corrections to Q are of the order of $4.8\times g^2/4\pi$ percent of the experimental value,[9] and for $g^2/4\pi$ of the order of 10, about 50 percent of the actual moment and of the opposite sign. We may also note here that the corrections coming from the reduction to Pauli functions in $\Delta E^{(1)}$ are much smaller than the effect treated above. We have

$$\Delta E_{\text{quad}}^{(1)} = -\tfrac{1}{4}eq\int \phi_{++}^*(3z^2-r^2)\phi_{++}d\mathbf{r}$$
$$= -\tfrac{1}{4}eq\int \phi_{\text{n.r.}}^*(3z^2-r^2)\phi_{\text{n.r.}}d\mathbf{r}$$
$$-\tfrac{1}{2}eq(2M)^{-2}\int \phi_{\text{n.r.}}^* p^2(3z^2-r^2)\phi_{\text{n.r.}}d\mathbf{r}. \quad (37)$$

[9] Compare the result of F. Villars, Phys. Rev. **86**, 476 (1952) that $\Delta Q = -3.7\times 10^{-29}(g^2/4\pi)$ cm^2. However, the two calculations differ in several respects. First, Villars did not employ the two-body equation. Second, he used the total interaction, rather than the retarded part, to compute the recoil effects. This counts the instantaneous contribution twice, since its effect is already taken into account through use of bound-state wave functions. Third, he employed different wave functions.

Any effects from the second term on the right-hand side of Eq. (37) are of the order of $(\mu/2M)^2$ of the leading term, for $p^2/(2M)^2$ represents less than the average value of $(v/c)^2$ in the deuteron, since the r^2 in Eq. (37) tends to weight the integral toward smaller p. Thus, the second term has a ratio to the first of less than 0.5 percent. The first term is the usual nonrelativistic expression for the quadrupole moment; it must now be increased to balance the negative sign of the correction term, which implies a rise in the required percentage of D state.

Although the large size of the correction obtained may be due, in part, to the particular form of the retarded interaction employed, the present considerations indicate that, in a correct treatment of the deuteron problem, the recoil effects will contribute appreciably to a calculation of the moments.

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Field Theory of Equations with Many Masses

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This paper develops the field theory of many mass equations with special attention to spin $\frac{1}{2}$ and the operator (1) of the author's previous paper on the irreducible volume character of events. The field is assumed to interact with the electromagnetic field which is introduced in a gauge-invariant way. General expressions for the charge-current four-vector and the symmetrical energy-momentum tensor are derived and are shown to satisfy the appropriate conservation theorems. According to a theorem of Leichter, the general solution is shown to be a superposition of nonorthogonal mass states which we designate as the root fields. Nevertheless, the physical quantities, such as the current four-vector, the energy-momentum tensor, etc., are shown to decompose into a sum over those of individual mass states but with an alternation of sign for consecutive roots. The Lagrangian takes the form of an alternating sum over the individual free-field Lagrangians for the mass states, plus the usual term $+(1/c)j_\mu A_\mu$ for the interaction with the electromagnetic field. The matter field may be quantized by treating the root fields as independent anticommuting fields. The transformation to the interaction representation is obviously unaltered and the charge and mass renormalization may be treated following Schwinger. To the order of approximation in Schwinger II these renormalizations are not affected. It would seem that these methods of quantization, together with the usual treatment of the electromagnetic field, are at variance with the manifest nonlocal nature of the theory for the irreducible volume character of events.

INTRODUCTION

THE earliest multiple-mass equations arose in an attempt to circumvent the divergence difficulties in electromagnetic theory and consisted in introducing besides the photon of zero rest mass one additional nonvanishing rest mass.[1] The first considerations of equations of infinite order with a continuous or discrete spectrum of masses were those of Blokhinzev, who developed the theory for scalar neutral fields with the view of their possible application to mesons.[2] He used Bose quantization based on a set of operators which decomposed the wave field into free fields satisfying the Schrödinger-Klein-Gordon equation for the individual masses contained in the mass spectrum of the operator. Born next introduced fields involving exponential operators [$\exp(a\Box)$, where $\Box = \sum_\lambda u_\lambda^2$] in connection with his method of mass quantization.[3] The present author, in connection with a theory of fundamental length, seems to be the first to propose an infinite-order differential equation for the Dirac field. In this theory

[1] F. Bopp, Ann. Physik **38**, 345 (1940); **42**, 573 (1943); A. Landé and L. H. Thomas, Phys. Rev. **60**, 121, 514 (1940); **65**, 175 (1944); B. Podolsky *et al.*, Phys. Rev. **62**, 68 (1942); **65**, 228 (1944); Revs. Modern Phys. **20**, 40 (1948); A. Green, Phys. Rev. **72**, 628 (1947); D. Montgomery, Phys. Rev. **69**, 117 (1947).

