The Elastic Scattering of Particles by Atomic Nuclei*

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The description of the elastic scattering of particles by atomic nuclei in terms of the optical model is studied. It is shown that the optical model does not represent only an approximation to the many-body problem, although the optical model potential must in general be considered to contain spin-orbit couplings. An explicit expression is obtained for the optical model potential in terms of the amplitudes for scattering of the incident particle by the individual neutrons and protons of the nucleus. The potential also depends upon correlations in the positions of the nuclear particles. A calculation of the parameters of the optical model seems to be in good agreement with some meson scattering experiments.

I. INTRODUCTION

IN a previous publication[1] a general theory of the multiple scattering of particles by complex systems was developed. The multiple scattering was described by the solution to a set of coupled integral equations, the number of equations being equal to the number of scattering centers in the scattering medium. When the number of scatterers is large a direct solution of the coupled integral equations is not feasible; however, in this case it was shown that the integral equations can be considerably simplified when the coherent (elastic) scattering is separated from the inelastic scattering. From this form of the equations it was possible to show that the inelastic scattering could be (approximately) described by a transport equation. The elastic scattering could be approximately described by the conventional "optical models"[2] when the energy of the incident particle was high.

In the present paper we consider the elastic scattering in more detail. The theory is applied specifically to the scattering of particles by atomic nuclei, but the formal arguments are quite general. In particular, we shall see that the "optical model" has in principle a wide range of validity and thus does not necessarily represent only an approximation to the many-body problem.

By the term "optical model," we mean that the elastic scattering can be obtained from the solution to a Schrödinger equation with an interaction potential which depends only upon the coordinate of the incident particle relative to (let us say) the center of mass of the nucleus. If the incident particle and the nucleus have spins, the potential may also depend upon these—but it does not depend upon individual coordinates of the neutrons and protons of the nucleus. In other words, the many-body problem is reduced to a two-body problem. This reduction is somewhat formal, since an analysis of the many-body problem is in principle required to find the equivalent *two-body* interaction. The advantage gained by such an approach is that in some cases the problem of finding the two-body potential is much simpler than is that of tackling directly the many-body Schrödinger equation. Also, for many problems it appears possible to make reasonable guesses concerning the form of the two-body interaction.

The use of such an "equivalent two-body potential" has been frequently made. Such an approach was employed by Ostrofsky, Breit, and Johnson[3] in calculating Coulomb barrier penetration. Bethe[4] has given a theory of nuclear reactions at low energies in terms of such an interaction. The model was also used by Feshbach, Peaslee, and Weisskopf[5] in the discussion of nuclear reactions. Fernbach, Serber, and Taylor[2] based their theory of nuclear reactions at high energies on an effective two-body potential. The latter model has been applied in particular to meson interactions.[6,7]

The purpose of the present paper is to study the relation of these models to the many-body problem. A special case of our results is an exact expression for the index of refraction for an extended scattering medium given in terms of the solution to an integral equation.

II. DEFINITIONS AND GENERAL PROPERTIES OF THE ELASTIC SCATTERING

Several useful characteristics of the elastic scattering of particles by atomic nuclei can be obtained quite generally without use of the more explicit multiple scattering theory of reference 1. In particular, we shall see that the elastic scattering can be obtained as the solution to a Schrödinger equation for a particle moving in the "potential" of a single heavy scatterer. That is, the potential involves only the coordinates and spin of the scattered particle, as well as the nuclear spin, but does not involve explicitly the coordinates of the neutrons and protons in the nucleus. Therefore, only the solution of a two-body (rather than a many-body) problem is required to determine the elastic scattering, once the "potential" is known.

To begin, we formulate the many-body problem

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[1] K. M. Watson, Phys. Rev. **89**, 575 (1953). This paper will hereafter be referred to as I.
[2] Fernbach, Serber, and Taylor, Phys. Rev. **75**, 1352 (1949).
[3] Ostrofsky, Breit, and Johnson, Phys. Rev. **49**, 22 (1936); Also B. Freeman and J. McHale, Phys. Rev. **89**, 223 (1953).
[4] H. A. Bethe, Phys. Rev. **57**, 1125 (1940).
[5] Feshbach, Peaslee, and Weisskopf, Phys. Rev. **71**, 145 (1947).
[6] H. A. Bethe and R. R. Wilson, Phys. Rev. **83**, 690 (1951).
[7] Brueckner, Serber, and Watson, Phys. Rev. **84**, 258 (1951).

which describes the interaction of the incident particle with an atomic nucleus. Let the nuclear hamiltonian be H_N and suppose that $g_\gamma(\xi)$ represent a complete set of nuclear wave functions, where ξ is some appropriate set of many-body coordinates and γ represents a specific nuclear state of energy W_γ. Then

$$H_N g_\gamma = W_\gamma g_\gamma. \tag{1}$$

We suppose that before bombardment the nucleus is in the initial state $\gamma_0 \equiv (A, M)$, where M is the azimuthal quantum number associated with the nuclear spin J. It is further assumed that the only degeneracy of this state is that due to different spin orientations (i.e., different values of M).

Let the incident particle have a spin s and a wave function $\lambda_q(\nu)$ when it is in a plane wave state with momentum $\mathbf{q}$ and has an azimuthal spin component ν. (We shall frequently omit the symbol "ν" from λ_q.) It will be convenient to consider the nucleus to be very heavy, so that $\mathbf{q}$ can be interpreted as the momentum of the incident particle *relative* to the nucleus. If we designate the kinetic energy operator and kinetic energy by h and ϵ_q, respectively, the Schrödinger equation for the incident particle is

$$h\lambda_q = \epsilon_q \lambda_q, \tag{2}$$

as long as it is far from the nucleus.

We define

$$H_0 \equiv H_N + h, \tag{3}$$

and V to be the many-body interaction between the incident particle and the nucleus. Then the Schrödinger equation which describes the interaction of the incident particle with the nucleus is

$$(H_0 + V)\Psi_a = E_a \Psi_a. \tag{4}$$

The solution Ψ_a has the boundary condition that, for large distances of the incident particle from the nucleus,

$$\Psi_a \rightarrow g_{(A,M)}\lambda_q(\nu). \tag{5}$$

We then have

$$E_a = W_a + \epsilon_q,$$

where ϵ_q is the energy of the incident particle [Eq. (2)]. A further boundary condition on Ψ_a is that it have only *outgoing* scattered waves. Associated with Ψ_a is a solution $\Psi_a^{(-)}$ to Eq. (4) which has *incoming* scattered waves. These solutions are related by the time reversal operator[8] K:

$$\mu_a \Psi_a^{(-)} = K\Psi_{(-a)}. \tag{6}$$

Here the state $(-a)$ is the state $(A, -M, -q, -\nu)$, and μ_a is a complex number of modulus one (i.e., a phase factor).[9]

We may formally write Ψ_a and $\Psi_a^{(-)}$ as

$$\begin{aligned} \Psi_a &= \sum_{\gamma,p,\nu'} (g_\gamma\lambda_p(\nu'), \Psi_a) g_\gamma\lambda_p(\nu'); \\ \Psi_a^{(-)} &= \sum_{\gamma,p,\nu'} (g_\gamma\lambda_p(\nu'), \Psi_a^{(-)}) g_\gamma\lambda_p(\nu'). \end{aligned} \tag{7}$$

We are interested in the elastic scattering by the nucleus. This is scattering which does *not* excite the nucleus. It is given by those terms in Eqs. (7) for which the energy W_γ of the nuclear state γ is equal to W_A, the energy of the initial state (A, M). The "elastic part" of Ψ_a and $\Psi_a^{(-)}$ is then obtained from Eqs. (7) as

$$\begin{aligned} \Phi_a &= \sum_{M',p,\nu'} (g_{(A,M')}\lambda_p(\nu'), \Psi_a) g_{(A,M')}\lambda_p(\nu'); \\ \Phi_a^{(-)} &= \sum_{M',p,\nu'} (g_{(A,M')}\lambda_p(\nu'), \Psi_a^{(-)}) g_{(A,M')}\lambda_p(\nu'). \end{aligned} \tag{8}$$

That is, the sum over nuclear states is a sum over spin orientations only, since it has been assumed that there is no further degeneracy of the state (AM). In analogy to Eq. (6) we have

$$\mu_a \Phi_a^{(-)} = K\Phi_a^{(-)}. \tag{9}$$

We wish now to see if the "elastic wave functions" Φ, as given by Eq. (8), represent solutions to a Schrödinger equation. For this purpose it is convenient to introduce the Møller wave matrix Ω_s and employ the operator algebra of Chew and Goldberger.[10] That is, the relations (7), when restated in formal matrix notation, read

$$\Psi_a = \Omega_s g_{(A,M)}\lambda_q(\nu), \quad \Psi_a^{(-)} = \Omega_s^{(-)} g_{(A,M)}\lambda_q(\nu). \tag{10}$$

The matrices Ω_s and $\Omega_s^{(-)}$ satisfy the Lippmann and Schwinger[11] integral form of the Schrödinger equation [see Eq. (4)]:

$$\Omega_s = 1 + a^{-1}V\Omega_s, \quad \Omega_s^{(-)} = 1 + (a^\dagger)^{-1}V\Omega_s^{(-)}, \tag{11}$$

where we have defined

$$a \equiv E_a + i\eta - H_0. \tag{12}$$

[Here η is a small positive parameter which is set equal to zero after the implied integrations in Eqs. (11) are done.]

