The Equations of Motion in Einstein's New Unified Field Theory

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It is shown that the field equations of Einstein's latest unified field theory do not lead to the Lorentz equations of motion for charged particles in an electromagnetic field, if these particles are considered to be singularities of the field. To a fourth-order approximation, the motion of such particles is not influenced by the electromagnetic field, no matter how much charge is placed on the particles.

INTRODUCTION

IT is well known that the field equations of general relativity (plus the Bianchi identities) lead to the correct equations of motion for a particle, represented as a point singularity. This point has been extensively discussed in the literature.[1–4] The situation is different in electromagnetism where additional hypotheses, such as the stress-energy tensor, have to be introduced before equations of motion can be obtained.[5] Here we examine the case of unified field theory, in which the use of the stress-energy tensor is abandoned, to see if the correct equations of motion for matter can be obtained. This paper is concerned only with Einstein's latest unified field theory.[6] A remark of Infeld[7] that one does not obtain the Lorentz equations of motion for a particle in an electromagnetic field in Einstein's previous unified field theory[8] is extended to apply in the latest theory. One still obtains the correct equations for the motion of an uncharged particle in a gravitational field.

We assume that particles are to be represented as singularities in the appropriate fields. We ask if the field equations place any restrictions on the motion of singularities, intending thereby to obtain the Lorentz equations of motion for the case of charged particles in an electromagnetic field. Einstein maintains,[9] however, that one must represent matter by nonsingular solutions of the field equations, and also that the fundamental nature of the Lorentz equations is open to question. He believes that in a rigorous solution of the field equations particles may interact with each other in such a way that the Lorentz equations will only represent a statistical, averaged effect. Since such solutions have not yet been obtained, we cannot comment on the possibility of obtaining equations of motion in the nonsingular case.

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METHOD

The calculation is somewhat involved and will only be sketched here. An adequate discussion of details can be obtained from references 3 and 7. In particular, this calculation follows that of reference 7.

The method rests on the solution of the field equations in a quasi-static approximation. This enables us to distinguish, by means of a parameter λ, quantities involving time derivatives as of smaller size. The essential steps in the calculation are as follows:

(1) We determine the relation between the fundamental tensor g_{ik} and the affinity $\Gamma^l{}_{ik}$ by an expansion in powers of λ. We are so far unable to determine this relation exactly, as is possible in general relativity.

(2) We identify the gravitational and electromagnetic fields by requiring that the field equations reduce to those of general relativity, and to Maxwell's equations in a suitable approximation.

(3) We calculate the field quantities in a quasi-static approximation.

(4) We ask whether any terms are added to the equations of motion of general relativity by the fact that the fundamental tensor g_{ik} is no longer symmetric. The equations of motion turn out to be given by the surface integral of certain quantities Λ_{ik}. Some elements of these Λ_{ik} appear in general relativity, and these we subtract out, since we are only interested in the additional terms. We then consider the surface integral of the remaining Λ_{ik} to see whether they contribute to the equations of motion.

One conclusion can be drawn very simply. If we make the assumption, which will later be justified, that the antisymmetric part of the fundamental tensor represents the electromagnetic field, we see that the correct equations of motion for charged matter in an electromagnetic field cannot be the geodesic equations, as is the case in general relativity. For the geodesics, the lines of shortest distance are determined from

$$\delta\int ds=0;$$

but

$$ds^2=g_{ik}dx^idx^k,$$

and any antisymmetric part of the g_{ik} will cancel in the summation. Thus the geodesics are not affected by the electromagnetic field.

We begin with the field equations. The fundamental tensor g_{ik} has a symmetric part $g_{\underline{ik}}$ and an antisymmetric part $g_{\underset{\smile}{ik}}$. Likewise the affinity $\Gamma^l{}_{ik}$ can be decomposed into symmetric and antisymmetric parts $\Gamma^l{}_{\underline{ik}}$ and $\Gamma^l{}_{\underset{\smile}{ik}}$; the latter is a tensor. The tensor R_{ik} is defined as in general relativity:

$$R_{ik}\equiv(\Gamma^t{}_{ik,\,t}-\Gamma^t{}_{sk}\Gamma^s{}_{it})-(\Gamma^t{}_{it,\,k}-\Gamma^t{}_{st}\Gamma^s{}_{ik}),$$

and it can also be decomposed into symmetric and antisymmetric parts $R_{\underline{ik}}$ and $R_{\underset{\smile}{ik}}$. The comma indicates ordinary differentiation. One defines

$$\Gamma^i{}_{\underset{\smile}{ki}}\equiv\Gamma_k.$$

This quantity is a vector. Since the affinity is nonsymmetric, there are several possibilities for covariant differentiation. For a covariant vector A_i these can be denoted as follows:

$$A_{\underset{+}{i};k}\equiv A_{i,\,k}-A_s\Gamma^s{}_{ik},\qquad A_{i;k}\equiv A_{i;k}-\begin{Bmatrix}s\\ik\end{Bmatrix}A_s.$$
$$A_{\underset{-}{i};k}\equiv A_{i,\,k}-A_s\Gamma^s{}_{ki},$$