[2] D. Blokhinzev, J. Phys. (U.S.S.R.) **11**, 72 (1947).

[3] M. Born, Revs. Modern Phys. **21**, 463 (1949).

the infinite-order operator is not arbitrary but is definitely determined as a finite displacement operator associated with the introduction of a fundamental length ω, and in terms of the operator z (defined in Sec. 1), it may be written

$$D(z)=[-2J_2(z)+\bar{k}J_1(z)]/z, \tag{1}$$

where $\bar{k}=\omega k=\beta e^2/\hbar c$, $\beta=1.6$, and J_1 and J_2 are the Bessel's functions of order one and two, respectively.[4] Pais and Uhlenbeck made an extensive investigation into operators of the form $f(\Box)$ for neutral scalar fields, where $f(x)$ is in general an integral function of x.[5] They used essentially the same quantization procedure proposed by Blokhinzev. In one short section of their paper they also applied the quantization method to a multiple-mass free Dirac field and made a few cursory remarks (without theoretical development) on the introduction of the electromagnetic potentials. However, the main conclusions which they state for the Dirac field in the presence of the electromagnetic potentials seem to be incorrect according to the results of Sec. 4 and Sec. 6 of the present paper. Finally, Heisenberg in a number of papers[6] has investigated in a general way Lagrangians with a mass spectrum and has suggested their connection with the possible existence of a fundamental length.

In this paper we present the theory of a spin-$\frac{1}{2}$ field ψ in interaction with the electromagnetic field. The field ψ will obey the gauge-invariant equation of motion,

$$D(Z)\psi=0, \tag{2}$$

where $D(z)$ is in general a real (for real argument) integral transcendental function of z with real roots (real mass spectrum). In particular, $D(z)$ may be a polynomial. The definitions of z and Z as operators are given at the beginning of Sec. 1. We shall base the quantum electromagnetics of this field on the methods of Schwinger;[7] consequently, we shall not carry along any of the electromagnetic field formulas since they will be the same as those of Schwinger. In order to conserve space we shall not give the formulas in charge-symmetric form, since these may be easily written down following Schwinger. For the same reason we also confine ourselves to the case where D has simple roots, but the method is easily extended to repeated roots; this limitation is not invoked until Sec. 4. Furthermore, if we consider the γ matrices to be those for other spins[8] (0 and 1 especially), it will be seen that up to Sec. 6 the formulas hold good. Our main concern is with our proposed theory of fundamental length, and it is for this reason that we are primarily concerned with spin $\frac{1}{2}$, as we shall explain in a later paper.

Considered as a function of z, D may be written as an everywhere convergent series:

$$D(z)=\textstyle\sum_r d_r z^r, \tag{3}$$

or, according to a theorem of Weierstrass[9], as the convergent infinite product

$$D(z)=D(0)\left[\exp\frac{D'(0)}{D(0)}\right]\prod_n\left\{\left(1-\frac{z}{z_n}\right)\exp\left(\frac{z}{z_n}\right)\right\}. \tag{4}$$

We wish at this place to emphasize the importance, to the development from Sec. 4 onward, of a theorem of Leichter (cf. Sec. 4) on the nature of the solutions for operators of the form (4). We also note that these sections contain a more general solution than the usual one to the quantum mechanical superposition problem posed by Landé[10] in his papers on his anti-Gibb's paradox principle. Landé shows that this principle necessitates a superposition principle (and hence quantization) for the description of the physical states of a system; however, although he suggests that a more general solution may be of great importance, the only solution available to him was a superposition of orthogonal states after the usual pattern in quantum theory. Contrary to this usual pattern the mass states ψ_n whose superposition constitutes the complete solution [see Eq. (25), Sec. 4] are *not* orthogonal.

