Proton-Proton Scattering for a Nucleon Isobar Model*

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High-energy proton-proton scattering is calculated for a static interaction which includes coupling to excited nucleon states. The spin and isotopic spin dependence is taken from the strong coupling approximation of the charge-symmetric pseudoscalar meson theory; for the space dependence the Yukawa potential is used. The parameters of the interaction are chosen to fit the low-energy p-p scattering data. The calculated high-energy scattering cross section is strongly anisotropic and thus in sharp conflict with the experimental results.

1. INTRODUCTION

A NUMBER of attempts have been made[1–5] to construct an interaction potential which will explain the observed proton-proton scattering. In spite of the large choice of possible potentials it is very hard to fit the scattering data over the entire energy range; most potentials which are adjusted to the low and intermediate energy data do not lead to the observed large isotropic cross section at high energies (between 3 and 5 mb/sterad in the energy interval of 100 to 350 Mev).[6]

Let us consider the triplet and singlet contributions to the cross section separately. In the absence of a tensor force, the triplet cross section must vanish at 90° in the center-of-mass system. Thus only a tensor force could possibly give rise to an isotropic triplet cross section. One might postulate a strongly singular tensor potential, since its effect would become more pronounced with increasing energy. Such a potential, originally proposed by Christian and Noyes[1] and recently re-examined by Goldfarb and Feldman[4] and Swanson,[5] does not lead to very good agreement with the observed results.

The "usual" potentials give rise to a strongly anisotropic singlet cross section at high energies. For a potential which is attractive at all distances both S and D phase shifts are positive; in the cross section this leads to destructive interference between S and D waves at 90° because $P_0(\cos\frac{1}{2}\pi)$ is positive and $P_2(\cos\frac{1}{2}\pi)$ is negative. Jastrow[2] has suggested a potential which becomes infinitely repulsive at a small distance. With such a "hard core" model, the S phase shift becomes negative at high energies, leading to constructive interference between S and D waves at 90°. This model fits the known p-p scattering best.

A more general p-p interaction than ordinarily considered can be obtained if nucleons are postulated to have excited isobaric states characterized by different spin and charge values.[7] Such an interaction is used in this paper. In order to narrow down the large choice of possible interactions, we shall limit ourselves to one type suggested by meson theory.

In the strong-coupling approximation of meson theory[8] (where the meson-nucleon coupling is assumed to be large) one does obtain an isobar model of the nucleon. In particular, according to the charge-symmetric pseudoscalar theory, a nucleon can exist in isobaric states with any half odd-integral spin and isotopic spin, the spin and isotopic spin being equal in a given state. The energy of an isobaric state is proportional to $s(s+1)$, where s is the spin (and isotopic spin). The interaction energy of two nucleons is of the form[9]

$$\Omega V(r)+T'U(r).$$

Here Ω and T' operate on the spins and isotopic spins of the two nucleons; T' also depends on the orientation of the relative position vector with respect to the spins; V and U are functions only of r, the distance between the two nucleons. For nonexcited states of the nucleons (spins and isotopic spins equal to $\frac{1}{2}$), the interaction reduces to that obtained in the weak coupling approximation.

A model of this type (with V and U square well potentials) was applied by Villars[10] to the neutron-proton problem. Villars found that if the parameters of the potentials were chosen to yield the correct deuteron binding energy and quadrupole moment, the excitation energy $\mathcal{E}$ of the first-excited isobar had to be of the order 300 Mev in order that there be no bound

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[1] R. S. Christian and H. P. Noyes, Phys. Rev. **79**, 85 (1950).

[2] R. Jastrow, Phys. Rev. **79**, 389 (1950); **81**, 636 (1951).

[3] K. M. Case and A. Pais, Phys. Rev. **80**, 203 (1950).

[4] L. J. B. Goldfarb and D. Feldman, Phys. Rev. **88**, 1099 (1952).

[5] D. R. Swanson, Phys. Rev. **89**, 740 (1953).

[6] See, for instance, the review article by R. S. Christian, Repts. Progr. Phys. **15**, 68 (1952).

[7] Several suggestions of this nature have been made: G. Breit, *Proceedings of the International Conference on Nuclear Physics and the Physics of Elementary Particles* (Institute for Nuclear Studies, Chicago, 1951), p. 107; R. B. Raphael and J. Schwinger, Phys. Rev. **90**, 373 (1953); J. Iwadare, Progr. Theoret. Phys. (Japan) **9**, 94 (1953); B. Feld, Bull. Am. Phys. Soc. **28**, No. 3, 30 (1953).

[8] G. Wentzel, Helv. Phys. Acta **13**, 269 (1940); **14**, 633 (1941); **16**, 222 (1943); **16**, 551 (1943); A. Houriet, Helv. Phys. Acta **18**, 473 (1945); W. Pauli and S. M. Dancoff, Phys. Rev. **62**, 85 (1942); R. Serber and S. M. Dancoff, Phys. Rev. **63**, 143 (1943); W. Pauli and S. Kusaka, Phys. Rev. **63**, 400 (1943).

[9] We are using the notation of M. Fierz, Helv. Phys. Acta **17**, 181 (1944); **18**, 158 (1945).

[10] F. Villars, Helv. Phys. Acta **19**, 323 (1946).

^1S state. We apply the same type of model to the proton-proton problem, taking $\mathcal{E}=265$ Mev and letting V and U be Yukawa potentials[11] with the same parametric range. The depths of the potentials and the parametric range are chosen to fit the observed low-energy p-p scattering data. Then the cross section is calculated for a high energy (300 Mev in the laboratory system).

The results obtained with the isobar model described above are quite similar to those obtained with most conventional interactions: The high-energy singlet cross section is strongly anisotropic due to S-D wave interference; the triplet cross section is flatter, but much too small to yield the observed cross section. We must conclude that at least the particular isobar model of nucleons used in this paper conflicts with the observed p-p scattering.

2. THE MODEL

The two-nucleon scalar and tensor interaction energies, $\Omega V(r)$ and $T'U(r)$, derived in the strong-coupling charge-symmetric pseudoscalar theory can be represented as matrices in the following variables:

s_1, s_2, spins of the nucleons;
K, total isotopic spin;
S, total spin;
l, orbital angular momentum;
J, total angular momentum.