The quantities $\begin{Bmatrix}s\\ik\end{Bmatrix}$ are the well-known Christoffel symbols.

With these preliminaries, we can state the field equations:

$$g_{\underset{+}{i}\,\underset{-}{k};\,l}=0,\quad \Gamma_i=0,$$
$$R_{\underline{ik}}=0,\quad R_{\underset{\smile}{ik},\,l}+R_{\underset{\smile}{kl},\,i}+R_{\underset{\smile}{li},\,k}=0. \tag{1}$$

The equation

$$g_{\underset{+}{i}\,\underset{-}{k};\,l}=\partial g_{ik}/\partial x^l-g_{sk}\Gamma^s{}_{il}-g_{is}\Gamma^s{}_{lk}=0 \tag{2}$$

relates the affinity and the metric. In general relativity, the $\Gamma^l{}_{ik}$ are replaced by the $\begin{Bmatrix}l\\ik\end{Bmatrix}$, and the symmetry of the Christoffel symbols makes solution simple. Here we have no such pleasant properties at our disposal, and we have to solve for the affinity by successive approximations. Following Infeld, we introduce the quantities $M^l{}_{ik}$ defined by

$$M^l{}_{ik}\equiv\Gamma^l{}_{ik}-\begin{Bmatrix}l\\ik\end{Bmatrix}$$

($M^l{}_{ik}$ is a tensor), the quantities $g^{\underline{ls}}$ defined by

$$g_{\underline{lk}}g^{\underline{ls}}=\delta_k{}^s,$$

and the tensor I_{ikl}:

$$I_{ikl}=\tfrac{1}{2}(g_{\underset{\smile}{ik},\,l}+g_{\underset{\smile}{kl},\,i}+g_{\underset{\smile}{li},\,k}).$$

Then we can show that[7] that

$$M_{ik}{}^l=g^{\underline{lt}}\{g_{\underset{\smile}{ik};\,t}-I_{ikt}-g_{\underset{\smile}{is}}M^s{}_{kt}-g_{\underset{\smile}{sk}}M^s{}_{ti}\}, \tag{3}$$

where

$$g_{\underset{\smile}{ik};\,t}\equiv g_{ik,\,t}-g_{\underset{\smile}{lk}}\begin{Bmatrix}l\\it\end{Bmatrix}-g_{\underset{\smile}{il}}\begin{Bmatrix}l\\kt\end{Bmatrix}.$$

If we put

$$g_{\underline{ik}}\equiv\eta_{\underline{ik}}+h_{\underline{ik}},$$

where η_{ik} represent the Minkowski metric with signature $+---$; we can assume that the $h_{\underline{ik}}$, $g_{\underset{\smile}{ik}}$, $M^l{}_{ik}$, and $\begin{Bmatrix}l\\ik\end{Bmatrix}$ are of the same order. Thus in the first approximation those terms on the right of (3) which contain $M^l{}_{ik}$ and $\begin{Bmatrix}l\\ik\end{Bmatrix}$ can be neglected. The $M^l{}_{ik}$ as given by the first approximation can be substituted back into (3), obtaining a second approximation, etc.

We can now discuss the identification of the symmetric and antisymmetric components of the fundamental tensor g_{ik}. Imagine a situation in which there is only a gravitational field in otherwise empty space. Then the gravitational field equations must apply, and the simplest way we can reduce our Eqs. (1) is to identify $g_{\underline{ik}}$ with the g_{ik} of general relativity. Now consider a weak electromagnetic field in otherwise free space. In this situation, the field equations (1) must reduce to Maxwell's equations.