The source of this divergence from the usual theory may be attributed: on the one hand, to the fact that mass, like energy but unlike momentum, angular momentum, etc., cannot be determined by an instantaneous observation (Sec. 5); on the other hand, to the fact that the mass, which is conjugate to proper time, unlike the energy, which is conjugate to time, does not enter the theory as an eigenvalue problem, but rather enters with a fixed spectrum. This property of possessing a universal spectrum is one that mass shares with such other basic observables as momentum, angular momentum, etc.

We believe that the relationships mentioned in the preceding paragraph for theories based on operators of the type of Eq. (4) are in essential agreement with the nature of mass, that they imply the existence of a universal proper-time interval, and hence that the basis (see Darling and Zilsel)[4] of the special form of the operator (1) is essentially correct.

1. HAMILTON'S INTEGRAL AND THE CHARGE-CURRENT FOUR-VECTOR

We introduce the following notation: $x_\lambda(\lambda=1\cdots4)$, $x_4=ict$, $u_\lambda=\partial/\partial x_\lambda$, A_λ the electromagnetic potentials, $a_\lambda=ieA_\lambda/\hbar c$, $U_\lambda=u_\lambda-a_\lambda$, $U_\lambda^*=u_\lambda+a_\lambda$, $z=-2\omega\gamma^\lambda u_\lambda$,

[4] B. T. Darling, Phys. Rev. **80**, 460 (1950); B. T. Darling and P. R. Zilsel, Phys. Rev. **91**, 1252 (1953).

[5] A. Pais and G. E. Uhlenbeck, Phys. Rev. **79**, 145 (1950).

[6] W. Heisenberg, Z. Naturforsch. **5**, 251, 367, 373 (1950).

[7] J. Schwinger, Phys. Rev. **74**, 1439 (1948); **75**, 651 (1949); **76**, 790 (1949).

[8] R. J. Duffin, Phys. Rev. **54**, 1114 (1938); N. Kemmer, Proc. Roy. Soc. (London) **A173**, 91 (1939); H. J. Bhabha, Revs. Modern Phys. **21**, 451 (1949); Harish-Chandra, Phys. Rev. **71**, 793 (1947).

[9] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Macmillan Company, New York, 1927), fourth edition, p. 136.

[10] A. Landé, Phys. Rev. **87**, 267 (1952); Am. J. Phys. **20**, 353 (1952); Philosophy of Science **20**, 101 (1953).

$\bar{z}=-z$, $a=2\omega\gamma^\lambda a_\lambda$, $Z=-2\omega\gamma^\lambda U_\lambda=z+a$, $\bar{Z}=2\omega\gamma^\lambda U_\lambda^*=\bar{z}+a$; the summation convention on repeated index is understood throughout, ω is a constant with the dimensions of a length, and the γ^λ are the Dirac matrices.[11]

The equation of motion (2) of the field ψ in interaction with the electromagnetic field may be obtained from the variation of φ in the Hamiltonian integral:

$$I=\int_{<S}\varphi\cdot D(Z)\psi dx. \tag{5}$$

The integral is extended throughout the region of the four-dimensional space $x(x_\lambda, \lambda=1, \cdots 4)$ interior to the closed hyper-surface S, and dx is the four-dimensional volume element. The *dot* will be used throughout this paper and is such that the operators to its right act to the right while those to its left act to the left. By varying ψ we obtain the adjoint equation:

$$\varphi D(\bar{Z})=0, \tag{6}$$

which is equivalent to the Hermitian adjoint of (2) if we set $\varphi=i\psi^*\gamma^4$, where ψ^* is the Hermitian conjugate of ψ.

We shall have much use for the following transference formula for the operator Z, namely,

$$\int_{<S}\varphi\bar{Z}^m\cdot Z^n\psi dx=-2\omega\int_S\varphi\bar{Z}^m\cdot\gamma^\lambda Z^{n-1}\psi dx^\lambda + \int_{<S}\varphi\bar{Z}^{m+1}\cdot Z^{n-1}\psi dx. \tag{7}$$

This formula can be established by a simple integration by parts: φ and ψ can be any functions, and $dx^\lambda=dx/dx_\lambda$ is a component of the element of area of the hypersurface S.

