manner as (13), by reducing the integral to loops around the poles. To evaluate the residues at the pole $p=p_j$ one can use the expansion:

$$\frac{1}{1-\bar{F}_0(p)}=\frac{1-\frac{1}{2}(p-p_j)\bar{F}_0''(p_j)/\bar{F}_0'(p_j)-\frac{1}{6}(p-p_j)^2\{\bar{F}_0'''(p_j)/\bar{F}_0'(p_j)-\frac{3}{2}[\bar{F}_0''(p_j)]^2/[\bar{F}_0'(p_j)]^2\}+\cdots}{[-\bar{F}_0'(p_j)](p-p_j)}. \quad (113)$$

The inverse transform of the sum of two terms $\bar{v}_0+\bar{v}_1$, is (35).

APPENDIX V. CONNECTION WITH THE LANDAU-SYMON THEORY

Landau[14] has given a theory of the energy straggling of charged particles which traverse thin layers of matter. This theory applies to total energy losses so small, that the collision probabilities $\mu(E)$ and $k(E,\epsilon)$ may be regarded as independent of E. Under this assumption the transform equation (46) can be solved by the method of Sec. 4. The initial energy of the particles is fixed at E_0, i.e., one takes $S(E)=\delta(E-E_0)$. The solution, equivalent to (13) is

$$\phi(E,\sigma)=(2\pi i)^{-1}\int_{-a-i\infty}^{-a+i\infty}\exp[-p(E_0-E)]\times\left[\mu-\sigma-\int_0^\infty k(\epsilon)\exp(p\epsilon)d\epsilon\right]^{-1}dp, \quad (114)$$

or, since $\mu=\int_0^\infty k(\epsilon)d\epsilon$,

$$\phi(E,\sigma)=(2\pi i)^{-1}\int_{-a-i\infty}^{-a+i\infty}\exp[-p(E_0-E)]\times\left\{\int_0^\infty k(\epsilon)[1-\exp(p\epsilon)]d\epsilon-\sigma\right\}^{-1}dp. \quad (115)$$

The Landau formula is obtained by entering (115) into the inverse transform (47) and carrying out the integration over σ before that over p. The integral over σ is given by the residue at

$$\sigma=\int_0^\infty k(\epsilon)[1-\exp(p\epsilon)]d\epsilon,$$

$$f(E,s)=(2\pi i)^{-1}\int_{-a-i\infty}^{-a+i\infty}\exp\left\{-p(E_0-E)-s\int_0^\infty k(\epsilon)[1-\exp(p\epsilon)]d\epsilon\right\}dp. \quad (116)$$

This is just Eq. (5) of Landau except for the replacement of p with $-p$.

In the evaluation of $\int_0^\infty k(\epsilon)[1-\exp(p\epsilon)]d\epsilon$, Landau took advantage of the fact that $k(\epsilon)=\text{const}/\epsilon^2$ over a large range of values of ϵ, for charged particles. For the sake of analytical convenience, Landau assumed that the ϵ^{-2} dependence extends to $\epsilon=\infty$ instead of stopping at some value $\epsilon_{\max}$. Thereby he introduced a spurious possibility of energy losses $>\epsilon_{\max}$, but the resulting error can be disregarded as long as the total probability of any such spurious loss over the pathlength s, s const$/\epsilon_{\max}$, remains much smaller than one. However, the possibility of the spurious losses raises to infinity the mean energy loss over a finite path length. Correspondingly, the saddle point of the integrand of (116), which must lie at $p=0$ for $E_0-E=\langle E_0-E\rangle$ (see p. 43) and at $p>0$ for $E_0-E>\langle E_0-E\rangle$, remains, in the Landau calculation, at $p<0$. Indeed the Landau value for $\int_0^\infty k(\epsilon)[1-\exp(p\epsilon)]d\epsilon$, namely const$p\log(-1.56p\epsilon')$, has a spurious singularity at $p=0$. It is interesting that the topography of the Laplace transform is thus seriously altered even though the Landau assumption is actually realistic over a broad range of conditions and even though the inverse transform remains unaffected, as it should.