The Eqs. (11) are more general than are required for the wave functions (10), since we may suppose Eqs. (11) to also define Ω_s and $\Omega_s^{(-)}$ for initial states whose energy is not equal to E_a. This generalization is useful for the description of "virtual scatterings" off the energy shell (a problem which one encounters in studying multiple scattering and also in field theory).[12]

If we write $T = V\Omega_s$, then

$$\Omega_s = 1 + a^{-1}T, \tag{13}$$

[8] E. P. Wigner, Gött. Nachr. **31**, 546 (1932).

[9] See, for instance, K. M. Watson, Phys. Rev. **88**, 1163 (1952). The reason for introducing Eqs. (6) and (9) is that these give simple relations between the solutions Ψ and $\Psi^{(-)}$, etc., making it unnecessary to solve the Schrödinger equation for these separately.

[10] G. F. Chew and M. L. Goldberger, Phys. Rev. **87**, 778 (1952).

[11] B. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950). If the incident particle has a charge, it is convenient to assume that the Coulomb potential is screened at large distances.

[12] K. Brueckner and K. Watson Phys. Rev. **90**, 699 (1953).

which satisfies the first of Eqs. (11) if T satisfies

$$T=V+Va^{-1}T. \tag{14}$$

The solution of this equation may be put into the form

$$\begin{aligned} T&=V+V\Omega_s a^{-1}V \\ &=V+Ta^{-1}V. \end{aligned} \tag{15}$$

This is easily verified by direct substitution into Eq. (14), making use of Eqs. (11) for Ω_s.

Forming the adjoint of the second of Eqs. (11), we have

$$\Omega_s^{(-)\dagger}=1+\Omega_s^{(-)\dagger}Va^{-1}, \tag{11'}$$

which is satisfied by

$$\Omega_s^{(-)\dagger}=1+Ta^{-1}. \tag{13'}$$

This is verified upon substitution into Eq. (11′), making use of Eq. (15).

In analogy to Eqs. (10) we may write the "elastic wave functions" Φ as

$$\Phi_a=\Omega_c g_{(A,M)}\lambda_q(\nu), \quad \Phi_a^{(-)}=\Omega_c^{(-)}g_{(A,M)}\lambda_q(\nu), \tag{16}$$

where

$$\Omega_c=1+a^{-1}T_c, \quad \Omega_c^{(-)\dagger}=1+T_c a^{-1}. \tag{17}$$

The quantities Ω_c and T_c are obtained from Ω_s and T [Eq. (13)] by keeping only those matrix elements of the latter which connect nuclear states of the same energy. That is, Ω_c and T_c are contained in Ω_s and T and are defined as follows:

$$\begin{aligned} \Omega_c&=(\gamma',\mathbf{k}',\nu'|\Omega_s|\gamma,\mathbf{k},\nu), \quad W_{\gamma'}=W_\gamma \\ &=0, \quad W_{\gamma'}\neq W_\gamma. \end{aligned} \tag{18}$$

Here $(\mathbf{k}',\nu')$ and $(\mathbf{k},\nu)$ represent any two plane wave states for the incident particle. T_c is obtained from T and $\Omega_c^{(-)}$ from $\Omega_s^{(-)}$ in the same manner.

The primary problem of our investigation is to find a potential $\mathcal{V}_c$, such that [see Eq. (14)]

$$T_c=\mathcal{V}_c+\mathcal{V}_c a^{-1}T_c. \tag{19}$$

If $\mathcal{V}_c$ exists, then T_c is the scattering amplitude which results from the potential $\mathcal{V}_c$. We may, however, consider Eq. (19) to be an integral equation to determine $\mathcal{V}_c$ in terms of T_c. If this equation has a solution then $\mathcal{V}_c$ exists.[13] In the next section we shall turn to the problem of actually calculating $\mathcal{V}_c$.

When the incident particle is a neutron or proton some modification of the above analysis is needed in order that the Pauli principle not be violated. If we employ the isotopic spin formalism, so that we may call either a neutron or proton a "nucleon," then the wave functions (10) must be made antisymmetric with respect to a permutation of any two "nucleon" coordinates. This may be easily done since Ψ_a [or $\Psi_a^{(-)}$], as given by Eqs. (10) is still a solution to Eq. (4) if we interchange the coordinates ξ_{A+1} of the incident nucleon with the coordinates ξ_α of any other nucleon ($\alpha=1, 2, \cdots A$, where A is the mass number of the nucleus). To obtain an antisymmetrized wave function, we then operate on Ψ_a [or $\Psi_a^{(-)}$] with the projection operator

$$\Lambda=\frac{1}{(A+1)^{\frac{1}{2}}}\left[I-\sum_{\alpha=1}^{A}P_\alpha\right]. \tag{20}$$

Here I is the identity permutation and P_α interchanges ξ_{A+1} and ξ_α.

If the "exchange scattering" terms in Eqs. (10) are unimportant, then our previous analysis needs no modification. (The exchange scattering refers to processes in which the incident nucleon changes places with one of the nucleons which was originally in the nucleus.) More generally, we must replace the T_c in Eq. (17) by the direct minus the exchange scattered amplitudes. If there still exists a potential $\mathcal{V}_c$ such that Eq. (19) is satisfied (with T_c in this equation including the exchange terms), then we can again describe the scattering by means of the "optical model," as developed in the next few paragraphs. However, the multiple scattering analysis of the next section is greatly complicated by the exchange scattering. For this reason we shall in the *subsequent* sections assume either that the incident particle is not a nucleon or that the exchange scattering is negligibly small.

The potential $\mathcal{V}_c$, which appears in Eq. (19), is the "optical model potential" which we have set out to find. We note that $\mathcal{V}_c$ is *not* a many-body interaction. It is obtained from T_c [Eq. (19)], which has the matrix elements

$$T_c=(A,M';\mathbf{k}',\nu'|T|A,M;\mathbf{k},\nu), \tag{21}$$

when operating on any nuclear state (A, M) which follows from Eq. (18). Consequently, $\mathcal{V}_c$ has the form

$$\mathcal{V}_c=(M',\mathbf{k}',\nu'|\mathcal{V}_c|M,\mathbf{k},\nu)=\mathcal{V}_c(\mathbf{J},\mathbf{S},\mathbf{k}',\mathbf{k}), \tag{22}$$

if we drop the symbol A. The last expression results from writing the $(M'\mu', M\mu)$ dependence in terms of the spin operators $\mathbf{J}$ and $\mathbf{S}$.

From Eqs. (17) and (19) it follows that

$$\Omega_c=1+a^{-1}\mathcal{V}_c\Omega_c, \tag{23}$$

and

$$\Omega_c^{(-)\dagger}=1+\Omega_c^{(-)\dagger}\mathcal{V}_c a^{-1}. \tag{24}$$

These are recognized as being the generalized Schrödinger equations for the elastic scattering [see Eqs. (11), for instance].

The nuclear wave functions can be eliminated from the Schrödinger equation for Φ_a by writing this as

$$\Phi_a=\sum_{M'}\varphi_q^{M'}g_{(A,M')}. \tag{25}$$

Then the Schrödinger equation for the column matrix $\varphi_q\equiv(\varphi_q^M)$ is

$$[h+\mathcal{V}_c(\mathbf{J},\mathbf{S})]\varphi_q=\epsilon_q\varphi_q. \tag{26}$$

[13] The formal Chew-Goldberger (reference 10) solution to Eq. (19) is $\mathcal{V}_c=T_c-T_c(a+T_c)^{-1}T_c$, so $\mathcal{V}_c$ exists at least formally.

This follows directly from Eq. (23), when it is noted that $a\equiv E_a+i\eta-H_0$ [see Eq. (12)] when operating on the state $g_{(A,M)}$ is equal to $\epsilon_q+i\eta-h$. The related wave function $\varphi_q^{(-)}$, obtained as in Eq. (25) from $\Phi_a^{(-)}$, satisfies the Schrödinger equation

$$[h+\mathcal{V}_c^\dagger]\varphi_q^{(-)}=\epsilon_q\varphi_q^{(-)}. \tag{27}$$

We emphasize once again that $\mathcal{V}_c$ depends only upon the coordinates of the incident particle and the spin of of the nucleus, so Eqs. (26) and (27) are not many-body equations. These equations represent the rigorous formulation of what may be called the "optical models" of nuclear interactions. They do not involve a high-energy approximation, nor do they represent approximations to the many-body problem.

The potential $\mathcal{V}_c$ is not in general hermitean, so there is no conservation of particle and current density associated with φ_q. This is, of course, necessary in order that there be conservation of probability for the original wave function Ψ_a. The state Φ_a describes the scattering in only one of the possible nuclear "channels" into which the reaction can go. As Φ_a is "feeding" the other channels, its normalization must become less than unity as one traces the wave through the scattering event.