It is well known that the charge-current four-vector may be obtained by varying the potentials in the integral I. If $\delta_A Z=-i\bar{e}\gamma^\lambda\delta A_\lambda(\bar{e}=2\omega e/\hbar c)$, then it is easy to establish that

$$\delta_A\int_{<S}\varphi\cdot Z^r\psi dx=\sum_{k=0}^{r-1}\int_{<S}\varphi\cdot Z^k\delta_A Z Z^{r-1-k}\psi dx,$$

and by repeated application of (7) we can transfer Z^k from operating to the right, so that

$$\delta_A\int_{<S}\varphi\cdot Z^r\psi dx=\int_S+\sum_{k=0}^{r-1}\int_{<S}\varphi\bar{Z}^k\cdot\delta_A Z Z^{r-1-k}\psi dx. \tag{8}$$

The surface integral may be taken zero by proper choice of δA on S. The current j_λ is the coefficient of $-i2\omega\delta A_\lambda/\hbar c^2$ in $\delta_A I$; hence by multiplying (8) by d_r and summing over r we have $j_\lambda=ecs_\lambda$, where

$$s_\lambda=\sum_{r=1}^{\infty}d_r\sum_{k=0}^{r-1}\varphi\bar{Z}^k\cdot\gamma^\lambda Z^{r-1-k}\psi. \tag{9}$$

We shall make frequent use of expressions of the type occurring in s_λ, so it would be valuable to introduce a special notation for them.[12] If X is any operator we define $\varphi D(\bar{Z}\cdot XZ)\psi$ by

$$\varphi D(\bar{Z}\cdot XZ)\psi=\sum_{r=1}^{\infty}d_r\sum_{k=0}^{r-1}\varphi\bar{Z}^k\cdot XZ^{r-1-k}\psi. \tag{10}$$

Thus we may write:

$$s_\lambda=\varphi D(\bar{Z}\cdot\gamma^\lambda Z)\psi.$$

Through repeated application of (7) we can prove that

$$\int_{<S}\varphi\cdot D(Z)\psi dx=\int_S\varphi D(\bar{Z}\cdot\gamma^\lambda Z)\psi dx^\lambda + \int_{<S}\varphi D(\bar{Z})\cdot\psi dx. \tag{11}$$

In virtue of (2) and (6) and the arbitrariness of S, we see that s_λ satisfies the conservation theorem

$$\partial s_\lambda/\partial x_\lambda=0. \tag{12}$$

If we replace ψ by $\delta\psi$ in (11) and choose $\delta\psi$ so that the surface integral vanishes, we establish the adjoint equation (6).

2. COMPLETE VARIATION OF HAMILTON'S INTEGRAL AND THE ENERGY-MOMENTUM TENSOR FOR THE FREE FIELD

We shall make use of the method of complete variation of Hamilton's integral where we also vary the bounding surface S in order to obtain the stress-energy tensor. This method was first applied by Weiss[13] to the case of fields, but it is a generalization of a method in particle dynamics.[14] The method of Weiss has also been applied to higher order equations by de Wet.[15]

However, since the method does not seem to be generally known or used, and since we depart slightly from Weiss, we shall make a brief sketch of it. Let L be the Lagrangian of a system of particles with generalized coordinates q (we omit subscripts and summation over them since this will be understood by the reader), and let both the final time and final position as well

[11] W. Pauli, *Handbuch der Physik* (Julius Springer, Berlin, 1933), Vol. 24, p. 220.

[12] We wish to note that these expressions could just as well be written in the product form according to Eq. (4) instead of the series form according to Eq. (3) as is done here.

[13] P. Weiss, Proc. Roy. Soc. (London) **A156**, 192 (1936); **A169**, 102 (1938).

[14] E. T. Whittaker, *Analytical Dynamics* (The Macmillan Company, New York, 1937), fourth edition, p. 246.