The Landau calculation breaks down when the probability of spurious losses, of the order of $\epsilon_{\max}$, becomes appreciable. This happens, for heavy charged particles, including mesons, before a very substantial fraction of the initial energy has been dissipated, owing to the smallness of $\epsilon_{\max}$. In the situation opposite to that of the Landau approximation, the energy degradation has proceeded so far that the frequency of actual losses of the order of $\epsilon_{\max}$ has become rather large instead of very low. In this situation the "transient" phase of the degradation is over, the "steady-state" analysis of part A, as applied in Secs. 13 and 12, becomes quite appropriate, and the straggling distribution approaches a Gaussian shape.

Symon[14] has bridged the gap between ranges of validity of the Landau analysis and of the steady-state analysis by a skilled and elaborate interpolation procedure.[30] He characterizes the distribution in energy, for a given path length, by: (1) the value of the most probable energy, (2) a factor which measures the scale of the fluctuations and which is related to the mean square energy fluctuation, (3) a skewness parameter λ whose value ranges from zero for a Gaussian distribution of straggling to 1.48 for the Landau distribution. The shape of the distribution depends only on λ. Symon constructs a standard set of distribution curves for $0<\lambda<1.48$, whose shapes vary gradually from Gaussian to Landau. The curves for $\lambda\leqslant 1$ are Gaussians modified

[30] Much of this procedure has been reproduced by B. Rossi, *High Energy Particles* (Prentice Hall, Inc., New York, 1952), p. 32ff.

by a suitably applied Edgeworth expansion. The curves for $\lambda>1$ are obtained from the Landau theory corrected to the first order for the error caused by the inclusion of the "spurious losses." For path lengths so small that $k(E, \epsilon)$ and $\mu(E)$ are effectively independent of E, the Symon parameters can be calculated directly. For large path lengths Symon fits his distribution to values of the mean, mean-square and mean-cube energy loss calculated to an accuracy equivalent to that of the first three cumulants in (76) and (77).

APPENDIX VI. UTILIZATION OF MOMENTS TO INVERT THE LAPLACE TRANSFORM

Lewis[11] has utilized a standard method of statistics, the Edgeworth series, which amounts to fitting initially a Gaussian distribution with the correct mean range and mean square deviation and then of improving this distribution on the basis of data on higher moments. This method is appropriate to the path length distribution of heavy charged particles, which involves rather small fluctuations and therefore is well approximated by the initial Gaussian.

Another type of approach has been developed to construct approximate distributions of x-rays in an infinite medium.[4] The idea is to represent $\phi(E, \sigma)$ for $\sigma\sim 0$ by an expansion especially chosen to converge rapidly over a broad interval of σ. Thus, for example, if ϕ is expanded into powers of $\sigma/(\mu_s-\sigma)$, each term of the expansion diverges as σ aproaches μ_s. Since we know that ϕ itself actually diverges in this limit, this expansion is more realistic than a simple power expansion. The coefficients of the first n terms of the expansion in powers of $\sigma/(\mu_s-\sigma)$ are linear combinations of the first n coefficients of the expansion in powers of σ and therefore their determination requires no additional information. The additional information, provided by independent knowledge on the behavior of $\phi(E, \sigma)$ far from $\sigma=0$, is embodied in the choice of the *type of expansion*. Many types of expansion allow analytical inversion of the transform term by term; e.g., the expansion in powers of $\sigma/(\mu_s-\sigma)$ yields an expansion of $f(E, s)$ in a series of Laguerre polynomials.[4]

APPENDIX VII. DISCUSSION OF THE LEWIS THEORY

A discussion of the range straggling theory of Lewis,[11] especially of his "simplified problem," serves to illustrate various points previously raised in the present paper.