It is just this feature, on the other hand, which permits us to use φ_q to calculate the cross section for inelastic scattering. From the general conservation theorem[11] which relates the imaginary part of the transition operator for forward scattering to the total transition rates for all scattering, we have

$$-2\,\mathrm{Im}\langle T\rangle_0=-2\,\mathrm{Im}\langle T_c\rangle_0=P_t \tag{28}$$

as the total rate at which the particle is scattered from its initial state.[14] The symbol "$\langle\cdots\rangle_0$" means the "diagonal matrix element of '$(\cdots)$' with respect to both nuclear states and the momentum and spin states of the incident particle." By means of a little algebra and the relation

$$\delta(E_a-H_0)=\frac{i}{2\pi}\left(\frac{1}{a}-\frac{1}{a^\dagger}\right),$$

we can easily express Eq. (28) as

$$P_t=2\pi\langle T_c^\dagger\delta(E_a-H_0)T_c\rangle_0-2\,\mathrm{Im}(\varphi_q,\mathcal{V}_c\varphi_q). \tag{29}$$

To derive this, we note that

$$-2\pi i T_c^\dagger\delta(E_a-H_0)T_c=T_c^\dagger\left(\frac{1}{a}-\frac{1}{a^\dagger}\right)T_c$$
$$=T_c^\dagger\Omega_c-\Omega_c^\dagger T_c+T_c-T_c^\dagger$$

by Eq. (17). If we write $\Omega_c^\dagger T_c=\Omega_c^\dagger\mathcal{V}_c\Omega_c$ and $\langle\Omega_c^\dagger\mathcal{V}_c\Omega_c\rangle_0=(\varphi_q,\mathcal{V}_c\varphi_q)$, etc., Eq. (29) follows.[15] In terms of cross sections, Eq. (29) may be written [see reference (14)]:

$$\sigma_t=\sigma_{\mathrm{el}}+\sigma_{\mathrm{ab}},$$

where

$$\sigma_{\mathrm{el}}=\frac{(2\pi)^4}{v_\pi}\langle T_c^\dagger\delta(E_a-H_0)T_c\rangle_0$$

and

$$\sigma_{\mathrm{ab}}=-\frac{2(2\pi)^3}{v_\pi}\,\mathrm{Im}(\varphi_q,\mathcal{V}_c\varphi_q). \tag{30}$$

Here σ_{el} is the total cross section for elastic scattering, and σ_{ab} is that for inelastic scattering by the nucleus. The relations (30) are equal to the cross sections which one would obtain from a solution to the *many-body* Schrödinger Eq. (4), but are obtained in terms of the solution to Eq. (26) only. In other words, the solution to Eq. (26) gives not only the detailed features of the elastic scattering but also the total cross section for inelastic scattering.

For incident particle energies above the resonance region it is quite plausible that the dependence of $\mathcal{V}_c$ (and of T_c) upon the nuclear spin may be weak. This is qualitatively understandable since the interaction at higher energies represents the effect of many interactions with individual nucleons, whose spins tend to take all orientations. If at such energies we approximate $\mathcal{V}_c$ by dropping $\mathbf{J}$, then the only possible dependence upon $\mathbf{S}$, the spin of the incident particle (if it has spin one-half), is a spin-orbit interaction of the form

$$\mathbf{S}\cdot(\mathbf{q}\times\mathbf{q}').$$

If this dependence is weak (or the energy high enough that the scattering tends to be in the forward direction), we may take $\mathcal{V}_c$ independent of spin interactions.

If furthermore the nuclear diameter is large compared to the wavelength of the incident particle, it seems quite plausible that $\mathcal{V}_c$ will be approximately constant inside the nucleus and vanish outside (in a coordinate representation). In this case $\mathcal{V}_c$ depends upon only two parameters at a given energy (its real and its imaginary parts), and Eq. (26) is quite simple. If we write $\epsilon_q=q^2/2\mu$ (μ is the mass of the incident particle), and $\varphi_q=e^{i\mathbf{k}\cdot\mathbf{x}}$ within the nucleus, Eq. (26) becomes

$$[k^2/2\mu+\mathcal{V}_c]=q^2/2\mu,$$

or

$$k^2/q^2\equiv n^2=1-(2\mu/q^2)\mathcal{V}_c. \tag{31}$$

We may interpret n to be the *index* of *refraction* of the nuclear medium. When Eq. (26) can be expressed in this form, the resulting analysis is usually referred to as "the optical model." To obtain the optical model in this form, we have neglected the dependence of $\mathcal{V}_c$ upon the spin of the nucleus and the incident particle (if it

[14] We normalize our wave functions in momentum space, so the total cross section is $\sigma_t=(2\pi)^3P_t/v_\pi$, where v_π is the velocity of the incident particle. For convenience we have supposed the nucleus to be infinitely heavy—otherwise it would be necessary to factor a δ function expressing momentum conservation from Eq. (28).

[15] This result, in a somewhat more restrictive context, has been obtained by M. Lax, Phys. Rev. 78, 306 (1950).

has a spin) and assumed that $\mathcal{V}_c$ is constant within the nucleus. These approximations seem quite reasonable except for low energies and the smaller nuclei. We emphasize, however, that Eq. (26) represents a very general formulation of the "optical model," and that Eq. (31) is only an approximation to this.

In the next section we shall calculate $\mathcal{V}_c$ directly in terms of the many-body interactions. As mentioned in the Introduction, however, it is possible for many applications to treat $\mathcal{V}_c$ just as a phenomenological parameter.[16]

III. CALCULATION OF THE POTENTIAL

In the present section we consider the calculation of the potential in terms of the interaction of the incident particle with the nucleus. To do this we employ the multiple scattering theory of reference 1. We suppose the potential V of Eq. (4), which describes the interaction of the incident particle with the nucleus, to be

$$V=\sum_{\alpha=1}^{A} V_\alpha, \tag{32}$$

where the summation runs over the A neutrons and protons of the nucleus. V_α is the interaction of the incident particle with the αth nucleon in the nucleus. [If we were to suppose that many-body forces were of importance, so that the incident particle interacts directly with clusters of nucleons, we could extend the summation in Eq. (32) to include these more general interactions. This would not modify the formal multiple scattering theory, since it would be only necessary to let the sum "over nucleons" have this more general interpretation in the multiple scattering Eqs. (33).] Then it was shown in I that the solution to Eqs. (11) for Ω_c is

$$\Omega_s=1+\frac{1}{a}\sum_{\alpha=1}^{A} t_\alpha\Omega_s(\alpha),\quad \Omega_s(\alpha)=1+\frac{1}{a}\sum_{\beta\neq\alpha}^{A} t_\beta\Omega_s(\beta). \tag{33}$$

The energy denominator a was defined by Eq. (12). The quantity t_α is the *effective scattering matrix* for the αth nucleon when it is *bound*, as described in I.

When true absorption of the incident particle can occur (as is the case with mesons), it was shown in I that a is to be replaced by

$$b=a-\Delta, \tag{34}$$

where

$$\Delta=(-iv_\pi/2\lambda_a)v(z).$$

v_π is the velocity of the incident particle, λ_a is the *mean free path for true absorption*, and $v(z)$ is the nucleon density in the nucleus normalized to

$$\int v(z)d^3z=V_A,$$

the nuclear volume.

Again following I, we decompose t_α into its matrix elements I_α which refer to inelastic scatterings (corresponding to a change in nuclear excitation energy) and those matrix elements C which correspond to elastic scattering.[17] That is

$$C=\langle t_\alpha\rangle. \tag{35}$$

The symbol $\langle\cdots\rangle$ means a matrix element between nuclear states having the same binding energy. In this notation, for instance,

$$\Omega_c=\langle\Omega_s\rangle$$

[see Eq. (18)]. If we use the isotopic spin formalism, so that the nuclear wave function is antisymmetric in all nucleon variables, it is clear that C is independent of the index α, and so is

$$t_c=AC. \tag{36}$$

Then it was shown in I that

$$\Omega_s=F[1+e^{-1}t_c], \tag{37}$$

where

$$e=b-t_c=a-\Delta-t_c, \tag{38}$$

and F is defined by the integral equations

$$F=1+\frac{1}{e}\sum_{\alpha=1}^{A} I_\alpha F_\alpha,\quad F_\alpha=1+\frac{1}{e}\sum_{\beta\neq\alpha}^{A} I_\beta F_\beta. \tag{39}$$

When true absorption of the particles is possible, we must replace Ω_s in Eq. (37) by

$$\Omega=[1+a^{-1}R]F[1+e^{-1}(t_c+\Delta)], \tag{40}$$

[see Eq. (34)] as was shown in I. The "absorption operator" R was described in reference 1. Equation (37) represents an approximation to Eq. (33) (which is rigorous), the relative error being of order $(1/A)$.

To calculate the elastic scattering [i.e., Eq. (16)], we need

$$\langle\Omega\rangle=\langle\Omega_s\rangle\equiv\Omega_c=\langle F\rangle[1+e^{-1}(t_c+\Delta)]. \tag{41}$$

This follows since the operator R, which absorbs the particle, can obviously not lead to elastic scattering (see reference 1 for a further discussion). The last step in Eq. (41) results from the fact that $(t_c+\Delta)$ is by definition diagonal in nuclear states, except for spin orientation. To obtain Ω_c we must calculate $\langle F\rangle$, which in I was approximated by

$$\langle F\rangle=1. \tag{42}$$

This is reasonable for high-energy scatterings. Indeed, the corrections to Eq. (42) result from several

[16] In a forthcoming publication the authors will present a theory of the deuteron-stripping reaction which makes use of the concepts of the present section. In this analysis $\mathcal{V}_c$ arises in a natural manner and can be treated as a phenomenological quantity.