[15] J. S. de Wet, Proc. Cambridge Phil. Soc. **44**, 546 (1948).

as the path be varied in Hamilton's integral. Then

$$\Delta\int_0^t Ldt=L(t)\Delta t+\int_0^t \delta L dt$$

$$=L(t)\Delta t+\left[\frac{\partial L}{\partial\dot q}\delta q\right]_t+\int_0^t\left[\frac{\partial L}{\partial q}-\frac{d}{dt}\frac{\partial L}{\partial\dot q}\right]\delta q dt,$$

after the usual integration by parts. The last term vanishes on account of Lagrange's equations. Examination of a graph q *vs* t will show that the variation Δq of the final position is composed of the sum of $\delta q(t)$ and $\dot q(t)\Delta t$. Hence, replacing δq by its equal $\Delta q-\dot q\Delta t$, we have

$$\Delta I=\left(L-\frac{\partial L}{\partial\dot q}\dot q\right)\Delta t+\frac{\partial L}{\partial\dot q}\Delta q=-H\Delta t+p\Delta q.$$

Thus in the total variation of I the coefficient of Δt is minus the energy, and the coefficient of Δq is the momentum.

Next consider a field ψ (which may be a multi-component field such as the Dirac or Maxwell fields) possessing a quadratic Lagrangian density $L(\psi,\psi_\lambda)$, where $\psi_\lambda=u_\lambda\psi$.[16] We consider the complete variation of $I=\int Ldx$ in which both ψ and the boundary is varied. Let S be a closed hyper-surface in the four-dimensional space-time continuum and let $\Delta x_\lambda(x_s)$ be an infinitesimal displacement of this surface such that each point x_λ on the surface is displaced to a point $x_\lambda+\Delta x_\lambda$. Then the complete variation $\Delta\psi$ at the boundary is the sum of $\delta\psi$ and $\Delta x_\lambda\psi_\lambda$ evaluated at the boundary. For the complete variation of I we have

$$\Delta I=\int_{<S+\Delta S}(L+\delta L)dx-\int_{<S}Ldx=\int_{<\Delta S}Ldx+\int_{<S}\delta Ldx$$

$$=\int_S L\Delta x_\sigma dx^\sigma+\int_S\frac{\partial L}{\partial\psi_\sigma}\delta\psi dx^\sigma$$

$$+\int_{<S}\left(\frac{\partial L}{\partial\psi}-\frac{\partial}{\partial x_\sigma}\frac{\partial L}{\partial\psi_\sigma}\right)\delta\psi dx,$$

where the integral throughout the volume between the two surfaces has been replaced by the equivalent surface integral, the first term of the last form for ΔI. On replacing $\delta\psi$ by $\Delta\psi-\Delta x_\lambda\psi_\lambda$ and remembering that the last integral vanishes on account of Lagrangian equations, the preceding expression may be written

$$\Delta I=\int_S\frac{\partial L}{\partial\psi_\sigma}\Delta\psi dx^\sigma-\int_S\left(\psi_\lambda\frac{\partial L}{\partial\psi_\sigma}-\delta_{\lambda\sigma}L\right)\Delta x_\lambda dx^\sigma. \quad (13)$$

From these two terms we identify

$$\pi_4=\partial L/\partial\psi_4$$

as the canonical momentum, and

$$T_{\sigma\lambda}=\psi_\lambda\partial L/\partial\psi_\sigma-\delta_{\lambda\sigma}L$$

as the canonical energy-momentum tensor.[17]

One easily deduces the conservation theorem

$$\partial T_{\sigma\lambda}/\partial x_\sigma=0 \quad (14)$$

from (13); for if the whole space is given a constant displacement Δx_λ, then $\Delta I=0$ and $\Delta\psi=0$; hence

$$\Delta x_\lambda\int_S T_{\sigma\lambda}dx^\sigma=0.$$

Since Δx_λ is arbitrary, one sees that the surface integral must vanish. Converting the surface integral to an integral throughout the volume enclosed by S, one obtains (14) since the surface S is arbitrary.

Now putting the potentials equal to zero in (5), we have for the complete variation of I:

$$\Delta I=\int_S\varphi\cdot D(z)\psi\Delta x_\sigma dx^\sigma+\int_{<S}\varphi\cdot D(z)\delta\psi dx.$$