Ω and T' are diagonal with respect to K and J, Ω also with respect to l and S. For given K and J, in the center-of-mass system, the Schrödinger function components $F_{Sls_1s_2}(r)$, satisfy the set of equations,

$$\Big\{-\frac{\hbar^2}{M}\Big(\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}\Big)$$
$$+\tfrac{1}{3}\mathcal{E}[s_1(s_1+1)+s_2(s_2+1)-\tfrac{3}{2}]\Big\}F_{Sls_1s_2}(r)$$
$$+\sum_{S'l's_1's_2'}\{(s_1s_2|\Omega|s_1's_2')\delta_{SS'}\delta_{ll'}V(r)$$
$$+(Sls_1s_2|T'|S'l's_1's_2')U(r)\}F_{S'l's_1's_2'}=EF_{Sls_1s_2};\quad(2.1)$$

here M is the proton mass, $\mathcal{E}$ the excitation energy of the first-excited isobar, and E the total energy. The isobaric energy term represents the energy difference between excited states and the ground state ($s_1=s_2=\frac{1}{2}$). The matrix elements of Ω and T' in this representation have been given by Fierz.[9] They obey the selection rules,

$$s_1'-s_1=0,\ \pm1;\quad s_2'-s_2=0,\ \pm1;$$
$$S'-S=0,\ \pm2;\quad l'-l=0,\ \pm2.$$

In addition, the vector inequalities,

$$|s_1-s_2|\leq S,\ K\leq(s_1+s_2);\quad |l-S|\leq J\leq(l+S)$$

and the Pauli principle must be satisfied. The latter requirement can be shown to be equivalent to

$$(S+K+l)=\text{odd integer}.$$

When the excited states are neglected, the interaction reduces to that obtained in the weak coupling approximation; i.e., for $s_1=s_2=s_1'=s_2'=\frac{1}{2}$,

$$\Omega=(1/9)(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2),\quad T'=(1/27)(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)S_{12},\quad(2.2)$$

where $\frac{1}{2}\boldsymbol{\sigma}_{1,2}=$spin of nucleon 1, 2; $\frac{1}{2}\boldsymbol{\tau}_{1,2}=$isotopic spin of nucleon 1, 2; and $S_{12}=3(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})r^{-2}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$.

The two potentials $V(r)$ and $U(r)$ are assumed to differ only by a constant factor,

$$V=V_0f(r/r_0),\quad U=U_0f(r/r_0),$$

where r_0 is the parametric range of the potentials. Equation (2.1) can now be written in dimensionless form

$$\Big\{-\Big(\frac{d^2}{dx^2}-\frac{l(l+1)}{x^2}\Big)$$
$$+\tfrac{1}{3}\epsilon[s_1(s_1+1)+s_2(s_2+1)-\tfrac{3}{2}]\Big\}F_{Sls_1s_2}(x)$$
$$+\{\sum_{S'l's_1's_2'}(Sls_1s_2|\Lambda|S'l's_1's_2')f(x)$$
$$\times F_{S'l's_1's_2'}(x)\}=k^2F_{Sls_1s_2}(x),\quad(2.3)$$

with

$$x=r/r_0,\quad \epsilon=(Mr_0^2/\hbar^2)\mathcal{E},\quad k^2=(Mr_0^2/\hbar^2)E,$$
$$\lambda=\tfrac{1}{9}(Mr_0^2/\hbar^2)V_0,\quad \gamma=\tfrac{1}{3}U_0/V_0,$$

and

$$\Lambda=\lambda(9\Omega+27\gamma T').$$

The constants are defined so that for nonexcited triplet p-p states $\Lambda=\lambda(1+\gamma S_{12})$. The values of the parameters obtained from low-energy scattering (Sec. 4) are $\lambda=0.35$, $\gamma=1.4$, and $r_0=2.0\times10^{-13}$ cm. With this value of r_0, one obtains $\epsilon\cong25$, $k^2\cong14$ for $\mathcal{E}=265$ Mev, $E=150$ Mev.

In the following, only the ground state ($s_1=s_2=\frac{1}{2}$) and the first-excited states ($s_1=\frac{1}{2}$, $s_2=\frac{3}{2}$ and $s_1=\frac{3}{2}$, $s_2=\frac{1}{2}$) of the system will be taken into account. The neglect of higher states seems reasonable if one considers that the excitation energy of the second-excited state ($s_1=s_2=\frac{3}{2}$) is already $2\mathcal{E}=530$ Mev and that the calculations are here carried out for a kinetic energy $E=150$ Mev. Estimates indicate that this approximation is really justified. We shall use the following notation: g for the ground state, ($s_1=\frac{1}{2}$, $s_2=\frac{1}{2}$); e for the symmetric combination of the two first-excited states $2^{-1/2}[(s_1=\frac{1}{2}, s_2=\frac{3}{2})+(s_1=\frac{3}{2}, s_2=\frac{1}{2})]$. The corresponding

[11] In both strong and weak coupling approximations meson theory yields $V(r)\sim e^{-\mu r}/r$; $U(r)\sim(3r^{-3}+3\mu r^{-2}+\mu^2r^{-1})e^{-\mu r}$. Even if one wanted to take meson theory seriously, one would have to modify $U(r)$ to eliminate the r^{-3} singularity.

TABLE I.

	Triplet (l odd) $S=1$	Singlet-quintet (l even) $S=0$ or 2
$J=0$	$l=1\begin{cases}g\\e\end{cases}$	$\begin{cases}l=0,\ S=0,\ g\\l=2,\ S=2,\ e\end{cases}$
even $J\neq 0$	$\begin{cases}l=J-1\begin{cases}g\\e\end{cases}\\l=J+1\begin{cases}g\\e\end{cases}\end{cases}$	$\begin{cases}l=J,\ S=0,\ g\\l=\begin{cases}J-2\\J\\J+2\end{cases},\ S=2,\ e\end{cases}$
odd J	$l=J\begin{cases}g\\e\end{cases}$	$l=J\pm1,\ S=2,\ e$

components of the Schrödinger function are

$$F_{Slg}\equiv F_{Sl(1/2)(1/2)},$$
$$F_{Sle}\equiv(1/\sqrt{2})(F_{Sl(1/2)(3/2)}+F_{Sl(3/2)(1/2)}).$$

In the g state, S is restricted to the values 0 and 1, in the e state to 1 and 2.