[17] In reference 1, only high-energy scatterings were considered, so C was defined to be diagonal in nuclear states. Here we wish also to consider the possibility of nuclear spin interactions and so give a more general definition of C.

inelastic scatterings, the net result of which is to leave the nucleus in its initial state. When the scatterings are very energetic, corresponding to high nuclear excitation, the probability that the *last* inelastic scattering should happen to leave the nucleus in its ground state would appear to be negligibly small, so Eq. (42) appears quite reasonable. However, for lower energies and especially for lighter particles such as mesons we may expect that Eq. (42) will require modification.

To calculate $\langle F\rangle$, we refer to Eq. (39) and write

$$\langle F\rangle = 1+e^{-1}L, \tag{43}$$

where

$$L=\sum_{\alpha=1}^{A}\langle I_\alpha F_\alpha\rangle. \tag{44}$$

We now introduce a new notation which will prove useful in the subsequent discussion. If Γ is a matrix involving nuclear states, we introduce

$$'\Gamma$$

to mean the matrix obtained from Γ by removing all nuclear matrix elements of the form

$$(A, M|\Gamma|\gamma),$$

for all M, where A refers to the initial state of the nucleus and γ is any nuclear state.[18] For example, in this notation we can write

$$L=\sum_{\alpha=1}^{A}\langle t_\alpha\, 'F_\alpha\rangle, \tag{44'}$$

since the prime notation tells us that there are to be no states (A, M')—for all M'—in the sum over states in the matrix product $\sum_\gamma \langle t_\alpha|\gamma)(\gamma|'F_\alpha\rangle$. If we let

$$'F_\alpha\rangle \equiv \sum_\gamma (\gamma|'F_\alpha|A, M)g_\gamma,$$

for some value of M, etc., we have from Eq. (39)

$$\begin{aligned} 'F_\alpha\rangle &= '1\rangle + \frac{1}{e}\sum_{\beta\neq\alpha} {'I_\beta F_\beta}\rangle \\ &= \frac{1}{e}\sum_{\beta\neq\alpha} {'I_\beta}\rangle\langle F_\beta\rangle + \frac{1}{e}\sum_{\beta\neq\alpha} {'I_\beta\, 'F_\beta}\rangle, \end{aligned} \tag{45}$$

since $'1\rangle$ vanishes identically because of the definition of the prime notation. The second step is just the explicit decomposition of $'I_\beta F_\beta\rangle$ into intermediate states (A, M') and those not equal to (A, M').

Now the quantity

$$F_c \equiv \langle F_\beta\rangle, \tag{46}$$

which occurs in Eq. (45), is independent of the index β for the same reason that C in Eq. (35) was. Thus Eq. (45) can be written as

$$'F_\alpha\rangle = \frac{1}{e}\sum_{\beta\neq\alpha} {'I_\beta}\rangle F_c + \frac{1}{e}\sum_{\beta\neq\alpha} {'I_\beta\, 'F_\beta}\rangle. \tag{47}$$

To solve this integral equation we write $'F_\alpha\rangle$ as

$$'F_\alpha\rangle = (G_\alpha - 1)\rangle F_c, \tag{48}$$

which when substituted into Eq. (47), leads to

$$\left[G_\alpha - 1 - \frac{1}{e}\sum_{\beta\neq\alpha} {'I_\beta G_\beta}\right]\Big\rangle F_c = 0. \tag{49}$$

This is satisfied when G_α is the solution to the set of equations

$$G_\alpha = 1 + \frac{1}{e}\sum_{\beta\neq\alpha} {'I_\beta G_\beta}. \tag{50}$$

If we substitute Eq. (48) into Eq. (44′), we obtain

$$\begin{aligned} L &= \left[\sum_{\alpha=1}^{A}\langle t_\alpha G_\alpha\rangle - t_c\right]F_c \\ &\equiv \mathcal{V}_1 F_c. \end{aligned} \tag{51}$$

Here the symbol $\mathcal{V}_1$, has been introduced for brevity as

$$\mathcal{V}_1 \equiv \sum_{\alpha=1}^{A}\langle t_\alpha G_\alpha\rangle - t_c. \tag{52}$$

From Eqs. (46) and (39) we have

$$\begin{aligned} \sum_{\alpha=1}^{A}\langle F_\alpha\rangle = AF_c &= A + \frac{1}{e}\sum_{\alpha=1}^{A}\sum_{\beta\neq\alpha}\langle I_\beta F_\beta\rangle \\ &= A + (A-1)\frac{1}{e}L, \end{aligned} \tag{53}$$

if we use Eq. (39) and the defining Eq. (44) for L. From Eq. (43), we have to relative order $1/A$ (the accuracy to which we are restricting ourselves)

$$F_c = 1 + e^{-1}L = \langle F\rangle. \tag{54}$$

The first Eq. (54) can be written as

$$F_c = 1 + e^{-1}\mathcal{V}_1 F_c, \tag{55}$$

since $L=\mathcal{V}_1 F_c$ by Eq. (51). The Chew-Goldberger[10] solution to Eq. (55) is

$$F_c = 1 + \frac{1}{e-\mathcal{V}_1}\mathcal{V}_1. \tag{56}$$

If we substitute this value of $\langle F\rangle$ into Eq. (41), we obtain after some algebra

$$\Omega_c = 1 + \frac{1}{e-\mathcal{V}_1}[t_c + \Delta + \mathcal{V}_1]. \tag{57}$$

[18] Expressed somewhat differently, the "prime" represents a projection operator standing to the left of Γ, which vanishes when operating on a ground state eigenfunction of the nucleus but which has the eigenvalue unity for all other nuclear states.

From the definition of e [Eq. (38)]

$$e-\mathcal{V}_1=a-[t_c+\mathcal{V}_1+\Delta],$$

so

$$\Omega_c=1+a^{-1}[t_c+\mathcal{V}_1+\Delta]\Omega_c, \tag{58}$$

as in Eq. (23). Therefore, the optical model potential $\mathcal{V}_c$ is

$$\mathcal{V}_c=t_c+\mathcal{V}_1+\Delta$$
$$=\sum_{\alpha=1}^{A}\langle t_\alpha G_\alpha\rangle+\Delta \tag{59}$$

[by Eq. (52)]. Comparison with Eq. (26) shows that φ_q satisfies that equation with $\mathcal{V}_c$ as given by Eq. (59). When there is no true absorption, $\Delta=0$, so

$$\mathcal{V}_c=\sum_{\alpha=1}^{A}\langle t_\alpha G_\alpha\rangle. \tag{59'}$$

The calculation of $\mathcal{V}_c$ is seen to involve the solution to the set of integral Eqs. (50).

IV. FURTHER DISCUSSION OF THE POTENTIAL $\mathcal{V}_c$

To obtain the potential $\mathcal{V}_c$ it is necessary to solve the integral Eqs. (50) for the quantities G_α. The possibility of doing this quantitatively evidently depends upon the complexity of the scattering medium. In the case of an atomic nucleus we are limited to semi-phenomenological calculations if it is necessary to improve the simpler version of the optical model which was given in I.

For purposes of making a definite calculation we shall suppose that the incident particle is a high-energy π meson and assume that $\mathcal{V}_c$ does not depend upon the nuclear spin (although some of our results, such as Eq. (80), are more general). We shall also suppose that the nuclear radius is large compared to the wavelength of the incident particle.

We define a "wave function" ψ_α as

$$\psi_\alpha=G_\alpha\lambda_k, \tag{60}$$

where $\lambda_k=(2\pi)^{-\frac{3}{2}}e^{i\mathbf{k}\cdot\mathbf{z}}$. In terms of ψ_α, Eq. (50) is

$$\psi_\alpha(z)=\lambda_k(z)+\frac{1}{e}\sum_{\beta\neq\alpha}{}'I_\beta\psi_\beta(z). \tag{61}$$

If we have solved Eq. (61), which is just Eq. (50) in an explicit representation, we can immediately obtain $\mathcal{V}_c$.

The first observation concerning Eq. (61) is that there is no attenuation or coherent modification of the "incident wave" λ_k. This results from

$$\langle\psi_\alpha\rangle=\lambda_k, \tag{62}$$

which follows from the *prime* condition on ${}'I_\beta$.

The next observation is that for a large homogeneous scattering medium

$$(\mathbf{k}'|\mathcal{V}_c|\mathbf{k})=B\delta(\mathbf{k}'-\mathbf{k}), \tag{63}$$

where B is a numerical function of k. This conclusion follows from a study of the individual terms in a series expansion of Eq. (50) in powers of ${}'I_\beta$. A somewhat improved form of Eq. (63) is

$$\mathcal{V}_c(z)=Bv(z), \tag{63'}$$

where $v(z)$ was defined in connection with Eq. (34) and $\mathbf{z}$ is the coordinate of the meson. In I, Eqs. (63) and (63') were referred to as approximations I and II, respectively.