In order to see how this occurs, we introduce the lowest approximation of (3) into the definition of R_{ik}, and find that $R_{\underset{\smile}{ik}}$ has only one nonvanishing term, $\Gamma^t{}_{\underset{\smile}{ik},\,t}$. Then we compute

$$R_{\underset{\smile}{ik},\,l}+R_{\underset{\smile}{kl},\,i}+R_{\underset{\smile}{li},\,k}=0,$$

and obtain

$$g^{\underline{st}}(g_{\underset{\smile}{ik},\,l}+g_{\underset{\smile}{kl},\,i}+g_{\underset{\smile}{li},\,k}),_{\,st}=0. \tag{4}$$

It is suggested that this be considered as the D'Alembertian of one of the two Maxwell equations. The other can be obtained since the equation

$$g^{\underset{\smile}{is}},_{\,s}=0 \tag{5}$$

holds in this approximation. This is true since if $\mathfrak{g}^{ik}$ is the tensor density associated with g^{ik}, the equation

$$\mathfrak{g}^{\underset{\smile}{ik}},_{\,k}=0 \tag{6}$$

is rigorous. We want the one of Maxwell's equations which is an identity in the fields to be true in general, so we are led to identify the $\mathfrak{g}^{\underset{\smile}{ik}}$ with the dual of the electromagnetic field tensor,

$$\mathfrak{g}^{\underset{\smile}{ik}}=\epsilon^{iklm}F_{lm}, \tag{7}$$

where $\epsilon^{0123}=-1$, and F_{lm} is the ordinary electromagnetic field tensor. In our approximation, this identification gives

$$g_{\underset{\smile}{12}}=kE_3,\quad g_{\underset{\smile}{13}}=-kE_2,\quad g_{\underset{\smile}{10}}=-kH_1,$$
$$g_{\underset{\smile}{23}}=kE_1,\quad g_{\underset{\smile}{20}}=-kH_2,\quad g_{\underset{\smile}{30}}=-kH_3, \tag{8}$$

where k is a constant depending on the system of units.

The quasi-static approximation in powers of λ is introduced and discussed extensively in the literature (see particularly reference 3). Through it we consider a situation in which time derivatives are an order smaller than space derivatives. The field equations are expanded as power series in λ in the following way. We distinguish time components by the index 0.

$$
\begin{aligned}
&g_{00}=1+\lambda^2\,{}_2g_{00}+\lambda^4\,{}_4g_{00}+\cdots,\\
&g_{\underset{\smile}{0n}}=\lambda^3\,{}_3g_{\underset{\smile}{0n}}+\lambda^5\,{}_5g_{\underset{\smile}{0n}}+\cdots,\\
&g_{\underline{0n}}=\lambda^3\,{}_3g_{\underline{0n}}+\lambda^5\,{}_5g_{\underline{0n}}+\cdots,\\
&g_{\underline{mn}}=-\delta_{mn}+\lambda^2\,{}_2g_{\underline{mn}}+\lambda^4\,{}_4g_{\underline{mn}}+\cdots,\\
&g_{\underset{\smile}{mn}}=\lambda^2\,{}_2g_{\underset{\smile}{mn}}+\lambda^4\,{}_4g_{\underset{\smile}{mn}}+\cdots.
\end{aligned}
\tag{9}
$$

The numerical subscript on the left indicates that this quantity is the coefficient of the particular power of λ; for instance, ${}_2g_{mn}$ multiplies λ^2. Reference to Eqs. (8) will show that it is appropriate to apply (9) to a system of slowly moving charged particles, where the principal interaction is electrostatic. The quantities $M^l{}_{ik}$ can be calculated using (8) and (9). We obtain

$$
\begin{aligned}
&{}_3M^0{}_{00}=0,\quad {}_2M^s{}_{00}={}_2M^0{}_{s0}={}_2M^0{}_{0s}=0,\\
&{}_3M^l{}_{0k}=-\,{}_3g_{\underset{\smile}{0k},\,l}+{}_3I_{0kl},\quad {}_3M^0{}_{lk}=+\,{}_3g_{\underset{\smile}{lk},\,0}-{}_3I_{lk0},\\
&{}_2M^l{}_{ik}=-\,{}_2g_{\underset{\smile}{ik},\,l}+{}_2I_{ikl},\\
&{}_4M^0{}_{0s}={}_4M^0{}_{s0}={}_4M^s{}_{00}=0,\\
&{}_4M^l{}_{\underline{ik}}={}_4M^l{}_{\underline{ki}}=-\,{}_2g_{\underset{\smile}{is}}(-\,{}_2g_{\underset{\smile}{kl},\,s}+{}_2I_{kls})\\
&\qquad\qquad -\,{}_2g_{\underset{\smile}{sk}}(-\,{}_2g_{\underset{\smile}{li},\,s}+{}_2I_{lis}),\\
&{}_4M^l{}_{\underset{\smile}{ik}}=-\,{}_4M^l{}_{\underset{\smile}{ki}}=-\,{}_4g_{\underset{\smile}{ik},\,l}+{}_4I_{ikl}+{}_2g^{\underline{nl}}({}_2g_{\underset{\smile}{ik},\,n}-{}_2I_{ikn})\\
&\qquad\qquad +\,{}_2g_{\underset{\smile}{is}}\,{}_2\begin{Bmatrix} s\\ lk\end{Bmatrix}+{}_2g_{\underset{\smile}{sk}}\begin{Bmatrix} s\\ il\end{Bmatrix}.
\end{aligned}
\tag{10}
$$