Since we are interested in p-p scattering, we want to consider the case $K=1$. Then the Pauli principle demands that $(S+l)$ be even. We obtain the scheme of linked states shown in Table I. This table indicates that for $J=0$ and odd J there is a coupling between two states; for even $J\neq 0$ there is coupling between four states. For instance, if $J=0$, $S=1$ (triplet), the components $F_{1,1,g}$ and $F_{1,1,e}$ satisfy a pair of coupled differential equations; for even $J\neq 0$, $S=1$, the components $F_{1,J-1,g}$, $F_{1,J-1,e}$, $F_{1,J+1,g}$, and $F_{1,J+1,e}$ satisfy a set of four coupled differential equations. When there is coupling between just two states, Eq. (2.3) reduces to

$$\left(\frac{d^2}{dx^2}-\frac{l(l+1)}{x^2}+k^2\right)F_{lg}=(lg|\Lambda|lg)fF_{lg}+(lg|\Lambda|l'e)fF_{l'e},$$
$$\left(\frac{d^2}{dx^2}-\frac{l'(l'+1)}{x^2}-\kappa^2\right)F_{l'e}=(l'e|\Lambda|lg)fF_{lg}+(l'e|\Lambda|l'e)fF_{l'e}; \tag{2.4}$$

here $\kappa^2=\epsilon-k^2$, and the index S may be dropped without ambiguity. The extension to cases of four coupled equations is straightforward.

The relevant matrix elements of Ω and T' are given in Appendix I.

3. METHODS OF CALCULATION

Two approximation methods, the uncoupling method and the Born approximation, were used in the integration of the sets of coupled equations. In the uncoupling method, the set of equations is replaced by a single equation with an "effective potential," which can be integrated numerically. This method was used in the treatment of the low-energy problem. For the high-energy scattering, the second Born approximation was found satisfactory. For several cases, the approximation methods were checked by exact numerical integration.

A. Numerical Integration

The point by point integration method of Manning and Millman[12] can be extended in a straightforward way to the case of two coupled equations. For any well-behaved function $u(x)$ we can express the second difference in terms of the second derivatives at 3 points,

$$u(x+h)+u(x-h)-2u(x)=\tfrac{1}{12}h^2[u''(x+h)+u''(x-h)+10u''(x)]+O(h^6). \tag{3A.1}$$

If we let $F_{lg}=y$, $F_{l'e}=z$, (2.4) takes on the form

$$y''(x)=a(x)y(x)+b(x)z(x),$$
$$z''(x)=b(x)y(x)+c(x)z(x). \tag{3A.2}$$

Applying Eq. (3A.1) to y and z in Eq. (3A.2), we can express the values of y and z at the point x_{n+1} in terms of their values at the points x_{n-1} and x_n. Solving the two equations for y_{n+1} and z_{n+1} yields

$$y_{n+1}\cong\frac{1}{A_{n+1}C_{n+1}}(\chi_nC_{n+1}+\xi_nB_{n+1});$$
$$z_{n+1}\cong\frac{1}{A_{n+1}C_{n+1}}(\xi_nA_{n+1}+\chi_nB_{n+1});$$
$$\chi_n=(12-10A_n)y_n+10B_nz_n-A_{n-1}y_{n-1}+B_{n-1}z_{n-1};$$
$$\xi_n=(12-10C_n)z_n+10B_ny_n-C_{n-1}z_{n-1}+B_{n-1}y_{n-1}, \tag{3A.3}$$

where

$$A_n=1-\tfrac{1}{12}h^2a(x_n);\quad C_n=1-\tfrac{1}{12}h^2c(x_n);$$
$$B_n=\tfrac{1}{12}h^2b(x_n).$$

The solutions were started at the origin by power series.[13]

B. The Uncoupling Method

The method here referred to as the uncoupling method has been used by several authors. Fierz and Wentzel[14] applied it to a problem closely related to the one treated here; they refer to it as the adiabatic method. Christian and Hart[15] used it to obtain effective potentials in the problem of the triplet n-p system; they apply the WKB method to the effective potentials and refer to the whole procedure as the WKB method for coupled equations.

The uncoupling method consists in transforming a set of coupled differential equations into a less strongly coupled set, and then neglecting the coupling. It has been found to give quite a good approximation even

[12] W. F. Manning and J. Millman, Phys. Rev. **53**, 673 (1938).
[13] A very similar example of power series integration is presented in detail in reference 5.
[14] M. Fierz and G. Wentzel, Helv. Phys. Acta **17**, 215 (1944).
[15] R. S. Christian and E. W. Hart Phys. Rev. **77**, 441 (1950).

when it was very hard to justify analytically. It will be shown later how the procedure can be justified in a certain special case.

Given a set of functions, $\mathbf{u}(x)\equiv[u_1(x), \cdots, u_N(x)]$, satisfying the differential equations

$$\frac{d^2}{dx^2}\mathbf{u}(x)=\mathbf{V}(x)\cdot\mathbf{u}(x), \tag{3B.1}$$

where $\mathbf{V}(x)$ is a symmetric matrix. For $\mathbf{u}=\begin{pmatrix}y\\z\end{pmatrix}$, $\mathbf{V}=\begin{pmatrix}a & b\\ b & c\end{pmatrix}$, Eq. (3B.1) reduces to Eq. (3A.2). Now let us transform $\mathbf{u}$ with an orthogonal transformation $\mathbf{S}(x)$ which brings $\mathbf{V}(x)$ into the diagonal form $\mathbf{W}(x)$,

$$\mathbf{S}^T\cdot\mathbf{V}\cdot\mathbf{S}=\mathbf{W}, \quad \mathbf{u}=\mathbf{S}\cdot\mathbf{v}; \quad \mathbf{S}^T\cdot\mathbf{S}=\mathbf{1}, \tag{3B.2}$$

where $\mathbf{S}^T$ is the transpose of $\mathbf{S}$. The transformed set of differential equations is

$$\frac{d^2}{dx^2}\mathbf{v}=\mathbf{U}\cdot\mathbf{v}+\mathbf{W}\cdot\mathbf{v}, \tag{3B.3}$$

where

$$\mathbf{U}=-\mathbf{S}^T\cdot\left[2\left(\frac{d}{dx}\mathbf{S}\right)\frac{d}{dx}+\left(\frac{d^2}{dx^2}\mathbf{S}\right)\right].$$

The diagonal elements of $\mathbf{U}$ can be simplified with the help of the orthogonality condition on $\mathbf{S}$,

$$U_{nn}=\sum_m\left(\frac{d}{dx}S_{mn}\right)^2. \tag{3B.4}$$

The uncoupling approximation now consists in neglecting the off-diagonal elements of $\mathbf{U}$; this leads to the uncoupled set of equations,

$$\frac{d^2}{dx^2}v_n(x)=(W_{nn}(x)+U_{nn}(x))v_n(x). \tag{3B.5}$$

The boundary conditions on the v_n follow from those on the u_n.