After an integration by parts this becomes

$$\Delta I=\int_S\varphi\cdot D(z)\psi\Delta x_\sigma dx^\sigma+\int_S\varphi D(\bar z\cdot\gamma^\sigma z)\delta\psi dx^\sigma$$

$$+\int_{<S}\varphi D(\bar z)\cdot\delta\psi dx.$$

Again, replacing $\delta\psi$ by $\Delta\psi-\Delta x_\lambda\psi_\lambda$ and taking account of (6) and the fact that u_λ commutes with z, the last result may be written

$$\Delta I=\int_S\varphi D(\bar z\cdot\gamma^\sigma z)\Delta\psi dx^\sigma$$

$$-\int_S[\varphi D(\bar z\cdot\gamma^\sigma u_\lambda z)\Delta x_\lambda\psi-\delta_{\lambda\sigma}\varphi\cdot D(z)\psi\Delta x_\lambda]dx^\sigma.$$

On remembering that ψ satisfies (2), and on taking Δx_λ constant, we have for the canonical stress tensor:

$$T_{\sigma\lambda}=(\hbar c/i)\varphi D(\bar z\cdot\gamma^\sigma u_\lambda z)\psi, \quad (15)$$

just as in the preceding case. One may show by direct calculation that the conservation theorem (14) for $T_{\sigma\lambda}$ is a consequence of the equations of motion, but we shall not do so at this point since we shall obtain later a more general result with the potentials included. The tensor (15) is not real and may be replaced by the real tensor

$$T_{\sigma\lambda}=(\hbar c/2i)[\varphi D(\bar z\cdot\gamma^\sigma u_\lambda z)\psi-\varphi D(\bar z u_\lambda\cdot\gamma^\sigma z)\psi]. \quad (16)$$

This last form may be obtained from the complete variation of $\int[\varphi\cdot D(z)\psi+\varphi D(\bar z)\cdot\psi]dx$. It also satisfies

[16] G. Wentzel, *Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1949), first English translation, p. 1.

[17] Reference 16, p. 9.

the conservation theorem in virtue of the equations of motion.

3. INTRODUCTION OF THE ELECTROMAGNETIC POTENTIALS AND THE SYMMETRICAL ENERGY-MOMENTUM TENSOR

A gauge-invariant stress-energy tensor may be obtained by introducing the electromagnetic potentials in the usual way. Thus we replace (16) by

$$T_{\sigma\lambda}'=(\hbar c/2i)[\varphi D(\bar{Z}\cdot\gamma^\sigma U_\lambda Z)\psi+\varphi D(\bar{Z}\bar{U}_\lambda\cdot\gamma^\sigma Z)\psi], \quad (17)$$

where $\bar{U}_\lambda$ stands for $-U_\lambda^*$. For the symmetrical tensor we take

$$T_{\sigma\lambda}=\tfrac{1}{2}(T_{\sigma\lambda}'+T_{\lambda\sigma}'), \quad (18)$$

and we shall verify that

$$\partial T_{\sigma\lambda}/\partial x_\sigma=-(\hbar c/i)s_\sigma F_{\sigma\lambda}=-es_\sigma f_{\sigma\lambda}, \quad (19)$$

where s_σ is the charge-current four-vector (9) and $F_{\sigma\lambda}=u_\sigma a_\lambda-u_\lambda a_\sigma$. $F_{\sigma\lambda}=(ie/\hbar c)f_{\sigma\lambda}$, where $f_{\sigma\lambda}$ are the electromagnetic field strengths. The conservation theorem (19) is the result that is to be expected.[18]

In order to demonstrate (19), first recall that

$$U_\sigma U_\lambda-U_\lambda U_\sigma=F_{\lambda\sigma}. \quad (20)$$

With this, one shows that

$$ZU_\lambda=U_\lambda Z+2\omega\gamma^\sigma F_{\sigma\lambda}. \quad (21)$$

One may put asterisks on the left side of (20) and change the sign of the right side and obtain a true relation. Also a bar may be put above the Z and an asterisk on the U in (21) to obtain another valid relation. Now perform u_σ on the first term of the bracket in the expression (17) for $T_{\sigma\lambda}'$ to get

$$\sum_r d_r\sum_{k=0}^{r-1}[\varphi\bar{Z}^k u_\sigma\gamma^\sigma\cdot U_\lambda Z^{r-1-k}\psi+\varphi\bar{Z}^k\cdot\gamma^\sigma u_\sigma U_\lambda Z^{r-1-k}\psi].$$