We can obtain the approximate value of $\mathcal{V}_c$ which was obtained in I by setting

$$\psi_\alpha(z)=\lambda_k(z),$$

so

$$(\mathbf{k}'|\mathcal{V}_c|\mathbf{k})=(\mathbf{k}'|t_c|\mathbf{k})+(\mathbf{k}'|\Delta|\mathbf{k}). \tag{64}$$

As in I, we write $(\mathbf{k}'|\Delta|\mathbf{k})=-(iv_\pi/2\lambda_a)\delta(\mathbf{k}'-\mathbf{k})$. [See Eq. (34).] Also, as in I,

$$(\mathbf{k}'|t_c|\mathbf{k})=\sum_\alpha\langle(\mathbf{k}'|t_\alpha|\mathbf{k})\rangle$$
$$\simeq B_s\delta(\mathbf{k}'-\mathbf{k})$$

[see Eqs. (35) and (37)], where B_s is given in terms of t_α. If we refer to Eq. (63), we see Eq. (64) implies the approximation

$$B\simeq B_0\equiv B_s-(iv_\pi/2\lambda_a). \tag{65}$$

If we assume the impulse approximation,[10] t_α is related to the meson scattering amplitude a_s from a free nucleon by

$$t_\alpha\simeq a_s/(2\pi)^2\epsilon',$$

where ϵ' is the meson energy (rest plus kinetic) evaluated in the nuclear medium. Then

$$B_s=\frac{A}{V_A}\frac{2\pi}{\epsilon'}[a_s(0)], \tag{66}$$

as was shown in I. Here $a_s(0)$ is the value of a_s for scattering in the *forward* direction and should be taken as the average scattering amplitude for the neutrons and protons in the nucleus. If we use the relationship between the imaginary part of the forward scattering amplitude and the total cross section, Eq. (66) can be rewritten as

$$B_s=\frac{A}{V_A}\frac{2\pi}{\epsilon'}\operatorname{Re}[a_s(0)]-\frac{iv_\pi}{2\lambda_s}. \tag{66'}$$

$\operatorname{Re}[a_s(0)]$ is the real part of $a_s(0)$, and $\lambda_s=(A\sigma_s/V_A)^{-1}$ is the mean free path for a single scattering from an individual nucleon in the nuclear medium, assuming that the cross section σ_s is the same as for an unbound nucleon (this follows from the impulse approximation). The velocity v_π is that which the meson has in the nuclear medium.

These results were obtained in I and provide a first approximation to $\mathcal{V}_c$. In order to test the validity of this approximation we can calculate B_s by means of

Eq. (66′) from the experimental phase shifts of Anderson *et al.*[19] These are available for the scattering of pions on protons at an energy of 78 Mev. The calculated value of B_s can be compared with the value of B [Eq. (63)] deduced by Lederman *et al.*[20] for the scattering of 65-Mev mesons by carbon.

From the Fermi phase shifts and Eq. (66′), we obtain

$$B_s = -[24+11i] \text{ Mev.} \tag{67}$$

The value of B obtained by Lederman *et al.* is

$$B = -[18+9i] \text{ Mev.} \tag{68}$$

In view of the experimental uncertainties these values seem to be quite compatible with each other. As is seen from Eq. (65), however, we must subtract $iv_\pi/2\lambda_a$ from B_s to obtain B_0. λ_a is known only roughly, but is expected to be at least no larger than λ_s at 65 Mev.[7] It thus seems that the imaginary parts of Eqs. (67) and (68) are in definite disagreement with each other.

The most apparent source of this error is associated with our use of the free nucleon scattering amplitude a_s in Eq. (66). As mentioned above this is the so-called "impulse approximation" as discussed by Chew and Goldberger[10] and others. However, the validity of the impulse approximation seems, perhaps, least certain for the calculation of the imaginary part of the forward scattering amplitude, since this is related to the total cross section.

To obtain a correction to the impulse approximation, we note that

$$-\mathrm{Im}\langle t_\alpha\rangle_0 = \pi\langle t_\alpha{}^\dagger \delta(E_a - H_0) t_\alpha\rangle_0. \tag{69}$$

[This is easily derived in the same manner as was Eq. (29).] Here t_α is the effective scattering operator for the αth nucleon when it is bound, as discussed in I [see also Eq. (33)]. According to the impulse approximation, the t_α's are to be replaced by the corresponding quantities for *free* nucleons, which also implies setting $E_a - H_0 = \epsilon_q - h$ in the δ function in Eq. (69).[21] For a first correction to the impulse approximation, it seems reasonable to replace the t_α's on the right hand side of Eq. (69) by the corresponding free nucleon t_α's, but to keep the energy of excitation of the nucleus in $\delta(E_a - H_0)$. This then corrects the total scattering cross section from the bound nucleon for binding effects in the final state.

Making a simplified evaluation of the right hand side of Eq. (69), using a degenerate Fermi gas model of the nucleus, we obtain a modified B_s, which seems to be in agreement with experiment:[20]

$$B_s \simeq -[24+5i] \text{ Mev.} \tag{67′}$$

[19] Anderson, Fermi, Martin, and Nagle, Phys. Rev. **91**, 155 (1953).

[20] Byfield, Kessler, and Lederman, Phys. Rev. **86**, 17 (1952).

[21] The energy ϵ_q should be replaced by the effective energy of the meson in the nuclear medium, according to the Appendix A of I.

Here again we have to use the impulse approximation for the $\mathrm{Re}[a_s(0)]$ so the real part of B_s is unchanged.

To obtain further corrections to Eq. (67′), we return to the integral Eqs. (50). A first "Born approximation" to these is

$$G_\alpha = 1 + \frac{1}{e}\sum_{\beta\neq\alpha}{}' I_\beta.$$

When this is substituted into Eq. (59) for $\mathcal{V}_c$, we obtain

$$\begin{aligned}\mathcal{V}_c &= \Delta + t_c + \sum_{\alpha_1\neq\alpha_2,\alpha_2}\left\langle I_{\alpha_1}\frac{1}{e}I_{\alpha_2}\right\rangle \\ &= \Delta + t_c + \sum_{\alpha_1\neq\alpha_2,\alpha_2}\langle I_{\alpha_1}(a - t_c - \Delta)^{-1} I_{\alpha_2}\rangle.\end{aligned} \tag{70}$$

The second equation follows from the expression (38) for e.

Now

$$\langle I_{\alpha 1} e^{-1} I_{\alpha 2}\rangle = \langle t_{\alpha 1} e^{-1} t_{\alpha 2}\rangle - \langle t_{\alpha 1}\rangle e^{-1}\langle t_{\alpha 2}\rangle \quad . \tag{71}$$

follows directly from the definition of I_α. If we neglect the energy of excitation of the nucleus in e, then

$$e \simeq \epsilon_q + i\eta - p_0 - B_0,$$

when operating on a meson wave having momentum $\mathbf{p}$. Here $p_0 = [p^2 + \mu^2]^{\frac{1}{2}}$, where μ is the rest mass of the meson, and B_0 is defined by Eq. (65). For simplicity we neglect the spin and isotopic spin dependence of t_α and write

$$t_\alpha = (\mathbf{p}|t^0|\mathbf{k}) \exp[-i(\mathbf{p}-\mathbf{k})\cdot\mathbf{z}_\alpha] \tag{72}$$

in a momentum representation. Then (also in a momentum representation),

$$\begin{aligned}\sum_{\alpha_1\neq\alpha_2}\left\langle I_{\alpha_1}\frac{1}{e}I_{\alpha_2}\right\rangle = \sum_{\alpha_1\neq\alpha_2}\int d^3z_{\alpha 1} d^3z_{\alpha 2} d^3p \\ \times[P(\mathbf{z}_{\alpha 1}, \mathbf{z}_{\alpha 2}) - P(\mathbf{z}_{\alpha 1})P(\mathbf{z}_{\alpha 2})] \\ \times\left[\frac{(\mathbf{k}'|t^0|\mathbf{p})(\mathbf{p}|t^0|\mathbf{k})}{\epsilon_q + i\eta - p_0 - B_0}\right] \\ \times\exp[-i(\mathbf{k}'-\mathbf{p})\cdot\mathbf{z}_{\alpha 1}] \\ \times\exp[-i(\mathbf{p}-\mathbf{k})\cdot z_{\alpha 2}].\end{aligned} \tag{73}$$

$P(\mathbf{z}_{\alpha 1}, \mathbf{z}_{\alpha 2})$ is the joint probability of finding nucleon α_1, at the point $\mathbf{z}_{\alpha 1}$, and nucleon α_2 at the point $\mathbf{z}_{\alpha 2}$. $P(\mathbf{z}_{\alpha 1})$ is the probability of finding nucleon α_1, at the point $\mathbf{z}_{\alpha 1}$, etc. The combination of the P's in Eq. (73) follows directly from Eq. (71).

