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A Theorem Concerning Angular Correlations

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It is shown that in a cascade of any number of basic transitions between states of monotonically increasing or decreasing angular momenta the angular correlation between any two of the radiations is independent of the spins, the number and character of the intermediate steps, and the order in which the radiations occur. It depends only on the character of the measured radiations.

IT will be shown here that for a series of basic[1] transitions of any character such that the spins of the levels form a monotonic sequence, the correlation depends only on the character of the particular two radiations being measured; it is independent of the number of transitions intervening between the two measured radiations and the character of the intervening transitions, and is unaltered if a constant is added to all the level spins.[2] That is, suppose we have the pure radiations of angular momentum $L_1, L_2\cdots L_n$ and the sequence of spins $j_0, j_1\cdots j_n$:

$$j_0 \xrightarrow{L_1} j_1 \xrightarrow{L_2} j_2 \cdots \xrightarrow{L_n} j_n,$$

such that $j_0=j_1+L_1$, $j_1=j_2+L_2$, $\cdots j_{n-1}=j_n+L_n$; then, the correlation between the L_1 and L_n radiations is independent of their position in the sequence and independent of the spins. The same result holds if the sequence obeys the rule: $j_0=j_1-L_1$, $j_1=j_2-L_2\cdots j_{n-1}=j_n-L_n$; we restrict ourselves here only to the first case.

This fact is of particular interest in the case of electromagnetic radiation. In the great majority of experimental cases a specific transition is both pure and basic; this is certainly not true in the case of the transition $j\rightarrow j$, which is rather an exceptional case and one with which we shall here assume we are not concerned. Let us suppose that we have a fast cascade of radiations $L_1, L_2, \cdots, L_n$. Then the level scheme remains undetermined because we have not fixed the order in which the radiations occur. With the assumption that all transitions are basic, it is, then, the point of the above remarks that angular correlation measurements alone will in no way fix the order in which the radiations occur.

The theorem applies, of course, to β radiation too. In practice we need consider the β only as the first radiation. However, the requirement that the transition be basic is not commonly validated. It would, for example, apply to the β transition characterized by

$$(\Delta J=2 \text{ yes});$$

it would apply to ($\Delta J=1$ yes), ($\Delta J=2$ no) only if $j_0=0$. Similar considerations apply to other radiations, which we do not further consider here.

The theorem is easily proved using the Racah techniques.[3] The matrix element for the $i\rightarrow(i+1)$ transition can be written in the form—for single particle radiation:

$$(j_i\|L_{i+1}\|j_{i+1})C(j_i, L_{i+1}, j_{i+1}; m_i, M_{i+1}) \times D(L_{i+1}, M_{i+1}, \tau_{i+1}; R_{i+1})c_{\tau i+1}(L_{i+1}),$$

where $C(j_i, L_{i+1}, j_{i+1}; m_i, M_{i+1})$ is the usual Clebsch-Gordan coefficient, $D(L_{i+1}, M_{i+1}\tau_{i+1}; R_{i+1})$ is the L_{i+1} irreducible representation of the rotation group, R_{i+1} is the rotation characterizing the directions that describe the L_{i+1} radiation, and $c_{\tau i+1}$ are the parameters that depend on the kind and character of the radiation. Thus, for α particles $c_\tau=\delta_{\tau,0}$; for circularly polarized γ rays $c_\tau=\delta_{\tau,\pm1}$; for linearly polarized γ rays $c_{+1}=+1$, $c_{-1}=\pm i$. Then, the correlation between the L_1 and L_n radiations is $W(1, n)$, where