Lewis assumes that the differential collision probability for heavy charged particles is

$$k(E, \epsilon)=k/2E\epsilon^2, \tag{117}$$

where $k=2\pi NZz^2e^4M/m$, as in (40). The values of the energy loss ϵ are supposed to vary from a lower limit δE to the upper limit $\gamma E\sim 4(m/M)E$ (γ is called ϵ by Lewis),

$$\delta E\leqslant\epsilon\leqslant\gamma E. \tag{118}$$

The assumption (117) is not realistic for low values of ϵ, but the value of δ is adjusted so that $M_1=\int_0^\infty k(E, \epsilon)\epsilon d\epsilon$ has the correct value (40). The error in the moments $M_2, M_3\cdots$ is also negligible and therefore the law (117) gives the correct results for the purpose at hand. The corresponding value of $\mu(E)$ is

$$\mu(E)=(k/2E^2)(\delta^{-1}-\gamma^{-1}). \tag{119}$$

The transport equation (44) is, then,

$$\partial f/\partial s=-(k/2E^2)(\delta^{-1}-\gamma^{-1})f(E, S)+(k/2)\int_{\delta E}^{\gamma E} f(E+\epsilon, s)d\epsilon/(E+\epsilon)\epsilon^2+\delta(E-E_0)\delta(s). \tag{120}$$

The main part of the Lewis treatment deals with the "simplified problem" in which he regards as constant the minimum fractional energy loss δ and therefore the stopping number $B=\log(\gamma/\delta)$. His calculation can be simplified by considering the moments of the spectrum, $m_n(s)=\int_0^{E_0} E^{2n}f(E, s)dE$, which coincide with the Lewis expression (L15). The $\int_0^{E_0}$ has to be understood as $\lim_{\Delta=0}\int_\Delta^{E_0}$, so that particles whose energy drops below Δ are effectively "absorbed." Multiplication of (120) by E^{2n} and integration over E yields

$$dm_n/ds=-kA_nm_{n-1}(s)+E_0^{2n}\delta(s), \tag{121}$$

equivalent to (L16), where

$$A_n=\tfrac{1}{2}\int_\delta^\gamma du u^{-2}[1-(1-u)^{2n}].$$

This expression of A_n, is obtained by setting $u=\epsilon/E=1-(1+\eta)^{-1}$ in Lewis's expression. The moments (58) of the range distribution, $\langle s^n\rangle_{[0]}$ are equal to $n\int_0^\infty dss^{n-1}m_0(s)$ according to (50). To evaluate them one can express m_0 in terms of d^nm_n/ds^n and of $\delta(s)$ by repeated application of (121) and then integrate by parts n times. The result is (L19).

The fluctuations of energy loss in the simplified problem remain *constant throughout the degradation*, if expressed in a logarithmic energy scale. If we take $x=\log(E_0/E)$ and $\xi=\log[E/(E-\epsilon)]=\log[(1-u)^{-1}]$, the collision probability becomes

$$k(E, \epsilon)d\epsilon=\tfrac{1}{2}kE_0^2\exp(2x)\exp(-\xi)[1-\exp(-\xi)]^{-2}d\xi=K(x, \xi)d\xi. \tag{122}$$

Similarly

$$\mu(E)=\mu(x)=\tfrac{1}{2}kE_0^{-2}\exp(2x)(\delta^{-1}-\gamma^{-1})=\mu(0)\exp(2x). \tag{123}$$

The transform equation (65) can then be solved, for $\sigma=0$, by the method of Sec. 4 without any approximation procedure. According to (13) and (7) the result is

$$y(x, 0)=[2\pi i\mu(x)]^{-1}\int_{-a-i\infty}^{-a+i\infty}\times\exp(-px)[1-\bar{F}(p)]^{-1}dp, \tag{124}$$

where

$$\bar{F}(p)=(\delta^{-1}-\gamma^{-1})^{-1}\int_{-\log(1-\delta)}^{-\log(1-\gamma)}\times\exp(-\xi)d\xi[1-\exp(-\xi)]^{-2}\exp(p\xi) \quad (125)$$

$$=(\delta^{-1}-\gamma^{-1})\int_{\delta}^{\gamma}duu^{-2}(1-u)^{-p}.$$

Evaluation of (123) by taking the residues at the poles gives the steady-state solution, valid for $x\gg\gamma$,