For the case of two coupled equations, the two eigenvalues of $V(x)$, $W_1(x)$ and $W_2(x)$, are

$$W_{1,2}=\tfrac{1}{2}(V_{11}+V_{22})\mp[\tfrac{1}{4}(V_{22}-V_{11})^2-V_{12}^2]^{1/2}. \tag{3B.6}$$

If the inequality

$$(V_{22}-V_{11})\gg|V_{12}| \tag{3B.7}$$

is satisfied, then it follows that

$$\begin{aligned} &W_1\cong V_{11}-V_{12}^2(V_{22}-V_{11})^{-1},\\ &W_2\cong V_{22}+V_{12}^2(V_{22}-V_{11})^{-1};\\ &S_{11}=S_{22}\cong 1, \quad S_{12}=S_{21}\cong V_{12}(V_{22}-V_{11})^{-1}. \end{aligned} \tag{3B.8}$$

Now consider the following example:

$$\begin{aligned} &V_{11}=\lambda_{11}e^{-x}-k^2;\\ &V_{22}=\lambda_{22}e^{-x}-k^2+\epsilon;\\ &V_{12}=\lambda_{12}e^{-x}. \end{aligned} \tag{3B.9}$$

Assume that the coefficients λ_{11}, λ_{22}, λ_{12} introduced here are all of the order of some constant λ; also assume

$$\epsilon>k^2; \quad \epsilon\gg\lambda; \quad \epsilon^{1/2}\gg 1. \tag{3B.10}$$

Then the expressions (3B.9) satisfy the condition (3B.7). We can show that the transformed Eqs. (3B.3) are less strongly coupled than the original Eqs. (3B.1). A reasonable criterion of the strength of the coupling is provided by comparing the off-diagonal elements with the difference of the diagonal elements. A rough estimate gives

$$|U_{12}|/(W_2-W_1)\lesssim\lambda/\epsilon^{3/2},$$

whereas

$$|V_{12}|/(V_{22}-V_{11})\sim\lambda/\epsilon.$$

Therefore the uncoupling method is justified here. Note also that the diagonal elements of $\mathbf{U}$, Eq. (3B.4), are of the order $(\lambda/\epsilon)^2$ and can therefore be neglected in Eq. (3B.5).

This example is hypothetical. In the cases arising in this paper, the situation is complicated by the appearance of centrifugal potential terms and by the use of Yukawa potentials rather than exponential potentials; in these cases the previously stated criterion for the uncoupling approximation is not satisfied for small x. Nevertheless, comparison with exact numerical solutions showed that the uncoupling approximation is satisfactory except when the potentials cross (i.e., when there is a value of x for which $V_{11}=V_{22}$; in the neighborhood of such a point the above analysis breaks down completely). The uncoupling approximation is actually only needed in the treatment of the low-energy ${}^1S-{}^5D$ system, where the potentials do not cross. The second Born approximation can be applied to all other states, since they only enter at higher energies.

For sufficiently high energy (large k^2), the Born approximation can supply another check for the validity of the uncoupling method (see below).

C. The Born Approximation

It was found satisfactory for high-energy scattering to treat the terms $(lg|\Lambda|l'g)f(x)$ of the interaction in first Born approximation only, and the terms $(lg|\Lambda|l'e)f(x)$ in second Born approximation. The formal development of the method follows.

Consider the Green's functions $G_k^{(l)}$, $G_\kappa^{(l)}$, satisfying the differential equations

$$\begin{aligned} &\left(\frac{d^2}{dx^2}-\frac{l(l+1)}{x^2}+k^2\right)G_k^{(l)}(x,x')=-\delta(x-x');\\ &\left(\frac{d^2}{dx^2}-\frac{l(l+1)}{x^2}-\kappa^2\right)G_\kappa^{(l)}(x,x')=-\delta(x-x'); \end{aligned} \tag{3C.1}$$

and the boundary conditions

$$G_k^{(l)}(x_<=0)=G_\kappa^{(l)}(x_<=0)=0; \quad G_\kappa^{(l)}(x_>=\infty)=0.$$

It is convenient to represent $G_k^{(l)}$, $G_\kappa^{(l)}$ in the following way:

$$G_k^{(l)}(x,x') = -(1/k)g_l(kx_<)m_l(kx_>); \qquad (3C.2)$$
$$G_\kappa^{(l)}(x,x') = (2/\pi)\int_0^\infty dp(p^2+\kappa^2)^{-1}g_l(px)g_l(px'),$$

where

$$g_l(\rho) = (\pi\rho/2)^{1/2}J_{l+(1/2)}(\rho), \quad m_l(\rho) = (-1)^{l+1}g_{-l-1}(\rho),$$
$$g_l(\rho\to\infty)\to\sin(\rho-\tfrac{1}{2}\pi l);$$

note that $g_l(\rho)/\rho = j_l(\rho)$ is the lth order spherical Bessel function.