If the nucleus is large we can write

$$\begin{aligned}&[P(\mathbf{z}_{\alpha 1}, \mathbf{z}_{\alpha 2}) - P(\mathbf{z}_{\alpha 1})P(\mathbf{z}_{\alpha 2})] \\ &\qquad = (1/V_A)^2 v(\mathbf{z}_{\alpha 1}) C(|\mathbf{z}_{\alpha 1} - \mathbf{z}_{\alpha 2}|),\end{aligned} \tag{74}$$

where $v(z)$ is defined in connection with Eq. (34). $C(|\mathbf{z}_{\alpha 1} - \mathbf{z}_{\alpha 2}|)$ is a function describing the correlation between nucleon positions. Then, if the nucleus is large

$$\sum_{\alpha_1\neq\alpha_2}\left\langle I_{\alpha_1}\frac{1}{e}I_{\alpha_2}\right\rangle=(2\pi)^3\left(\frac{A}{V_A}\right)^2\delta(\mathbf{k}'-\mathbf{k})\times\int d^3r d^3p\frac{(\mathbf{k}'|t^0|\mathbf{p})(\mathbf{p}|t^0|\mathbf{k})}{\epsilon_q+i\eta-p_0-B_0}\times\exp[i(\mathbf{p}-\mathbf{k})\cdot\mathbf{r}]C(\mathbf{r}),\quad(75)$$

where we have written $\mathbf{r}=\mathbf{z}_{\alpha1}-\mathbf{z}_{\alpha2}$.

Equation (75) represents a correction to $t_c+\Delta$ due to correlation in nucleon positions (or, in other words, to the mutual interaction of two nucleons). It corresponds to two inelastic scatterings, the second of which returns the nucleus to its original state. To calculate this effect more carefully, we note that by Eq. (59),

$$\mathcal{V}_c=\Delta+\sum_{\alpha_1}\langle t_{\alpha1}G_{\alpha1}\rangle \simeq\Delta+t_c+\sum_{\alpha_1\neq\alpha_2}\langle I_{\alpha1}\Gamma_{\alpha1\alpha2}\,'I_{\alpha2}\rangle,\quad(76)$$

where the quantity $\Gamma_{\alpha1\alpha2}$ results from expanding the solution to Eq. (50) in a series in the $'I_\alpha$'s; that is,

$$\Gamma_{\alpha1\alpha2}=1+\sum_{(\alpha)}\left\{\frac{1}{e}\,'I_{\alpha3}+\frac{1}{e}\,'I_{\alpha3}\frac{1}{e}\,'I_{\alpha4}+\cdots\right\}.\quad(77)$$

The sum over (α) is such that no two indices on adjacent $'I_\alpha$'s are the same when $\Gamma_{\alpha1\alpha2}$ is substituted into Eq. (76). We have neglected in Eq. (76) those terms for which $\alpha_1=\alpha_2$.

Now if we neglect all correlations except those between pairs of nucleons—that is, except those of the form given by Eq. (75)—then the nuclear states of the $'I_\alpha$'s must be so paired. Since the nuclear ground state occurs in the matrix elements of only the first and last I_α in Eq. (76), these two must be paired and we can write $\mathcal{V}_c$ as

$$\mathcal{V}_c=t_c+\Delta+\sum_\gamma\sum_{\alpha_1\neq\alpha_2}(A,M'|I_{\alpha1}|\gamma)(\gamma|\Gamma_{\alpha1\alpha2}|\gamma)\times\left(\gamma\left|\frac{1}{e}\,'I_{\alpha2}\right|AM\right),\quad(78)$$

where the "γ" represent a complete set of nuclear states. (If there is spin dependent scattering, we should interpret $(\gamma|\Gamma_{\alpha1\alpha2}|\gamma)$ to be off diagonal in the nuclear spin orientation, as before.) Now a comparison of the quantity $(\gamma|\Gamma_{\alpha1\alpha2}|\gamma)=\langle\Gamma_{\alpha1\alpha2}\rangle$ with the $\langle F\rangle$ of Eq. (41) shows that these have the same structure. There are only two differences: (1) the indices α_1 and α_2 are suppressed for the first and last scatterings, respectively, in $\Gamma_{\alpha1\alpha2}$. This should not be important if the mass number A is large. (2) $\langle F\rangle$ contains I_α's while $\langle\Gamma_{\alpha1\alpha2}\rangle$ contains $'I_\alpha$'s. Since we have started the system in an excited state γ in $(\gamma|\Gamma_{\alpha1\alpha2}|\gamma)$, the ground state has lost its preferred role here, so this distinction is probably not of great importance. Therefore, we set

$$(\gamma|\Gamma_{\alpha1\alpha2}|\gamma)=\langle F\rangle=F_c\quad(79)$$

in Eq. (78). Since by Eq. (56)

$$F_c=1+(1/e-\mathcal{V}_1)\mathcal{V}_1,$$

we have, on doing a little algebra,

$$F_c(1/e)=1/(e-\mathcal{V}_1)=1/(a-\mathcal{V}_c).$$

If we substitute this into Eq. (78), we obtain

$$\mathcal{V}_c=t_c+\Delta+\sum_{\alpha_1\neq\alpha_2,\alpha_2}\left(I_{\alpha1}\frac{1}{a-\mathcal{V}_c}I_{\alpha2}\right).\quad(80)$$

This is an integral equation for $\mathcal{V}_c$ and has a simple interpretation. It differs from Eq. (70) only in that the approximate propagation function $1/e$ is replaced by the "correct function" $(a-\mathcal{V}_c)^{-1}$. If we write

$$(\mathbf{k}'|\mathcal{V}_c|\mathbf{k})=B\delta(\mathbf{k}'-\mathbf{k}),$$

it is evident that Eq. (75) is modified in that B_0 is replaced by B in the energy denominator. When the integrals in Eq. (75) are evaluated, Eq. (80) becomes an *algebraic* equation to be solved for B (if we can neglect the dependence of B on the energy when evaluating the integral).

For a qualitative evaluation of B we shall assume that

$$C(r)=\pm1,\quad r<r_0;\qquad =0,\quad r>r_0.$$

If, for instance, we choose the negative sign, we can interpret $C(r)$ as describing the net effect of the Pauli principle and a possible short range repulsion in the nuclear interaction. A choice of the positive sign would imply a net "bunching together" of nucleons as short distances. We interpret r_0 to be a parameter measuring the strength of this correlation.

To obtain a rough estimate of $\mathcal{V}_c$ from Eq. (80), we assume that and that the nucleons are point scatterers. Then the integral (75) is easily evaluated and we obtain

$$B\simeq\frac{B_0\mp iB_s^2\epsilon_q r_0/q}{1\pm iB_s^2\epsilon_q^2 r_0/q^3},\quad(81)$$

the sign depending upon that of $C(r)$.

Equation (81) provides us with an additional correction to Eq. (67'). For $r_0\lesssim(\hbar/\mu c)$ the correction to B_0 is no more than 10–20 percent and is thus negligible compared to the experimental uncertainties. It is reassuring to find this correction small, since it implies that Eq. (65) is a useful approximation for the optical model. On the other hand, one may hope eventually to have sufficiently precise measurements so that it is necessary to use Eq. (80). In this case the results might be used to obtain a measure of nucleon correlations in nuclear structure.

We finally observe that by keeping more terms in the expansion of Eq. (50) it is possible to obtain the effect of three-particle and four-particle correlations, etc., on $\mathcal{V}_c$.

V. APPLICATIONS TO MESON INTERACTIONS

By way of further application of the theory which has been developed, we shall briefly discuss the scattering and photoproduction of π mesons from complex nuclei. We shall employ the same simple model of the photoproduction which was used in I (i.e., that the matrix elements for the photoproduction from a bound nucleon are the same as for a free nucleon). From the point of view of the formal theory this means that there is a uniformly distributed source density in a median which can both scatter and reabsorb.

We shall arrive at essentially the same conclusions which were obtained in I. However, we are now able to ascribe a much more general validity to our results. Indeed, we shall obtain some specific predictions which may provide a test of the model for photoproduction (that the mechanism is the same for a bound as for an unbound nucleon, as discussed in more detail in I).

The scattering of mesons by nuclei is described by the theory already developed. The meson wave function $\varphi_q=\Omega_c\lambda_q$ satisfies Eq. (26). $\mathcal{V}_c$ does not of course now depend upon the spin $\mathbf{S}$ since the meson has no spin and we may also suppose it to not depend upon the nuclear spin $\mathbf{J}$.

As was shown in I, the transition operator T for photomeson production is

$$T=T_\pi+T_s+T_a,$$

where

$$T_a=RFe^{-1}H', \tag{82}$$

and

$$T_\pi+T_s=[1+(t_c+\Delta)e^{-1}][1+\textstyle\sum_\alpha I_\alpha F_\alpha e^{-1}]H'. \tag{83}$$

The quantity H', as defined in I, is the sum of the matrix elements for photomeson production from the individual nucleons in the nucleus. R is the absorption operator introduced in Eq. (40). Thus T_a is the transition operator for photomeson production followed by reabsorption. $T_\pi+T_s$ describes photoproduction with possible scattering of the meson before it leaves the nucleus. T_π is now explicitly defined as the transition amplitude for producing mesons which are *not* subsequently scattered *inelastically* or absorbed by the nucleus. That is,

$$T_\pi=[1+(t_c+\Delta)e^{-1}][1+\textstyle\sum_\alpha\langle I_\alpha F_\alpha\rangle e^{-1}]H'. \tag{84}$$

The quantity

$$\textstyle\sum_\alpha\langle I_\alpha F_\alpha\rangle\equiv L=\mathcal{V}_1+\mathcal{V}_1(e-\mathcal{V}_1)^{-1}\mathcal{V}_1,$$

as was shown in Sec. III. On introducing this into Eq. (84) and on rearranging, we obtain

$$\begin{aligned}T_\pi&=\left[1+\mathcal{V}_c\frac{1}{a-\mathcal{V}_c}\right]H'\\&=\Omega_c^{(-)\dagger}H'\end{aligned} \tag{85}$$

[see Eq. (24)]. This represents the rigorous form of the T_π which was given in the approximation of Eq. (64) in I. The matrix element of T_π for photoproduction of a meson of momentum $\mathbf{q}$ when the nucleus is excited to a state γ is then

$$\begin{aligned}(\gamma q|T_\pi|A,M)&=(\lambda_q,\Omega_c^{(-)\dagger}\langle\gamma|H'|A,M\rangle)\\&=(\varphi_q^{(-)},\langle\gamma|H'|A,M\rangle).\end{aligned} \tag{86}$$

The symbol $\langle\gamma|H'|A,M\rangle$ means the matrix element of H' with respect to the *nuclear* states (A,M) and γ. Equation (86) differs from a simple lowest order perturbation calculation of T_π only in that the "distorted wave" $\varphi_q^{(-)}=\Omega_c^{(-)}\lambda_q$ [see Eq. (27)] is used for the final state rather than the plane wave λ_q.

