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Yukawa Forces with Added Attractions*

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A "Yukawa potential" is defined here as the conventional second-order potential of pseudoscalar meson theory with pseudoscalar coupling, cut off by a hard repulsive core. We have carried out many calculations of the properties of the neutron-proton system at low energies assuming that the interaction is this Yukawa potential with various additional spin independent central wells. The results indicate that interactions of this type (including the Levy potential) do not give a satisfactory fit to the low-energy data.

I. THE LÉVY POTENTIAL

We have developed a computing program for the University of Illinois Electronic Digital Computer (The ILLIAC) which solves the differential equations for the $^3S+{}^3D$ ground state of the deuteron, and also computes quantities characteristic of the $^3S+{}^3D$ scattering state and of the ^1S scattering state. We have applied this code to a study of nuclear potentials, among them the potential suggested by Lévy.[1]

We write Lévy's potential in the form

$$(\mu c\hbar)^{-1}V(r)=+\infty \quad \text{for} \quad r<r_{\text{core}},$$

$$\begin{aligned}(\mu c\hbar)^{-1}V(r)=&-(G^2/4\pi)^2(3/2\pi)(Mr)^{-2}K_1(2\mu r)\\&-(g^2/4\pi)(\mu/2M)^2(\mu r)^{-1}\exp(-\mu r)\\&\times\{1+S_{12}[1+(3/\mu r)+3/(\mu r)^2]\}\\&\qquad\text{for}\quad r>r_{\text{core}}.\end{aligned}\qquad(1)$$

This is essentially the form proposed by Lévy; we have introduced two coupling constants, G and g, the first for the fourth-order force, the second for the usual second-order force. Lévy imposed the condition $g=G$, but other work[2] has shown that this condition is not necessarily a consequence of the pseudoscalar meson theory of nuclear forces with pseudoscalar coupling (which underlies Lévy's treatment). We have also dropped Lévy's $V_4^{(b)}$ from the fourth-order force [the first term in (1)], following arguments presented by Drell[3] and Klein.[4]

Table I gives a list of the runs we have made so far. In each case the binding energy of the deuteron has been fitted (by adjusting g^2). Table II gives interpolated cases designed to fit the singlet state scattering length. Table III gives interpolated cases designed to fit the quadrupole moment of the deuteron.

In interpreting such results, it is necessary to know how sensitively various experimental numbers depend upon the parameters of the theory. Assuming always that we adjust things so as to fit the binding energy of the deuteron, the most sensitive experimental number is the singlet scattering length. This is because of the well-known "resonance" close to zero energy. A small change in the potential produces large changes in a_s. The next experimental number is the quadrupole moment. This is primarily determined by, and therefore fixes, the "outside" behavior of the tensor potential. This can be seen from Table III, for example, by observing that the coupling constant $g^2/4\pi$ is almost the same for all core radii. $g^2/4\pi$ determines the outside behavior of the tensor force, and the fact that $g^2/4\pi$ is nearly independent of the core radius (for cases which fit the quadrupole moment) indicates that the quadrupole moment does not depend strongly upon the close-in behavior of the tensor force. Finally, the least sensitive experimental number is the triplet effective range. It takes large changes in the potential to produce appreciable changes in the effective range. Qualitatively speaking, we may say that the following discrepancies should be considered serious: a factor of 3 in the singlet scattering length, 20 percent in the quadrupole moment, 10 percent in the triplet effective range. The singlet effective range changes rather rapidly as a function of the singlet scattering length. But if the singlet scattering length has been fitted (Table II), a 20 percent discrepancy in the singlet effective range is also serious.

Lévy's original program consisted in setting $g=G$ and adjusting r_{core} and g so that the potential fits the binding energy of the deuteron and the singlet scattering length. Table II shows that this is impossible if the fourth-order potential is taken as $V_4^{(a)}$ rather than as $V_4^{(a)}+V_4^{(b)}$. The values of $G^2/4\pi$ are systematically larger than the values of $g^2/4\pi$, for all core radii investigated by us.

Since the condition $G=g$ may well be too stringent, it is worth while to try to see how good a fit can be obtained by leaving G and g as two free parameters. Tables II and III show that the agreement is poor. In order to fit the binding energy of the deuteron, the quadrupole moment, and the singlet scattering length, the core radius must be chosen to be $r_{\text{core}}=0.189R$ $=0.814\times10^{-13}$ cm. The triplet effective range is then $r_{0t}=0.484R=2.08\times10^{-13}$ cm. The singlet effective range is $r_{0s}=0.792R=3.41\times10^{-13}$ cm. The experimental values

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[1] M. M. Lévy, Phys. Rev. **88**, 72, 725 (1952).
[2] G. Wentzel, Helv. Phys. Acta **15**, 111 (1942); **25**, 569 (1952); Phys. Rev. **86**, 802 (1952).
[3] S. Drell (private communication).
[4] A. Klein, Phys. Rev. **89**, 1158 (1953); **90**, 1101 (1953).