Using (3C.1) we can rewrite Eq. (2.4) as a pair of coupled integral equations,

$$F_{lg}(x) = g_l(x) - \int_0^\infty dx' G_k^{(l)}(x,x')f(x') \times[(lg|\Lambda|lg)F_{lg}(x') + (lg|\Lambda|l'e)F_{l'e}(x')], \qquad (3C.3)$$
$$F_{l'e}(x) = -\int_0^\infty dx' G_\kappa^{(l')}(x,x')f(x') \times[(l'e|\Lambda|lg)F_{lg}(x') + (l'e|\Lambda|l'e)F_{l'e}(x')].$$

Using (3C.2) we find

$$F_{lg}(x\to\infty)\sim\sin(kx-\tfrac{1}{2}\pi l+\delta_l),$$

where

$$-k\tan\delta_l = \int_0^\infty dx g_l(kx)f(x) \times[(lg|\Lambda|lg)F_{lg}(x) + (lg|\Lambda|l'e)F_{l'e}(x)].$$

In the approximation specified previously, we obtain

$$-k\tan\delta_l \cong (lg|\Lambda|lg)\int_0^\infty dx[g_l(kx)]^2 f(x) - (lg|\Lambda|l'e)^2\int_0^\infty\int_0^\infty dxdx' G_\kappa^{(l')}(x,x') \times g_l(kx)g_l(kx')f(x)f(x'). \qquad (3C.4)$$

Introduce the notations

$$I_{l,l'}(k,p) = \int_0^\infty dx f(x)g_l(kx)g_{l'}(px);$$
$$I_{l,l'}(k) = I_{l,l'}(k,k);$$
$$\mathcal{I}_{l,l'}(k) = \int_0^\infty dx[f(x)]^2 g_l(kx)g_{l'}(kx);$$
$$\mathfrak{B}_{l,l'}(k) = (2/\pi)\int_0^\infty dp(\epsilon+p^2-k^2)^{-1}[I_{l,l'}(k,p)]^2; \qquad (3C.5)$$
$$\mathfrak{B}_{l,l'}(k) = (2/\pi)\int_0^\infty dp(\epsilon+p^2-k^2)^{-1} \times I_{l,l'}(k,p)I_{l',l'}(p,k).$$

With the help of Eqs. (3C.2) and (3C.5), Eq. (3C.4) can be written as

$$-k\tan\delta_l = (lg|\Lambda|lg)I_{l,l}(k) - (lg|\Lambda|l'e)^2\mathfrak{B}_{l,l'}(k). \qquad (3C.6)$$

In the singlet J even ($\neq 0$) case, a g-state component, F_{Jg}, is linked to three e-state components, $F_{J-2,e}$, $F_{J,e}$, and $F_{J+2,e}$; the immediate extension of the above method gives

$$k\tan\delta_J = -(J,g|\Lambda|J,g)I_{J,J}(k) + (J,g|\Lambda|J-2,e)^2\mathfrak{B}_{J,J-2}(k) + (J,g|\Lambda|J,e)^2\mathfrak{B}_{J,J}(k) + (J,g|\Lambda|J+2,e)^2\mathfrak{B}_{J,J+2}(k). \qquad (3C.7)$$

In the triplet even $J(\neq 0)$ case, the four components $F_{J-1,g}$, $F_{J+1,g}$, $F_{J-1,e}$, and $F_{J+1,e}$ enter into a set of four coupled differential equations. There are two independent proper solutions, i.e., solutions for which all four components vanish at the origin, and $F_{J-1,e}$ and $F_{J+1,e}$ vanish asymptotically; $F_{J-1,g}$ and $F_{J+1,g}$ will be sinusoidal for large x. The problem of calculating the triplet cross section is the same as that for an ordinary tensor force, since only the ground-state components, $F_{J-1,g}$ and $F_{J+1,g}$, appear in the asymptotic wave function. For the purpose of calculating the cross section it is convenient to choose the two independent solutions so that in each solution both ground-state components have the same phase shift; i.e., the asymptotic behavior of one solution will be given by

$$F_{J-1,g}\to\sin[kx-\tfrac{1}{2}\pi(J-1)+\delta_J],$$
$$F_{J+1,g}\to\eta_J\sin[kx-\tfrac{1}{2}\pi(J+1)+\delta_J]; \qquad (3C.8)$$

η_J is called the admixture parameter, δ_J the eigenphase shift. The other solution will have the same form but different values of η_J and δ_J. The previously used approximation can be applied to this case in a straightforward manner; the η_J and δ_J are then expressed in terms of the integrals (3C.5).

The integrals $I_{l,l'}$ and $\mathcal{I}_{l,l'}$ defined in (3C.5) can be evaluated in terms of elementary functions if $f(x)$ is the Yukawa function (see Appendix II). $\mathfrak{B}_{l,l'}$ and $\mathfrak{B}_{l,l'}$ cannot be so evaluated and must in general be calculated by numerical quadrature; however, in certain cases one can find approximate expressions for these integrals. Consider, for instance,

$$\mathfrak{B}_{0,0}(k) = (2/\pi)\int_0^\infty dp(\epsilon+p^2-k^2)^{-1} \times\left[\int_0^\infty dx f(x)\sin kx\sin px\right]^2, \quad k^2<\epsilon. \qquad (3C.9)$$

If $k^2\gg 1$, the integral in the curly brackets is sharply peaked about $p=k$. Then the term $(\epsilon+p^2-k^2)^{-1}$ in the integrand can be approximated by ϵ^{-1}; for given k this

approximation improves with increasing ϵ.

$$\mathfrak{B}_{0,0}(k)\cong\epsilon^{-1}(2/\pi)\int_0^\infty dp\left[\int_0^\infty dx f(x)\,\sin kx\,\sin px\right]\times\left[\int_0^\infty dx' f(x')\,\sin kx'\,\sin px'\right].$$

In this expression, the integration over p gives a δ function, and we have

$$\mathfrak{B}_{0,0}(k)\cong\epsilon^{-1}\int_0^\infty dx[f(x)]^2\sin^2 kx. \quad (3C.10)$$

The same arguments hold for $\mathfrak{B}_{l,l}$ in general, although the approximation deteriorates with increasing l,

$$\mathfrak{B}_{l,l}(k)\cong\epsilon^{-1}\int_0^\infty dx[f(x)]^2[g_l(kx)]^2 = \epsilon^{-1}\mathcal{J}_{l,l}. \quad (3C.11)$$

This approximate expression for $\mathfrak{B}_{l,l}$ is closely related to the uncoupling approximation applied to the equations coupling F_{lg} and F_{le}; it is the contribution of the e state to $k\tan\delta_l$, when the first Born approximation is applied to the effective potential. We have thus confirmed that, when both the uncoupling and the Born approximation are justified, they both lead to the same result.