When the photoproduction from a given nucleon is localized at the point of production, the photoproduction cross section, as obtained from Eq. (86), can be written approximately as[22,23]

$$\frac{d\sigma_\pi}{d\Omega}=A\frac{d\sigma_f}{d\Omega}\eta\frac{(2\pi)^3}{V_A}\int d^3x|\varphi_q^{(-)}(x)|^2. \tag{87}$$

$d\sigma_f/d\Omega$ is the cross section for photomeson production from a free nucleon (actually, a suitably weighted average for the neutrons and protons in the nucleus, taking into account the charge of the meson). η is a parameter introduced to account for the effects of nuclear binding on the photoproduction and is expected to be independent of A to a fair approximation.[23]

We consider $d\sigma_\pi/d\Omega$ to correspond to the experimentally observed photomeson cross section when the very low energy (i.e., inelastically scattered mesons) are not observed. This conclusion is further justified since true absorption seems to predominate over the inelastic scattering for meson energies below 100 Mev.

From Eq. (30) we obtain the cross section for inelastic scattering or absorption of mesons by a nucleus to be

$$\sigma_{\mathrm{ab}}=-2\frac{(2\pi)^3}{v_\pi}\operatorname{Im}[(\varphi_q,\mathcal{V}_c\varphi_q)]. \tag{88}$$

If $\mathcal{V}_c$ is considered to be a constant within the nucleus and to vanish outside, the integrals of Eqs. (87) and (88) are the same and can be eliminated from these two equations to give

$$\frac{d\sigma_\pi}{d\Omega}=\left[\frac{d\sigma_f}{d\Omega}\eta\lambda\frac{A}{V_A}\right]\sigma_{\mathrm{ab}}, \tag{89}$$

[22] This was discussed in reference 1 and in more detail in reference 23.

[23] N. Francis and K. Watson, Phys. Rev. **89**, 328 (1953).

a result discussed in more detail in reference 23. To obtain Eq. (89), $\mathrm{Im}\mathcal{V}_c$ was set equal to $-v_\pi/2\lambda$, where λ is the mean free path for scattering or absorption.

Since we have defined T_π, T_s is also defined. This corresponds to photomeson production followed by inelastic scattering in the nucleus in which it was produced. We now wish to calculate the cross section σ_{star}, which corresponds to the photoproduction of a meson which is either reabsorbed or scattered inelastically. σ_{star} is expected to lead to considerable nuclear excitation. We first calculate the cross section σ_t for the production of a meson, irrespective of its subsequent behavior.

σ_t can most easily be evaluated by calculating the transition amplitude for the elastic, forward scattering of a high energy γ ray by the nucleus:

$$\langle T_\gamma\rangle_0=\langle H'Fe^{-1}H'\rangle_0 \tag{90}$$

as shown in I [the notation is that of Eq. (28)]. Now photomeson production seems to involve considerable nuclear excitation (recoil nucleons seem to have been observed)[24] and may reasonably be thought not to interfere coherently with the subsequent scattering of the meson. In this case, we can write Eq. (90) as

$$\langle T_\gamma\rangle_0=\sum\nolimits_\gamma(A,M|H'|\gamma)(\gamma|F|\gamma)(\gamma|e^{-1}H'|A,M), \tag{91}$$

where the "γ" represent a complete set of nuclear states. [Equation (91) is similar to Eq. (78).] Now $(\gamma|F|\gamma)=\langle F\rangle$, as given by Eq. (56). But

$$\langle F\rangle\frac{1}{e}=\frac{1}{a-\mathcal{V}_c},$$

so

$$\langle T_\gamma\rangle_0=\left\langle H'\frac{1}{a-\mathcal{V}_c}H'\right\rangle_0. \tag{92}$$

The cross section σ_t is then[11]

$$\sigma_t=-\frac{2(2\pi)^3}{v_\pi}\mathrm{Im}\left\{\left\langle H'\frac{1}{a-\mathcal{V}_c}H'\right\rangle_0\right\}. \tag{93}$$

For the same reasons which led to Eq. (91) it seems reasonable that the coherent interference of T_π and T_s can be neglected, so

$$\sigma_{\mathrm{star}}=\sigma_t-\sigma_\pi. \tag{94}$$

As argued in I, the quantity $\mathcal{V}_c$ is not expected to be of much importance in Eq. (93), so

$$\sigma_t\simeq A\sigma_f\eta, \tag{95}$$

where σ_f is the total weighted cross section for the protons and neutrons of the nucleons if they were free. We can take into account the effect of $\mathcal{V}_c$ in Eq. (93) by rewriting Eq. (95) as

$$\sigma_t=A\sigma_f\eta\eta', \tag{96}$$

where η' is the factor correcting Eq. (95). η' can be calculated from the optical model. Integrating the differential cross section (89) gives

$$\sigma_\pi=A\sigma_f\eta(\lambda\sigma_{\mathrm{ab}}/V_A),$$

so Eq. (94) becomes

$$\sigma_{\mathrm{star}}=A\sigma_f\eta[\eta'-(\lambda\sigma_{\mathrm{ab}}/V_A)], \tag{97}$$

on using Eq. (96). Equations (89) and (97) represent two relations between the four cross sections σ_f, σ_{ab}, σ_π and σ_{star}, each of which has been studied experimentally.

An approximate evaluation of η', if we use Eq. (93), and assume that the nucleus is large, gives

$$\eta'=1-\frac{3v_\pi}{8A}\left(\frac{a_0}{\lambda}\right)\left(\frac{B_R}{\epsilon_q}\right)\frac{1}{[(B_Ra_0)^2+(v_\pi a_0/2\lambda)^2]^2}, \tag{98}$$

where a_0 is related to the nuclear radius R by

$$a_0=R/A^{\frac{1}{3}}.$$

B_R is the real part of the part of B [Eq. (65)], the well depth. Taking $\sigma_{\mathrm{ab}}=\pi R^2$ and the values of B_R and λ obtained by Lederman *et al.*[20] for 65-Mev mesons, we obtain

$$\eta'=1+4.4/A,$$

and

$$\sigma_{\mathrm{star}}=A\sigma_f\eta[1+4.4/A-1.86/A^{\frac{1}{3}}]. \tag{99}$$

This is in qualitative agreement with the magnitude of the observed high energy photostar cross section.[24] These phenomena are complicated, however, and it appears that the meson reabsorption may be just one of several contributing causes. The relation (89) seems to be well satisfied experimentally.[23]

VI. FINAL COMMENTS

Our discussion indicates that the optical models have a wide range of validity. For applications one may ignore the problem of actually calculating the potential $\mathcal{V}_c$ and attempt to describe it by means of parameters which are determined by experiment. On the other hand, the arguments of Sec. IV suggest that it is not unfeasible to try calculating $\mathcal{V}_c$ directly in terms of free nucleon scattering amplitudes and certain gross parameters describing nuclear structure. In this case it is reasonable to hope that the theory can be useful in deducing such nuclear properties as density and strength of correlations.

The role played by spin-orbit interactions is of interest, although we have made no attempt to explicitly calculate such effects. If the spin dependence of the t_α's is known, then the methods of Sec. IV can be applied to calculate the spin dependence of $\mathcal{V}_c$. For the elastic scattering of slow nucleons by nuclei it is tempting to relate $\mathcal{V}_c$ to the "one-body potential" used in the shell model. In this case one may very well expect appreciable spin-orbit interactions.

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In the Appendix we give an alternate derivation of the potential $\mathcal{V}_c$. This alternate derivation has the advantage that it does not involve an approximation which assumes that the number of scatterers is large, as was done in Sec. III. The corrections for a finite number of scatterers appear in a somewhat curious and surprisingly simple manner—and permit us to define the optical model even for the somewhat singular (and trivial) case of a single bound scatterer.

We are indebted to Professor K. A. Brueckner for several helpful discussions.

APPENDIX

An Alternate Derivation of the Potential $\mathcal{V}_c$

We shall give an alternate development of the optical model potential $\mathcal{V}_c$ which has an advantage over that of Sec. III in that it is both simpler and of greater rigor. (It does not make the approximation that the number of scatterers is large.) For simplicity we shall assume that there is no true absorption, so $\Delta=0$.