TABLE I. This table contains the results of all runs with the Lévy potential (1). All results fit the binding energy of the deuteron. The core radius, as well as all other lengths, is measured in units of the radius of the deuteron (4.3×10^{-13} cm). s is the nondimensional well depth of the fourth-order potential. $s=1$ corresponds to a fourth-order potential which, by itself, gives rise to a scattering resonance at exactly zero energy. s is proportional to $(G^2/4\pi)^2$ for any one core radius, but the proportionality constant depends upon the core radius. $G^2/4\pi$ and $g^2/4\pi$ are defined by (1); p_D is the D-state probability, Q is the quadrupole moment, ρ_t is the triplet effective range (evaluated from the ground-state wave functions); ρ_s is the singlet effective range, and a_s is the singlet scattering length. The experimental values, in our units, are: $Q=0.0147\pm0.0002$, $\rho_t=0.395\pm0.015$, $a_s=-5.488\pm0.02$, $\rho_s=0.63\pm0.1$. The uncertainties stated here are somewhat larger than the ones claimed in the various experimental papers. Even though the machine finds the wave functions in every run, we did not ask it to print out all that information for every run. The runs for which wave functions are available are marked with an asterisk.

r_{core}	s	$G^2/4\pi$	$g^2/4\pi$	p_D	Q	ρ_t	ρ_s	a_s
0.123	0.3	6.532	13.382	0.06635	0.01484	0.4060	9.47	−0.1015
	*0.319167	6.738	13.175	0.06546	0.01470	0.4052	8.16	−0.1109
	0.6	9.238	10.020	0.05040	0.01238	0.3954	1.643	−0.3629
	0.9	11.314	6.174	0.02826	0.008863	0.3881	0.6137	−3.192
	*0.992	11.452	5.853	0.02628	0.008519	0.3877	0.5836	−5.132
	1.0	11.926	4.634	0.01879	0.007121	0.3865	0.4985	5.457
	1.170	12.899	0	0	0	...	...	...
0.140	0.3	7.570	15.894	0.07217	0.01683	0.4346	10.14	−0.1065
	*0.544	10.193	12.688	0.05907	0.01470	0.4253	2.237	−0.3075
	0.6	10.705	11.917	0.05555	0.01414	0.4234	1.737	−0.3923
	0.9	13.111	7.392	0.03203	0.01029	0.4153	0.6609	−3.760
	*0.917	13.234	7.104	0.03040	0.009998	0.4150	0.6362	−5.530
	1.0	13.820	5.600	0.02183	0.008377	0.4137	0.5396	5.183
	1.179	15.008	0	0	0	...	...	...
0.185	0.3	10.753	23.598	0.08614	0.02218	0.5000	12.27	−0.1162
	0.6	15.207	17.739	0.06822	0.01895	0.4877	1.984	−0.4635
	*0.880	18.417	11.630	0.04392	0.01467	0.4792	0.8129	−3.790
	*0.898	18.604	11.198	0.04199	0.01431	0.4788	0.7807	−5.434
	0.9	18.625	11.150	0.04177	0.01427	0.4787	0.7773	−5.702
	1.0	19.632	8.598	0.02989	0.01194	0.4769	0.6406	4.726
	1.201	21.514	0	0	0	...	...	...
0.189	*0.89698	19.142	11.601	0.04295	0.01470	0.4839	0.7924	−5.493
0.240	*0.878	26.848	17.752	0.05585	0.02008	0.5438	0.9516	−5.347
	0.9	27.182	16.980	0.05320	0.01956	0.5433	0.9073	−9.676
	1.0	28.652	13.275	0.03971	0.01676	0.5415	0.7524	4.420
	*1.057	29.458	10.950	0.03078	0.01471	0.5409	0.6893	2.610
	1.1	30.051	9.022	0.02331	0.01278	0.5406	0.6504	2.045

of the latter two are $(1.70\pm0.05)\times10^{-13}$ cm and $(2.6\pm0.4)\times10^{-13}$ cm respectively. (The effective range in proton-proton scattering is $(2.65\pm0.07)\times10^{-13}$ cm.) Conversely, if we try to fit the triplet data only (i.e., fit the binding energy, quadrupole moment, and triplet effective range), the core radius necessary is $r_{\text{core}}=0.114R$ $=0.491\times10^{-13}$ cm, and the singlet state results are utterly unreasonable: $a_s=-0.045R=0.19\times10^{-13}$ cm compared to the experimental value $a_s=-5.49R=$ -23.68×10^{-13} cm.