The quantity inside this bracket may be written

$$\frac{1}{2\omega}[\varphi\bar{Z}^{k+1}\cdot U_\lambda Z^{r-1-k}\psi-\varphi\bar{Z}^k\cdot ZU_\lambda Z^{r-1-k}\psi]$$

on addition of $(a/2\omega)\varphi\bar{Z}^k\cdot U_\lambda Z^{r-1-k}\psi$ to the first term and the subtraction of the same from the second term. The Z and the U may be commuted in the second term by use of (21), so that we have

$$\frac{1}{2\omega}[\varphi\bar{Z}^{k+1}\cdot U_\lambda Z^{r-1-k}\psi-\varphi\bar{Z}^k\cdot U_\lambda Z^{r-k}\psi-2\omega(\varphi\bar{Z}^k\cdot\gamma^\sigma Z^{r-1-k}\psi)F_{\sigma\lambda}].$$

On summing over k, the first two terms telescope and we have, after multiplying by d_r and summing over r,

$$\frac{1}{2\omega}[\varphi D(\bar{Z})\cdot U_\lambda\psi-\varphi\cdot U_\lambda D(Z)\psi]-s_\sigma F_{\sigma\lambda}.$$

[18] Reference 16, p. 189.

But the quantity in brackets vanishes on account of the equations of motion. Similar operations may be performed on the second term in the bracket of (17); consequently $T_{\sigma\lambda}'$ satisfies Eq. (19).

Next we show that $T_{\sigma\lambda}'-T_{\lambda\sigma}'$ has vanishing divergence. Now

$$(2i/\hbar c)(T_{\sigma\lambda}'-T_{\lambda\sigma}')=\varphi D[\bar{Z}\cdot(\gamma^\sigma U_\lambda-\gamma^\lambda U_\sigma)Z]\psi-\varphi D[\bar{Z}(\gamma^\sigma U_\lambda^*-\gamma^\lambda U_\sigma^*)\cdot Z]\psi.$$

But the right side is equal to

$$\frac{1}{2}\frac{\partial}{\partial x_\mu}\varphi D[\bar{Z}\cdot(\gamma^\mu\gamma^\lambda\gamma^\sigma+\gamma^\lambda\gamma^\sigma\gamma^\mu)Z]\psi.$$

To prove this, one first proves by integration by parts that

$$\int_{<S}\varphi\cdot\gamma^\lambda\gamma^\sigma D(Z)\psi dx =-2\omega\int_S\varphi D(\bar{Z}\cdot\gamma^\mu\gamma^\lambda\gamma^\sigma Z)\psi dx^\mu +4\omega\int_{<S}\varphi D[\bar{Z}\cdot(\gamma^\sigma U_\lambda-\gamma^\lambda U_\sigma)Z]\psi dx +\int_{<S}\varphi D(\bar{Z})\cdot\gamma^\lambda\gamma^\sigma\psi dx,$$

in virtue of the relation

$$\gamma^\lambda\gamma^\sigma Z=Z\gamma^\lambda\gamma^\sigma+4\omega(\gamma^\sigma U_\lambda-\gamma^\lambda U_\sigma);$$

and

$$\int_{<S}\varphi D(\bar{Z})\cdot\gamma^\lambda\gamma^\sigma\psi dx =2\omega\int_S\varphi D(\bar{Z}\cdot\gamma^\lambda\gamma^\sigma\gamma^\mu Z)\psi dx^\mu +4\omega\int_{<S}\varphi D[\bar{Z}(\gamma^\sigma U_\mu^*-\gamma^\mu U_\sigma^*)\cdot Z]\psi dx +\int_{<S}\varphi\gamma^\lambda\gamma^\sigma\cdot D(Z)\psi dx,$$

in virtue of the relation

$$\bar{Z}\gamma^\lambda\gamma^\sigma=\gamma^\lambda\gamma^\sigma\bar{Z}+4\omega(\gamma^\sigma U_\lambda^*-\gamma^\lambda U_\sigma^*).$$