4. DETERMINATION OF THE PARAMETERS λ, γ, AND r_0

For the determination of low-energy scattering, only the $^1S_0$ component, $F_{S=0,\,l=0,\,g}$, of the Schrödinger function enters directly; it is, however, coupled to $F_{2,2,e}$. The effect of this coupling was considered in the uncoupling approximation by means of the appropriate effective potential $W(x)$. Writing $u_0(x)$ for the solution of the problem at zero energy, and introducing the p-p Coulomb interaction,[16] we have

$$\left(\frac{d^2}{dx^2}-\frac{r_0}{R}\frac{1}{x}\right)u_0(x)=W(x)u_0(x),$$

where

$$R=(\hbar^2/Me^2)=2.88\times10^{-12}\text{ cm}, \quad (4.1)$$

$$W(x)=\tfrac{3}{4}\lambda(5\gamma-3)f(x)+3x^{-2}+\tfrac{1}{2}\epsilon - \{[\tfrac{3}{4}\lambda(5\gamma+1)f(x)+3x^{-2}+\tfrac{1}{2}\epsilon]^2+[12\lambda\gamma f(x)]^2\}^{1/2},$$

$$f(x)=e^{-x}/x, \quad x=r/r_0.$$

[16] A consistent way of introducing the Coulomb term would be to add to the nuclear interaction, $\Omega V(r)+T'U(r)$, a Coulomb interaction, $V_c=(e^2/r)(K_3^{(1)}+\frac{1}{2})(K_3^{(2)}+\frac{1}{2})$, where $K_3^{(1)}$, $K_3^{(2)}$ are the third components of the isotopic spins of the two nucleons. The interaction is now no longer independent of the direction of the total isotopic spin $\mathbf{K}$. The p-p system is represented by the state $(K=1, K_3=1, g)$; for the case $K=1$, $K_3=1$, one has

$$(g|V_c|g)=e^2/r, \quad (g|V_c|e)=0, \quad (e|V_c|e)=\tfrac{1}{4}e^2/r.$$

The incorporation of these matrix elements into the original Schrödinger equation and the application of the uncoupling method leads to Eq. (4.1) with a $W(x)$ differing by a negligible amount from the one there given.

Use has here been made of Eqs. (2.4) and (3B.4) and of the matrix elements in Appendix I.

For a given set of values λ, γ, r_0, Eq. (4.1) can be integrated, and the solution u_0 can then be used to determine the scattering length a, and the effective range r_{eff} (see Jackson and Blatt[17]).

The experimentally determined values are

$$a=-7.67\pm0.05\times10^{-13}\text{ cm},$$
$$r_{\text{eff}}=2.65\pm0.07\times10^{-13}\text{ cm}. \quad (4.2)$$

Since there are three parameters, they are not completely determined by the given values of a and r_{eff}. This arbitrariness should be disposed of so that the best possible agreement is attained for high-energy scattering. It is clear that γ must be fairly large; for when $\gamma=0$, i.e., when the tensor force term is absent, the scattering cross section vanishes at 90° in sharp contradiction with experiment.

The following set was found to lead to values of a and r_{eff} quite close to the experimental values (4.2),

$$\lambda=0.35; \quad \gamma=1.4; \quad r_0=2.0\times10^{-13}\text{ cm}. \quad (4.3)$$

The parametric range r_0 turns out to be much larger than that obtained for an "ordinary" Yukawa potential (i.e., the potential in the absence of linking to the $F_{2,2,e}$ component). The reason for this can be understood by study of the effective potential $W(x)$ in (4.1); if we expand the radical, considering the last term small, we obtain

$$W(x)\cong-3\lambda\frac{e^{-x}}{x}-\frac{(12\lambda\gamma)^2}{\epsilon+6x^{-2}}\left(\frac{e^{-x}}{x}\right)^2, \quad x=r/r_0. \quad (4.4)$$

In the absence of the second term in Eq. (4.4), $W(x)$ would be the Yukawa potential, and the value of the parametric range r_0 which would give the correct effective range would be 1.13×10^{-13} cm.[17] Since the second term in Eq. (4.4) is a steeper function of x than the first term,[18] its presence results in shortening the effective range r_{eff} for a given parametric range r_0; therefore, in order to get the correct value of r_{eff}, r_0 must be taken larger than 1.13×10^{-13} cm.

5. RESULTS

Both singlet and triplet phase shifts were calculated up to $J=2$. The singlet $J=0$ phase shift was calculated by the uncoupling method; all the others were calculated in the second Born approximation. At a c.m. energy of 150 Mev, and with the parameters given in (4.3), the results were

$$\text{(Singlet)}\quad \tan\delta_0=0.50_0, \quad \tan\delta_2=0.14_7;$$
$$\text{(Triplet)}\quad \tan\delta_0=0.34_4, \quad \tan\delta_1=-0.17_5,$$
$$\tan\delta_2{}^{\alpha}=0.009_4, \quad \tan\delta_2{}^{\gamma}=0.062_6,$$
$$\eta_2{}^{\alpha}=0.98_5;$$

[17] J. D. Jackson and J. M. Blatt, Revs. Modern Phys. **22**, 77 (1950).

[18] More precisely, the magnitude of its logarithmic derivative is larger everywhere except for very small x, a region not important for the low-energy scattering.

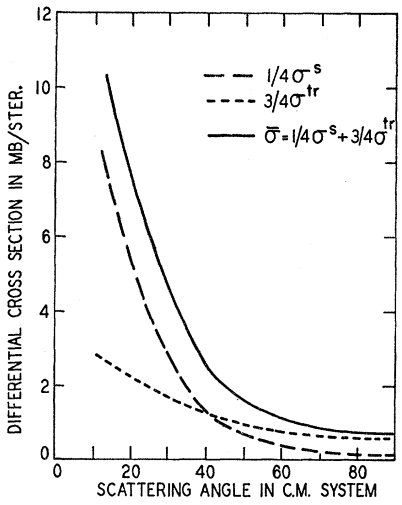

FIG. 1. Differential p-p cross sections at 300 Mev obtained with the isobar model.

here the subscripts denote the total angular momentum J.

In the calculation of the cross sections, the effect of the higher phase shifts was taken into account by means of the first Born approximation.[19] The resulting differential cross sections are presented in Fig. 1.