We conclude that the Lévy potential does not give an acceptable fit to the experimentally observed properties of the neutron-proton system at low energies.[5]

II. YUKAWA FORCES WITH ADDED ATTRACTIONS

Since the Lévy potential does not give an adequate fit to the neutron proton data, we have investigated to what extent the disagreement is a result of the particular form of the Lévy potential. We define a "Yukawa force with added attraction" as follows:

$$V=+\infty \quad \text{for} \quad r<r_{\text{core}};$$
$$V=V_a(r-r_{\text{core}})-(g^2/4\pi)(\mu/2M)^2(\mu c\hbar)(\mu r)^{-1} \times\exp(-\mu r)\{1+S_{12}[1+3/\mu r+3/(\mu r)^2]\} \quad \text{for} \quad r>r_{\text{core}}. \qquad (2)$$

The "added attraction" V_a is a completely arbitrary central force. It is written as a function of the difference $r-r_{\text{core}}$ in order to avoid introducing a (spurious) strong well shape dependence caused by the different ways a hard repulsive core cuts off the central attractive region of various potentials. The only conditions imposed on the potential (2) are:

(a) The singlet state force equals the central force in the triplet state (no $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$ term).

(b) The quantity μ is determined from the experimental mass of the π meson, and is therefore not an adjustable parameter.

We believe that forces of this type represent a reasonable generalization of the Lévy potential.

The following well shapes have been tried for V_a: Gauss well, Yukawa well, and Morse well, in addition to the fourth-order potential of Lévy discussed in the

[5] The results of this section are in qualitative agreement with those of R. Jastrow, Phys. Rev. 91, 749 (1953), whose procedure is similar to our own. He also examines the case $g\neq G$, but allows a different core radius for the singlet potential. Not all his numerical results agree with ours.

TABLE II. Interpolated cases which fit the singlet scattering length and the binding energy of the deuteron. Note that a core radius slightly less than 0.140, which fits ρ_t and ρ_s, gives rise to much too low a quadrupole moment Q; conversely, a core radius slightly in excess of 0.185, which fits Q, gives rise to much too large values of ρ_t and ρ_s.

r_{core}	s	$G^2/4\pi$	$g^2/4\pi$	p_D	Q	ρ_t	ρ_s	a_s
0.123	0.9244	11.478	5.818	0.02606	0.008482	0.3876	0.5804	*−5.488*
0.140	0.9167	13.232	7.108	0.03041	0.01000	0.4150	0.6366	*−5.488*
0.185	0.8984	18.615	11.188	0.04194	0.01431	0.4788	0.7800	*−5.488*
0.240	0.8792	26.867	17.709	0.05571	0.02005	0.5438	0.9490	*−5.488*

TABLE III. Interpolated cases which fit the quadrupole moment. A core radius slightly less than 0.123 fits the triplet data (binding energy, Q, ρ_t) reasonably well, but gives utter nonsense for the singlet state. Notice how nearly constant $g^2/4\pi$ is as a function of the core radius. This indicates that the quadrupole moment is determined primarily by the "exterior" behavior of the tensor force.

r_{core}	s	$G^2/4\pi$	$g^2/4\pi$	p_D	Q	ρ_t	ρ_s	a_s
0.123	0.31876	6.736	13.180	0.06548	*0.01470*	0.4053	8.185	−0.1107
0.140	0.54395	10.193	12.689	0.05908	*0.01470*	0.4253	2.238	−0.3074
0.185	0.8780	18.401	11.666	0.04408	*0.01470*	0.4792	0.8157	−3.713
0.240	1.057	29.460	10.944	0.03075	*0.01470*	0.5409	0.6892	2.607

preceding section. The Gauss and Yukawa well shapes represent extremes in well shapes: the former is highly concentrated; the latter is the most "long-tailed" of the commonly used potentials. The Yukawa shape for V_a is written as a function of $r-r_{core}$. Hence, it actually diverges at the core radius. The Morse well was employed primarily to make sure that there is nothing special about the sharply cut-off core employed for the other runs. There is not.