$$\begin{aligned}W(1,n)=\sum_{\substack{m_0, m_1, \cdots m_f\\ m_1', \cdots m_{n-1}'\\ \tau_1, \cdots \tau_n\\ \tau_1', \cdots \tau_n'}} \int dR_2\cdots dR_{n-1}
&\times c_{\tau1}(1)c_{\tau1'}(1)D(L_1, M_1, \tau_1; R_1)\\
&\times D^*(L_1, M_1', \tau_1'; R_1)C(j_0, L_1, j_1; m_0, M_1)\\
&\qquad\times C(j_0, L_1, j_1; m_0, M_1')\\
&\times c_{\tau2}(2)c_{\tau2'}(2)D(L_2, M_2, \tau_2; R_2)\\
&\times D^*(L_2, M_2'; \tau_2'; R_2)C(j_1, L_2, j_2; m_1, M_2)\\
&\qquad\times C(j_1, L_2, j_2; m_1', M_2')\\
&\times \cdots\\
&\times c_{\tau n}(n)c_{\tau n'}(n)D(L_n, M_n, \tau_n; R_n)\\
&\times D^*(L_n, M_n', \tau_n'; R_n)\\
&\times C(j_{n-1}, L_n, j_f; m_{n-1}, m_f-m_{n-1})\\
&\qquad\times C(j_{n-1}, L_n, j_f; m_{n-1}', m_f-m_{n-1}').\end{aligned}$$

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[1] A *basic transition* between two levels the spins of which are j_1, j_2, respectively, is defined as the emission of radiation of angular momentum L, where $L=|j_1-j_2|$

[2] Special case of this for dipole and quadrupole transitions are given in: Biedenharn, Arfken, and Rose, Phys. Rev. 83, 586 (1951); Arfken, Biedenharn, and Rose, Oak Ridge National Laboratory Report 1103 (unpublished).

[3] Use has been made throughout of the simplified and unified approach formulated by: L. C. Biedenharn and M. E. Rose, Revs. Modern Phys. (to be published). The notation of these authors has been used throughout. We wish to thank these authors for making preprints available to us.

Using the facts that:

$$\int D(L,M,\tau;R)D^*(L,M',\tau';R)dR = 8\pi^2/(2L+1)\delta_{M,M'}\delta_{\tau,\tau'},$$

$$D(L,M,\tau;R)D^*(L,M',\tau';R) = (-1)^{M'-\tau'}\textstyle\sum_\nu C(L,L,\nu;M,-M') \times C(L,L,\nu;\tau,-\tau')D(\nu,M-M',\tau-\tau';R),$$

and the Racah relation:

$$\sum_{m_0}(-1)^{m_0}C(j_0,L,j_1;m_0,m_1-m_0) \times C(j_0,L,j_1;m_0,m_1'-m_0)C(L,L,\nu;m_0-m_1,m_1'-m_0) = -j_0-2L(-1)(2j_1+1)^{\frac{1}{2}}C(j_1,j_1,\nu;m_1,-m_1') \times W(j_1,j_1,L,L;\nu,j_0),$$

and carrying out first the m_0, m_f summation, then the M_2, $M_3\cdots M_{n-1}$, m_{n-1} summation, in that order, we immediately obtain, after dropping irrelevant factors:

$$\sum_q(-1)^q[\sum_{\tau_1} c_{\tau_1}{}^{(1)}c_{\tau_1'}{}^{(1)}(-1)^{\tau_1'}C(L_1,L_1,\nu;\tau_1,-\tau_1') \times D(\nu,q,\tau_1-\tau_1';R_1)]W(j_1j_1L_1L_1;\nu j_0) \times W(j_2j_2j_1j_1;\nu L_1)\cdots W(j_{n-1}j_{n-1}j_{n-2}j_{n-2};\nu L_{n-1}) \times[\sum_{\tau_n} c_{\tau_n}{}^{(n)}c_{\tau_n'}{}^{(n)}(-1)^{\tau_n'}C(L_n,L_n,\nu;\tau_n,-\tau_n') \times D(\nu;-q,\tau_n-\tau_n';R_n)]W(j_{n-1}j_{n-1}L_nL_n;\nu j_n),$$

or, in the more usual form:

$$[\sum_{\tau_1,\tau_n,\tau_1',\tau_n'}(-1)^{\tau_1'+\tau_n}c_{\tau_1}{}^{(1)}c_{\tau_1'}{}^{(1)}c_{\tau_n}{}^{(n)}c_{\tau_n'}{}^{(n)} \times C(L_1,L_1,\nu;\tau_1,-\tau_{1'})C(L_n,L_n,\nu;\tau_n,-\tau_{n'}) \times D(\nu,\tau_n'-\tau_n,\tau_1-\tau_1';R)]W(j_1j_1L_1L_1;\nu j_0) \times W(j_2j_2j_1j_1;\nu L_2)\cdots W(j_{n-1}j_{n-1}j_{n-2}j_{n-2};\nu L_{n-1}) \times W(j_{n-1}j_{n-1}L_nL_n;\nu j_f),$$