6. DISCUSSION OF RESULTS

The high-energy cross section obtained here (Fig. 1) is in sharp conflict with the observed cross section. The theoretical cross section is strongly peaked in the forward and backward directions, and at 90° it is too small by a factor of about 8. Let us examine the results in detail.

First it should be noted that coupling to excited states always increases the phase shifts. (In the uncoupling approximation, the excited states make a negative contribution to the effective potential, and this results in a positive contribution to the phase shifts.)

The singlet cross section almost vanishes at 90° because of S-D wave interference. The S- and D-phase shifts are, in fact, quite close to those obtained with an ordinary Yukawa potential adjusted to produce the observed low-energy scattering. The S-phase shift is determined by the effective potential given in Eq. (4.4). Although the second term in Eq. (4.4), representing coupling to the excited state, is important, the effective potential $(\hbar^2/Mr_0^2)W(r/r_0)$ is roughly the same function of r as an ordinary Yukawa potential, if both potentials are adjusted to the same low-energy scattering data; therefore it is not surprising that both potentials lead to approximately the same high-energy S-phase shift. In the approximation which we have used, the D-phase shift is given by Eq. (3C.7). All four terms in Eq. (3C.7) are positive, the last three terms because they represent coupling to excited states, the first term because $(J, g|\Lambda|J, g)=-3\lambda$ is negative and the integral $I_{2,2}$ is necessarily positive. The D-phase shift resulting from the sum of the four terms in Eq. (3C.7) turns out to be slightly larger than that obtained from an ordinary potential adjusted to the low energy data.

The triplet cross section at 90° (about 0.6 mb/sterad) is larger than the singlet cross section but still much too small to lead to agreement with the experimental data. The discussion of the triplet case is not so simple as that of the singlet case because of the complexity of the expression for the cross section. First it should be noted that although the phase shifts $\delta_2{}^{\alpha}$, $\delta_2{}^{\gamma}$ (corresponding to $J=2$) are much smaller in magnitude that δ_0 and δ_1 (the phase shifts corresponding to $J=0$ and $J=1$), their effect is not negligible. However, one can say roughly that the smallness of the cross section at 90° is due to the smallness of the phase shifts δ_0 and δ_1. A stronger tensor force $T'U(r)$ would be needed to increase δ_0 and δ_1 in magnitude.

As was pointed out in Sec. 4, the low-energy scattering data are insufficient to determine uniquely the three parameters: parametric range r_0, depth of scalar potential λ, and depth of tensor potential $\lambda\gamma$. The ratio γ of the depths of the tensor and scalar potentials was fixed more or less arbitrarily. It is clear that a change in γ cannot affect the singlet results radically; since S- and D-phase shifts are necessarily always positive, there will always be destructive S-D wave interference at 90°. On the other hand, the triplet cross section at 90° would increase with γ. Estimates indicate, however, that even in the absence of the scalar term (i.e., $\gamma=\infty$, $\lambda=0$, $\lambda\gamma$ finite), the triplet cross section at 90° would still be much too small.

7. CONCLUSIONS

The particular isobar model of nucleons used in this paper does not lead to agreement with the observed p-p scattering.[20] The present investigation also seems to indicate that in general no agreement can be attained so long as highly singular interactions are excluded. This may be seen roughly as follows: For the most important states, ${}^1S_0$, ${}^3P_0$, ${}^3P_1$, the influence of coupling to excited states can be expressed fairly accurately in terms of effective potentials. Thus the problem is approximately reducible to a conventional potential scattering problem. For nonsingular interactions, the effective potentials are also nonsingular; hence one would expect disagreement with the observed p-p scattering. In other words, the presence of excited states does not simulate anything like a hard core or a strongly singular tensor force. In this sense, isobar models offer no advantage over the conventional types of interaction.

[19] See, for instance, reference 5, for details.

[20] A similar disagreement with experiment has been found by J. Iawadare (private communication), who used square-well potentials for $V(r)$, $U(r)$.

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APPENDIX I. MATRIX ELEMENTS

Triplet

$$(g|\Omega|g)=1/9;\quad (g|\Omega|e)=8/9;\quad (e|\Omega|e)=11/18.$$

For given l, l',

$$(g|T'|g)=-(g|T'|e)=-8(e|T'|e).$$

J odd:

$$l=l'=J,\quad (g|T'|g)=2/27;$$

J even:

$$l=l'=J-1,\qquad (g|T'|g)=-\frac{1}{27}\frac{2(J-1)}{(2J+1)};$$

$$l=l'=J+1,\quad (g|T'|g)=-\frac{1}{27}\frac{2(J+2)}{(2J+1)};$$

$$\text{or}\quad \begin{matrix} l=J-1,\ l'=J+1, \\ l=J+1,\ l'=J-1, \end{matrix}\qquad (g|T'|g)=\frac{2}{9}\frac{[J(J+1)]^{1/2}}{(2J+1)}.$$

Singlet-Quintet

$$(g|\Omega|g)=-\tfrac{1}{3},\quad (e|\Omega|e)=-\tfrac{1}{6}.$$

For $J=0$, $(0,0,g|T'|2,2,e)=4/9$,
$(2,2,e|T'|2,2,e)=5/18$;

for $J=2$, $(0,2,g|T'|2,2,e)=-4\sqrt{2}/9\sqrt{7}$,
$(0,2,g|T'|2,0,e)=4/9\sqrt{5}$,
$(0,2,g|T'|2,4,e)=4\sqrt{2}/3\sqrt{35}$.