$$y(x,0)=[\mu(x)\bar{F}'(0)]^{-1}$$

$$=E_0^2 2\exp(-2x)\left\{k\int^{\gamma}duu^{-2}\log[(1-u)^{-1}]\right\}^{-1}$$

$$=E_0^2 2\exp(-2x)/k[B+\tfrac{1}{2}(\gamma-\delta)+\cdots]. \quad (126)$$

As anticipated in Sec. 6, the exact solution obtained in this manner is equivalent to the corresponding solution (43), if the expression

$$\{\textstyle\int_{\delta}^{\gamma}duu^{-2}\log[(1-u)^{-1}]\}^{-1}=[B+\tfrac{1}{2}(\gamma-\delta)+\ldots]^{-1} \quad (127)$$

is expanded into powers of γ/B. The fraction δ can be disregarded in this expansion (as it is effectively disregarded in (41)), because it is extremely small. The term γ/B^2 of (43) cannot appear here since it arises from the variation of B.

It was pointed out in Sec. 13 that an evaluation of the mean path length by the formula $\langle s\rangle_{[x]}=\int_0^x y(x',0)dx'$ yields an incorrect result if one utilizes the steady-state expression (126) for y even in the interval of integration $x'\sim 0$. In this manner one would find only the integral of (84), without the additional term in front of it. To obtain the correct value of $\langle s\rangle_{[x]}$ one must take into consideration the exact solution (124) of the simplified problem, which includes the transient effect,

$$\langle s\rangle_{[x]}=\int_0^x y(x',0)dx'=\int_0^x dx'\exp(-2x')[2\pi i\mu(0)]^{-1}$$

$$\times\int_{-a-i\infty}^{-a+i\infty}\exp(-px')[1-\bar{F}(p)]^{-1}dp=\frac{1}{\mu(0)}\int_{-a-i\infty}^{-a+i\infty}dp\frac{1-e^{-(p+2)x}}{p+2}\frac{1}{1-\bar{F}(p)}.$$

Since x itself is large, the factor $\exp[-(p+2)x]$ becomes very small in the residues at all poles of $1-\bar{F}$ except at $p=0$. Therefore we may apply the steady-state approximation to the portion of the integral which contains $\exp[-(p+2)x]$. This portion yields

$$-\exp(-2x)/2\mu(0)\bar{F}'(0)=-[2\mu(x)F'(0)]^{-1}$$
$$=-E^2/k[B+\cdots]. \quad (128)$$

The remaining part of the integral, which does not contain the exponential, converges for p large and negative and reduces to the residue at $p=-2$ plus a half contour at $p=-\infty$, that is, to

$$[1/\mu(0)]\{[1-F(-2)]^{-1}-\tfrac{1}{2}\}=E_0^2/kA_1-1/2\mu(0). \quad (129)$$

Here $A_1=\frac{1}{2}(\delta^{-1}-\gamma^{-1})[1-F(-2)]$ is just the parameter which appears in the Lewis range formula (L20), $1/2\mu(0)\sim(E_0^2/k)\delta$ is a very small distance, of the order of the mean distance between collisions.

At $E=0$, (128) vanishes and (129) coincides with the Lewis value of the mean range, to within the insignificant term $1/2\mu(0)$. For purpose of comparison with (84) one may add $E_0^2/k[B+\cdots]$ to (128) and subtract it from (129). With this addition (128) becomes essentially equal to the integral in (84), i.e., to the integral over the steady state (126), and (129) becomes $(E_0^2/k)\{A_1^{-1}-[B+\cdots]^{-1}\}$ and coincides with the corrective term in front of the integral in (84).

It may be concluded from the discussion that, whereas the Lewis mean range E_0^2/kA_1 is correct, its interpretation as the integral of contributions from different energy intervals $\int_0^{E_0}2EdE/kA_1$ is not quite correct. End-effects, as discussed in Sec. 13, are primarily responsible for the fact that the range is not $\int_0^{E_0}2EdE/k[B+\frac{1}{2}\gamma+\cdots]$.