where $R=R_n^{-1}R_1$ represents the directions between the L_1 and L_n radiations. We have, then, only to examine the products of the Racah coefficients. This is easily done since for the special case here considered the usual sum becomes a single, simple term;[4] dropping irrelevant factors, we have:

$$W(j_1j_1L_1L_1;\nu j_0)\rightarrow \left\{\frac{1}{(2j_1-\nu)!(2L_1-\nu)!(2j_1+\nu+1)!(2L_1+\nu+1)!}\right\}^{\frac{1}{2}},$$

$$W(j_{n-1}j_{n-1}L_nL_n;\nu,j_f)\rightarrow \left\{\frac{(2j_{n-1}-\nu)!(2j_{n-1}+\nu+1)}{(2L_n-\nu)!(2L_n+\nu+1)!}\right\}^{\frac{1}{2}}(-1)^{-\nu},$$

$$W(j_2j_2j_1j_1;\nu L_2)\rightarrow \left\{\frac{(2j_1-\nu)!(2j_2+2L_2+\nu+1)!}{(2j_2-\nu)!(2j_2+\nu+1)!}\right\}^{\frac{1}{2}}(-1)^{\nu}.$$

The product of the Racah coefficients becomes, after again dropping irrelevant factors,

$$\left\{\frac{1}{(2L_1-\nu)!(2L_1+\nu+1)!}\frac{1}{(2L_n-\nu)!(2L_n+\nu+1)!}\right\}^{\frac{1}{2}}.$$

Since this is the same as

$$W(jjL_1L_1;\nu j_1)W(jjL_nL_n;\nu j_f)$$

if $j_0=j+L_1$ and $j=L_n+j_f$, we have proved the desired theorem.

It is interesting to consider from a different viewpoint how this result comes about. Suppose we choose the Z axis along the direction of the first quantum. The relative population of the different m_1 sublevels of j_1 is, then:

$$\textstyle\sum_\nu[(-1)^{\tau_1}C(L_1,L_1,\nu;-\tau_1,\tau_1)c_{\tau_1}{}^2(1)](-1)^{m_1'} \times C(j_1,j_1,\nu;m_1,-m_1) \times\left\{\frac{1}{(2j_1-\nu)!(2L_1-\nu)!(2j_1+\nu+1)!(2L_1+\nu+1)!}\right\}^{\frac{1}{2}}.$$

The relative population of the m_2 sublevels of j_2, following the unobserved L_2 radiation, turns out to be the same function with j_1, m_1 replaced by j_2, m_2. This "standard population" is transmitted down to the j_{n-1} level. That is, the population in the j_{n-1} level, when we have observed the L_1 radiation emitted along the Z axis, is the same as the population produced by the following hypothetical transition: an L_1 radiation is emitted along the Z axis in a transition from an isotropic state of spin $(j_{n-1}-L_1)$ to the state j_{n-1}. The cascade thus reduces in effect to a single basic-basic correlation, which is known[5] to be independent of the j values; this implies the theorem.

The extension to transitions in which two particles are emitted—as in beta decay—is only slightly more difficult. We omit the details, but it is sufficient to point out that

$$\int(j_0,m_0|\quad|j_1,m_1)(j_0,m_0|\quad|j_1,m_1')^*d\omega_\nu,$$

which replaces the factor

$$(j_0m_0|L_1|j_1m_1)(j_0m_0|L_1|j_1m_1')^*,$$

can be written in the form

$$(-1)^{m_1'}C(j_1,j_1,\nu;m_1,-m_1') \times W(j_1j_1L_1L_1;\nu j_0)(\text{function of } L_1).$$

The derivation then goes through as before.

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[4] G. Racah, Phys. Rev. **62**, 438 (1942), especially Eq. (36′).

[5] S. P. Lloyd, Phys. Rev. **83**, 716 (1951).