APPENDIX II. EVALUATION OF BORN INTEGRALS

The integrals $I_{l,l}(k,p)$, $\mathcal{I}_{l,l}(k)$ can be evaluated simply by means of Gegenbauer's addition theorem:[21] Given two vectors $\mathbf{k}$ and $\mathbf{p}$, θ the angle between $\mathbf{k}$ and $\mathbf{p}$, then

$$\frac{\sin|\mathbf{k}-\mathbf{p}|r}{|\mathbf{k}-\mathbf{p}|r}=\frac{1}{kpr^2}\sum_l(2l+1)g_l(kr)g_l(pr)P_l(\cos\theta). \quad \text{(II.1)}$$

Multiplying by $P_{l'}$ and integrating over $\cos\theta$,

$$g_l(kr)g_l(pr)=\tfrac{1}{2}kp\int_{-1}^{1}dxP_l(x)r(k^2+p^2-2kpx)^{-1/2}\times\sin r(k^2+p^2-2kpx)^{1/2}. \quad \text{(II.2)}$$

$$I_{l,l}(k,p)=\int_0^\infty dr(e^{-r}/r)g_l(kr)g_l(pr)$$
$$=\tfrac{1}{2}kp\int_{-1}^{1}dxP_l(x)(k^2+p^2-2kpx)^{-1/2}\times\int_0^\infty dre^{-r}\sin r(k^2+p^2-2kpx)^{1/2}$$
$$=\tfrac{1}{4}\int_{-1}^{1}dxP_l(x)\{[(1+k^2+p^2)/2kp]-x\}^-$$
$$=\tfrac{1}{2}Q_l[(1+k^2+p^2)/2kp]; \quad \text{(II.3)}$$

here Q_l is the lth order Legendre function of the second kind.[22]

$$\mathcal{I}_{l,l}(k)=\int_0^\infty dr(e^{-r}/r)^2[g_l(kr)]^2$$
$$=\tfrac{1}{2}k^2\int_{-1}^{1}dxP_l(x)[2k^2(1-x)]^{-1/2}\times\int_0^\infty dr(e^{-2r}/r)\sin r[2k^2(1-x)]^{1/2}$$
$$=\tfrac{1}{4}k^2\int_{-1}^{1}dxP_l(x)[\tfrac{1}{2}k^2(1-x)]^{-1/2}\times\tan^{-1}[\tfrac{1}{2}k^2(1-x)]^{1/2},$$

or, by change of variable,

$$\mathcal{I}_{l,l}(k)=\int_0^k dyP_l(1-2k^{-2}y^2)\tan^{-1}y. \quad \text{(II.4)}$$

If $P_l(1-2k^{-2}y^2)$ is expressed as a polynomial in y, each term in Eq. (II.4) can be integrated in an elementary manner.

For $l\neq l'$, $I_{l,l'}(k,p)$ can be evaluated in the following way: if $l>l'$, express $g_{l'}(px)$ in terms of sines and cosines, i.e.,

$$g_0(px)=\sin(px),\quad g_1(px)=(1/px)\sin(px)-\cos(px),\quad \text{etc.}$$

Then $I_{l,l'}(k,p)$ reduces to a sum of integrals of the form

$$\int_0^\infty dx(e^{-x}/x^{m+1})g_l(kx)\sin(px),$$
$$\int_0^\infty dx(e^{-x}/x^{m+1})g_l(kx)\cos(px),\ 0\leq m\leq l'. \quad \text{(II.5)}$$

Now consider the identity[23]

$$\int_0^\infty dx(e^{-ax}/x)g_l(x)=(i)^{l+1}Q_l(ia),\ \text{for R.P. } a>0. \quad \text{(II.6)}$$

[21] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Macmillan Company, New York, 1941), second edition, p. 366.

[22] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Cambridge, 1927), fourth edition, p. 320.

[23] Reference 22, p. 317, and reference 21, p. 385.

Successive integration of (II.6) yields

$$\int_0^\infty dx(e^{-ax}/x^{m+1})g_l(x) = (i)^{l+1}\int_a^\infty da^{(m)}\cdots\int_{a^{(3)}}^\infty da''\int_{a''}^\infty da' Q_l(ia'), \quad m\leq l. \quad \text{(II.7)}$$

If the explicit expressions for $Q_l(ia')$ are used, each integral on the right side of (II.7) is elementary. Now note that the integrals (II.5) can be expressed in terms of the left side of (II.7),

$$\int_1^\infty dx(e^{-x}/x^{m+1})g_l(kx)\begin{Bmatrix}\sin px\\ \cos px\end{Bmatrix} = \begin{Bmatrix}\text{I.P.}\\ \text{R.P.}\end{Bmatrix} k^m\int_0^\infty dx(e^{-ax}/x^{m+1})g_l(x), \quad \text{(II.8)}$$

where $a=(1-ip)/k$.

Several special cases which were required in this paper are listed below:

$$I_{2,0}(k,p)=\frac{3}{2k}p-\frac{3}{2k^2}p[\tan^{-1}(p+k)-\tan^{-1}(p-k)]+\frac{1}{8}\left(1+\frac{3}{k^2}-\frac{3}{k^2}p^2\right)\ln\left[\frac{1+(p+k)^2}{1+(p-k)^2}\right],$$

$$I_{3,1}(k,p)=\frac{1}{8k^2}[-(5+k^2)+15p^2]-\frac{5}{2k^3}p^2[\tan^{-1}(p+k)-\tan^{-1}(p-k)]+\frac{1}{32pk^3}[(5+6k^2+k^4)+(30+6k^2)p^2-15p^4]\times\ln\left[\frac{1+(p+k)^2}{1+(p-k)^2}\right];$$

$$\mathcal{J}_{1,1}=\frac{k}{3}\tan^{-1}k+\frac{1}{3}-\left(\frac{1}{2}+\frac{1}{3k^2}\right)\ln(1+k^2),$$

$$\mathcal{J}_{2,2}=\frac{k}{5}\tan^{-1}k+\frac{7}{10}+\frac{3}{5k^2}-\left(\frac{1}{2}+\frac{1}{k^2}+\frac{3}{5k^4}\right)\ln(1+k^2),$$

$$\mathcal{J}_{3,3}=\frac{k}{7}\tan^{-1}k+\frac{41}{42}+\frac{16}{7k^2}+\frac{10}{7k^4}-\left(\frac{1}{2}+\frac{2}{k^2}+\frac{3}{k^4}+\frac{10}{7k^6}\right)\ln(1+k_2),$$

$$\mathcal{J}_{4,4}=\frac{k}{9}\tan^{-1}k+\frac{43}{36}+\frac{143}{27k^2}+\frac{145}{18k^4}+\frac{35}{9k^6}-\left(\frac{1}{2}+\frac{10}{3k^2}+\frac{9}{k^4}+\frac{10}{k^6}+\frac{35}{9k^8}\right)\ln(1+k^2).$$