This analysis determines the $M^l{}_{ik}$ up to fourth order. In the above, summations run from 1 to 3 only, time being distinguished by the index 0. This convention prevails through the remainder of the paper.

We employ (9) and (10) to calculate R_{ik}. Through the second and third order, which is all we need here, the field equations split into equations of the gravitational and the electromagnetic field separately [as in (4)]. The following field equations are then obtained to second and third order:

$$
\begin{aligned}
&\nabla^2({}_2g_{\underset{\smile}{ik},\,l}+{}_2g_{\underset{\smile}{kl},\,i}+{}_2g_{\underset{\smile}{li},\,k})=0,\\
&\nabla^2({}_3g_{\underset{\smile}{0k},\,l}+{}_3g_{\underset{\smile}{kl},\,0}+{}_3g_{\underset{\smile}{l0},\,k})=0,
\end{aligned}
\tag{11}
$$

and from the equation $\Gamma_i=0$,

$$
\begin{aligned}
&\operatorname{div}\,{}_2g_{\underset{\smile}{ik}}={}_2g_{\underset{\smile}{ik},\,k}=0,\\
&\operatorname{div}\,{}_3g_{\underset{\smile}{0k}}={}_3g_{0k,\,k}=0.
\end{aligned}
\tag{12}
$$

Since our object is to determine the equation of motion of particles, we look for solutions of (11) and (12) which will reduce, at distances large compared to the gravitational radius of the particle, yet small compared to the wavelength of any radiation being emitted, to the ordinary electric and magnetic fields of slowly moving charges. Because the motion is assumed to be slow, however, radiative effects are assumed to be of higher order. It is true that in the immediate neighborhood of these singularities, the weak-field approximation employed above will not be valid, and it will not be possible to separate the gravitational and electromagnetic fields in such a simple fashion. But our technique in finding the equations of motion is, following Infeld, to surround each charge by a large closed surface. The distance of the surface from the particle is taken to lie within the above-mentioned limits, but is otherwise arbitrary.

Specifically, we consider the particles to be electric charges with a $1/r^2$ singularity in the electromagnetic field. The nature of the gravitational singularity is immaterial. For generality we consider n particles, and to the order of approximation in question we introduce the electric potential of these n particles:

$$
{}_2\varphi=\sum_{i=1}^{N}{}_2\varphi(i).
$$

We have, according to electrostatics,

$$
{}_2\varphi(k)=e(k)/r(k),
$$

where $r(k)$ is the distance (a magnitude, not a vector) from the charge to the field point, and e is a constant proportional to the charge. Thus, in accord with (8),

$$
{}_2g_{\underset{\smile}{mn}}=\epsilon_{mns}\,{}_2\varphi,_{s},
\tag{13}
$$

where ϵ_{mns} is a completely antisymmetric three-dimensional Cartesian tensor. This choice satisfies the field equations to the second order.

To satisfy the field equations in the third order, we let the motion of the kth singularity be given by three functions of time $\xi^m(k,t)$, velocities of order λ, $\dot{\xi}^m(k,t)$ and accelerations of order λ^2, $(\partial^2/\partial t^2)\xi(k,t)$.

Then we set

$$
g_{\underset{\smile}{0k}}=-\,\epsilon_{0kij}\Big(\sum_{l=2}^{N}{}_2\varphi(l)\dot{\xi}_i(l,t)\Big)_{,j},
\tag{14}
$$

where ϵ_{0kij} are the components of a completely antisymmetric Cartesian tensor in four dimensions. Equation (14) satisfies the field equations in the third order.