We cannot give a table of these runs because we ran well over 100 cases for a complete exploration. For both the Yukawa and the Gauss shape, we ran intrinsic ranges of $R/10$, $2R/10$, and $3R/10$ (R=radius of the deuteron), core radii of $0.09R$, $0.123R$, $0.140R$, $0.185R$, $0.240R$ and either 3 or 4 choices of intrinsic depths between -0.3 and 1.0; in addition we ran a large number of interpolated cases designed to fit either the quadrupole moment or the singlet state scattering length or both. Each "run" would take at least a week by hand computation (this is an extremely conservative estimate).

We shall confine ourselves to the main results here. First of all, we found, in agreement with Jastrow, that the original Yukawa potential with a repulsive core (i.e., the choice $V_a=0$) gives a good fit to the *triplet* data. Furthermore, the core radius necessary is very close to twice the Compton wavelength of the nucleon; this is a reasonable *a priori* value for r_{core}. The results for a pure Yukawa potential with repulsive core but no added attraction (interpolated to fit the quadrupole moment of the deuteron) are

$$\begin{aligned} r_{core}&=0.106R=0.456\times10^{-13}\text{ cm},\\ g^2/4\pi&=13.6,\\ p_D&=7.18\text{ percent},\\ r_{0t}&=0.386R=1.66\times10^{-13}\text{ cm}. \end{aligned} \tag{3}$$

However, the singlet data implied by the same potential are completely unreasonable ($a=-0.00411R=-1.8\times10^{-15}$ cm). This is understandable because here practically all the binding in the triplet state is due to the tensor force.

With the additional freedom associated with the introduction of V_a, one might think that it should be easy to fit all the low-energy data. We were rather surprised to find, instead, that it is not possible no matter how V_a is adjusted. Indeed, it is impossible to fit simultaneously as few as 4 experimental numbers:

(a) binding energy of deuteron,
(b) quadrupole moment of deuteron, (4)
(c) triplet effective range,
(d) singlet scattering length.

If we fit the triplet data (the first three quantities in (4)), the singlet scattering length is completely unreasonable (too low by more than a factor of 10). If we fit (a), (b), and (d), the triplet effective range is at least 15 percent too large. This 15 percent discrepancy corresponds to an (extrapolated) zero-range V_a, i.e., to an infinitely strong, zero-range attraction just outside an infinitely high repulsive core. The discrepancies for physically reasonable V_a are larger. For a range (outside the core) of V_a equal to $R/5=0.86\times10^{-13}$ cm, the triplet effective range is 21 percent too large for a Yukawa well V_a, 19 percent too large for a Gauss well V_a. Finally, if we try to fit (a), (c), and (d), the quadrupole moment is considerably less than $\frac{2}{3}$ of its experimental value in all cases.

Qualitatively speaking, the difficulty can be traced to the singular nature of the tensor force in the second-order pseudoscalar meson theory. The value of the quadrupole moment determines $g^2/4\pi$ to a first approximation (independently of the core radius and of the nature of V_a). The resulting strong tensor force contributes so much to the binding energy of the deuteron that the central force is much too weak to fit the singlet data. If we restrict ourselves to V_a with intrinsic ranges (outside the core) larger than $R/10$, the potentials which fit the triplet data have central forces which are too weak by more than a factor of 3 to give the required singlet state resonance.

The relativistic corrections to the magnetic moment of the deuteron are probably of the same order of magnitude as the correction due to the D state. Since these relativistic corrections cannot be computed reliably on the basis of present theory, we have taken the "experimental" D-state probability to be 4 ± 4 percent, i.e., $p_D<0.08$. All the theoretically computed values lie within that range.

The well shape of V_a makes very little difference in the results. Strong well shape dependences are spurious and arise only because usually one does not consider V_a as a function of $r-r_{\mathrm{core}}$, but rather as a function of r. In that case a portion of V_a is "cut off" by the core, and the strength of the remaining part depends strongly on the well shape and range of V_a. This dependence was eliminated in our work by choosing to consider V_a as a function of $r-r_{\mathrm{core}}$.

We conclude that the poor fit obtained with the Lévy potential is not a particular property of the fourth-order potential of Lévy, but applies equally to all Yukawa forces with added attractions. Of course, a good fit can be obtained to the triplet data alone (a), (b), and (c) of (4), and the singlet data can then be fitted by adding an appropriate amount of $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$ force. However, such a fit means very little because the number of theoretical parameters used exceeds the number of experimental data.

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