The present derivation begins with the original (rigorous) multiple scattering Eqs. (33):

$$\Omega_s=1+\frac{1}{a}\sum_{\alpha=1}^{A} t_\alpha\Omega_s(\alpha),\quad \Omega_s(\alpha)=1+\frac{1}{a}\sum_{\beta\neq\alpha}^{A} t_\beta\Omega_s(\beta). \tag{A-1}$$

If we use the notation of Sec. III,

$$\Omega_c\equiv\langle\Omega_s\rangle=1+\frac{1}{a}\sum_{\alpha=1}^{A}\langle t_\alpha\Omega_s(\alpha)\rangle. \tag{A-2}$$

Define

$$\mathcal{L}\equiv\sum_{\alpha=1}^{A}\langle t_\alpha\Omega_s(\alpha)\rangle \tag{A-3}$$

and

$$\mathfrak{F}_c\equiv\langle\Omega_s(\alpha)\rangle. \tag{A-4}$$

Compare with the notation of Eqs. (44) and (46). Then using the "prime" notation and that of Eq. (45), the second set of Eqs. (A-1) lead to

$$\begin{aligned}{}'\Omega_s(\alpha)\rangle&={}'1\rangle+\frac{1}{a}\sum_{\beta\neq\alpha}^{A}{}'t_\beta\Omega_s(\beta)\rangle\\&=\frac{1}{a}\sum_{\beta\neq\alpha}^{A}[{}'t_\beta\rangle\mathfrak{F}_c+{}'t_\beta\,{}'\Omega_s(\beta)\rangle].\end{aligned} \tag{A-5}$$

Setting

$${}'\Omega_s(\alpha)=(\mathcal{G}_\alpha-1)\rangle\mathfrak{F}_c. \tag{A-6}$$

Equation (A-5) is satisfied if the $\mathcal{G}_\alpha$, satisfy the set of integral equations

$$\mathcal{G}_\alpha=1+\frac{1}{a}\sum_{\beta\neq\alpha}^{A}{}'t_\beta\mathcal{G}_\beta. \tag{A-7}$$

Then, from Eqs. (A-3) and (A-6), we find that

$$\mathcal{L}=\mathcal{U}_c\mathfrak{F}_c, \tag{A-8}$$

where $\mathcal{U}_c$ is defined as

$$\mathcal{U}_c\equiv\sum_{\alpha=1}^{A}\langle t_\alpha\mathcal{G}_\alpha\rangle. \tag{A-9}$$

From the second of Eqs. (A-1), we obtain

$$\langle\Omega_s(\alpha)\rangle=1+\frac{1}{a}\sum_{\beta=\alpha}^{A}\langle t_\beta\Omega_s(\beta)\rangle. \tag{A-10}$$

Since $\mathfrak{F}_c\equiv\langle\Omega_s(\alpha)\rangle$, and is independent of α, we obtain on summing Eq. (A-10) over α,

$$A\mathfrak{F}_c=A+(A-1)a^{-1}\mathcal{L}. \tag{A-11}$$

If we divide by A (we do not here drop the $1/A$ term as was done in Sec. III) and use Eq. (A-8), we obtain an integral equation for $\mathfrak{F}_c$:

$$\mathfrak{F}_c=1+a^{-1}\delta\mathcal{U}_c\mathfrak{F}_c \tag{A-12}$$

where we have defined

$$\delta\equiv1-1/A. \tag{A-13}$$

The approximation used in Sec. III was to neglect terms of relative order $1/A$. If we did that there we could set $\delta=1$ and obtain the optical model directly. More generally, let us define a "pseudopotential" $\mathcal{U}_\delta$ as

$$\mathcal{U}_\delta=\delta\mathcal{U}_c. \tag{A-14}$$

Then Eq. (A-12) can be written as

$$\mathfrak{F}_c=1+a^{-1}\mathcal{U}_\delta\mathfrak{F}_c. \tag{A-12$'$}$$

In a formal sense, Eq. (A-12′) defines a scattering problem with the pseudopotential $\mathcal{U}_\delta$. The coefficient of $1/a$ is the "transition operator" T_δ for this scattering:

$$T_\delta=\mathcal{U}_\delta\mathfrak{F}_c. \tag{A-15}$$

We return now to the actual problem, Eq. (A-2) is

$$\begin{aligned}\Omega_c&=1+a^{-1}\mathcal{L}\\&=1+a^{-1}\mathcal{U}_c\mathfrak{F}_c,\end{aligned} \tag{A-16}$$

by Eq. (A-8). The actual transition operator T_c is the coefficient of $1/a$ in Eq. (A-16), or

$$\begin{aligned}T_c&=\mathcal{U}_c\mathfrak{F}_c\\&=\delta^{-1}T_\delta.\end{aligned} \tag{A-17}$$

The last step follows upon comparison with Eqs. (A-14) and (A-15). When the number of scatterers A is large enough that one can set $\delta=1$, then $\Omega_c=\mathfrak{F}_c$ [see Eqs. (A-12) and (A-16)] and $T_c=T_\delta$. (We may also verify that $T_c=\mathcal{U}_c\mathfrak{F}_c$ holds even when $A=1$!)

In general, however, the actual transition operator differs from that obtained with the pseudopotential only by the factor δ^{-1}. The scattering cross sections are then related by δ^{-2}. This implies that in comparing an observed cross section with the predictions from the model with $\mathcal{U}_c\simeq t_c$, for instance (as was done in Sec. IV), one should multiply the observed cross section by

δ^2. The potential deduced from the experimental cross section is then $\mathcal{U}_\delta$, which should be divided by δ to obtain $\mathcal{U}_c$.

It appears that the $1/A$ corrections occur in a rather simple manner and indeed tend to cancel each other somewhat.

Equations (A-7) seem to be more difficult to tackle than Eqs. (50). That is, Eqs. (A-7) includes *elastic* scatterings in excited nuclear states. These are already taken into account in Eqs. (50) by the appearance in that equation of $1/e$ rather than $1/a$.

Another (related) difference between Eqs. (50) and (A-7) is that the scattering operators t_α are different in these two equations. The incident energy in the t_α of Eqs. (50) is corrected to be the energy of the particle *in the medium.* [This point was discussed in Appendix (A) of reference 1.] As was also shown in I, the fact that this energy has a small imaginary part does not lead to significant corrections—at least, where the imaginary part is no larger than it seems to be for mesons. It thus appears that certain complicated properties of Eqs. (A-7) appear in a more natural and simple manner in Eqs. (50).

As a final remark, we note that our rather free use of operators such as $(a-\mathcal{V})^{-1}$ is justified if there exists the Møller wave matrix,

$$\omega=1+a^{-1}\mathcal{V}\omega,$$

since $(a-\mathcal{V})^{-1}$ can be defined to be

$$\omega a^{-1}.$$

Therefore, all the results of this paper and of reference 1 depend upon the existence of such quantities ω. In particular, the entire derivation may be carried out in terms of the solution to integral equations and without the use of the more formal algebraic treatment which was actually employed.

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The Nuclear Moments of Technetium-99

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The hyperfine structure in the optical spectrum of technetium has been investigated in the region 3600–7000A with a Fabry-Perot interferometer crossed with a quartz prism spectrograph. The light source was a liquid-nitrogen-cooled hollow cathode discharge tube. The nuclear-spin, magnetic moments previously reported by one of us are confirmed ($I=9/2\hbar$ and $\mu=5.5\pm0.3$ nm) and are in agreement with a more recent nuclear induction measurement. The quadrupole moment of Tc^{99} is found to be: $Q=(+0.34\pm0.17)\times10^{-24}$ cm^2.

I. INTRODUCTION

ELEMENT 43 was first prepared in 1937 in the Berkeley cyclotron by neutron bombardment of molybdenum, and was named technetium (Tc) by Perrier and Segrè.[1,2] In 1948, milligram amounts of Tc^{99} were separated from uranium fission products by Parker, Reed, and Ruch.[3] A 3-mg sample of Tc^{99} was received by the spectroscopy laboratory of the National Bureau of Standards in January, 1949, for a preliminary investigation of the arc and spark spectra of the element.[4,5] In July, 1950, an additional amount of 3 mg was received from the Atomic Energy Commission for further work on the spectra of Tc. Wavelength measurements on 2121 lines in Tc I and Tc II were published by Meggers and Scribner,[6] and a preliminary analysis identifying 20 terms of Tc I and 4 terms of Tc II was published by Meggers.[7]

Of the second sample received, 2 mg were saved for an investigation of the hyperfine structure (hfs) of Tc. The nuclear spin (9/2) was determined by Kessler and Meggers[8] and a preliminary value (5.2 ± 0.5 nm) of the magnetic moment of the Tc nucleus was reported.[9] More recently a 7-mg sample of Tc^{99} has been received from the Oak Ridge National Laboratory. Four mg have been used to improve the descriptions of the arc and spark spectra, and the remainder was devoted to the investigation of the hfs of Tc reported below.

A nuclear induction experiment on 156 mg of Tc^{99} by Walchli, Livingston, and Martin[10] yielded for the magnetic moment a value of 5.6805 ± 0.0004 nm, consistent with our preliminary value.

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