As stated previously, we wish to compare the equations of motion obtained in unified field theory with those from general relativity. In general relativity, the equations of motion are determined from

$$R_{ik}=0,$$

where R_{ik} is a symmetric tensor. In unified field theory, we must consider the two equations

$$R_{\underline{ik}}=0, \quad R_{\underset{\smile}{ik},l}+R_{\underset{\smile}{kl},i}+R_{\underset{\smile}{li},k}=0.$$

The second of these equations states that $R_{\underset{\smile}{ik}}$ can be written as the curl of a vector: there exists a vector B_i such that

$$R_{\underset{\smile}{ik}}=B_{k,i}-B_{i,k}.$$

If we form a surface integral containing (linear combinations of) R_{ik}, this integral can be transformed by Stokes' theorem into the line integral of the vector B_i around a closed path bounding the surface. But for a closed surface, this path is of zero length, and the integral must vanish. Thus the quantities $R_{\underset{\smile}{ik}}$ can play no role in determining the equations of motion. The equations of motion must be obtained from the equation

$$R_{\underline{ik}}=0.$$

It is shown in reference 3 that the equations of motion are to be found from the surface integral of certain quantities Λ_{ik} simply related to the tensor $R_{\underline{ik}}$. The determination of Λ_{ik} will be discussed subsequently. If

$$\Lambda_{ik,k}=\mathrm{div}\Lambda_{ik}=0, \tag{15}$$

then a two-dimensional surface integral over a closed surface will not depend on the shape of the surface. If we choose surfaces enclosing individual singularities at the proper distances so that (13) and (14) are good approximations, the surface integral will be independent of the surface. But then it can be a function only of the coordinates of the singularities and their time derivatives, thus giving a contribution to the equations of motion.

It can be shown[3,7] that in order to obtain additional terms in the fourth-order equations of motion, we must evaluate the surface integral of the following:

$$-{}_4\Lambda_{ik}=({}_4R_{\underline{mn}}-{}_4P_{mn})+\tfrac{1}{2}\delta_{mn}({}_4R_{00}-{}_4P_{00}) -\tfrac{1}{2}\delta_{mn}({}_4R_{\underline{ss}}-{}_4P_{ss}), \tag{16}$$

in which P_{mn} is the Ricci tensor of general relativity (R_{mn} formed using the Christoffel symbols). In order to evaluate these quantities to the fourth order, it turns out that one needs only the second- and third-order solutions of the field equations as given by (13) and (14). The result of this computation is[7]

$$-{}_4\Lambda_{mn}=\{{}_2g_{\underset{\smile}{ms}}\,{}_2g_{\underset{\smile}{np},s}+\delta_{pm}\,{}_2\varphi_{,rn}\,{}_2\varphi_{,r} -\delta_{mn}\,{}_2\varphi_{,pm}\,{}_2\varphi_{,r}\}_{,p}. \tag{17}$$

The quantity inside the braces is antisymmetric in n and p. Thus,

$${}_4\Lambda_{mn,n}=0. \tag{18}$$

We are now required to form the surface integrals[3,7]

$$\frac{1}{2\pi}\int_4 \Lambda_{mk}n_k ds={}_4C_m(k), \tag{19}$$

in which n_k is the normal vector to the surface surrounding the kth singularity. The additional equation of motion is then

$${}_4C_m(k)=0. \tag{20}$$

We can write $-{}_4\Lambda_{mn}=F_{mnp,p}$, where $F_{mnp}=-F_{mpn}$ and $F_{mnp,pn}=0$. Thus we have to evaluate

$$-\frac{1}{2\pi}\int F_{mkp}n_k ds=0. \tag{21}$$

This expression is essentially the surface integral of a curl (reference 3, p. 213) and therefore vanishes. Thus we do not obtain contributions from the electromagnetic field to the equations of motion in the fourth order. We know from general relativity[3] that terms in this order should contain products of charges like $e(1)e(2)$ so that the proper equations of motion cannot be obtained.

CONCLUSIONS

Although we can make the equations of Einstein's new unified field theory reduce to something like Maxwell's equations in a sufficiently low approximation, we cannot obtain the Lorentz equation of motion, if we represent particles with charge as singularities in the field. This result is the same as that obtained by Infeld[7] for Einstein's previous unified field theory.

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Note added in proof.—B. Kursunoglu [Proc. Phys. Soc. (London) **65A**, 81 (1952)] has explained the failure to obtain the correct equations of motion for a charged particle in an electromagnetic field in Einstein's 1950 theory (see reference 8) as due to the vanishing of the stress-energy tensor.