The surface integrals may be transformed by the divergence theorem to integrals throughout the volume. After using (2) and (6) and on subtracting the latter result from the former and remembering that the surface S is arbitrary, we obtain the above-stated expression for the difference of the tensors. We are interested in the nondiagonal terms $\sigma\neq\lambda$, in which case $\gamma^\mu\gamma^\lambda\gamma^\sigma+\gamma^\lambda\gamma^\sigma\gamma^\mu$ vanishes if $\mu=\sigma$ or λ, and is $2\gamma^\lambda\gamma^\sigma\gamma^\mu$ if $\mu\neq\sigma$, $\mu\neq\lambda$; these may be shown from the commutation properties of the γ matrices. Finally,

$$\frac{2i}{\hbar c}\frac{\partial}{\partial x_\sigma}(T_{\sigma\lambda}'-T_{\lambda\sigma}')=\sum_{\substack{\mu\neq\sigma\\ \mu\neq\lambda}}\frac{\partial^2}{\partial x_\sigma\partial x_\mu}\varphi D(\bar{Z}\gamma^\lambda\gamma^\sigma\gamma^\mu Z)\psi$$

vanishes because of the anticommutativity of γ^σ and γ^μ.[19] This completes the demonstration of (19).

4. ROOT FIELDS AND THE DECOMPOSITION OF THE PRECEDING PHYSICAL QUANTITIES

Let z_n be a root of the equation

$$D(z_n)=0. \tag{22}$$

For the sake of simplicity in the formulas we assume that all of these roots are simple. This is true for the infinite-order operator (1) of the theory of fundamental length which holds our primary interest. The case in which the roots are not simple may be treated by a direct extension of the method. Let ψ_n and $\varphi_n=i\psi_n{}^*\gamma^4$ be the *root fields* defined through the equations of motion

$$(Z-z_n)\psi_n=0, \tag{23}$$

$$\varphi_n(\bar{Z}-z_n)=0. \tag{24}$$

Then these root fields are also solutions of (2) and (6) in virtue of (22).

We now make use of a general theorem due to Leichter.[20] For the special case in which $D(Z)$, whether of finite or of infinite order, has no exponential factors and has simple roots, his theorem states that the general solution of (2) is a superposition of the root fields. We write the solution in the form

$$\psi=\sum_n \frac{1}{|b_n|^{\frac{1}{2}}}\psi_n, \tag{25}$$

where b_n are constants to be chosen at our convenience later. Furthermore, the solution has the same form when exponential factors of the type in (4) are present, provided the equation

$$e^{AZ}\psi=0 \tag{26}$$

has only the solution $\psi\equiv 0$; A is a constant. He also proves that $\psi\equiv 0$ is the only solution of (26), provided the A_λ and ψ satisfy certain conditions of analyticity. Thus, for the operator (1) of the theory of fundamental length, the assumption that the A_λ and ψ are analytic except for isolated singularities in a strip obtained by surrounding each point of the space-time continuum by a region of radius 2ω in the complex space allows one to conclude that the exponential operators have no nonvanishing solutions and that the general solution is of the form (25).

Assuming then that ψ has the form (25) we may obtain the individual ψ_n. Let us define the operator $D_n(Z)$ by

$$D_n(Z)=D(Z)/(Z-z_n). \tag{27}$$

Then, if $z_m\neq z_n$, we have

$$D_n(z_m)=0,$$

whereas

$$D_n(z_n)=D'(z_n),$$

where the prime denotes the derivative of $D(z)$ with respect to z. The $D'(z_n)$ are nonvanishing real numbers (in virtue of the reality of the roots) and we choose our b_n so that $b_n=D'(z_n)$. Then,

$$D_n(Z)\psi=(b_n/|b_n|^{\frac{1}{2}})\psi_n=\beta_n|b_n|^{\frac{1}{2}}\psi_n, \tag{28}$$

where $\beta_n=$ sign of b_n.

The Lagrangian density decomposes into a sum over those for the root fields

$$\varphi\cdot D(Z)\psi=\sum_n \beta_n\varphi_n\cdot(Z-z_n)\psi_n;$$

for the left side may be written

$$\sum_n \frac{1}{|b_n|^{\frac{1}{2}}}\varphi_n\cdot D(Z)\psi=\sum_n \frac{1}{|b_n|^{\frac{1}{2}}}\varphi_n\cdot(Z-z_n)D_n(Z)\psi$$

$$=\sum_n \beta_n\varphi_n\cdot(Z-z_n)\psi_n$$

by use of (28).

The current s_λ and the stress tensor $T_{\sigma\lambda}$ also decompose, namely

$$s_\lambda=\sum_n \beta_n s_{n;\lambda},\quad T_{\sigma\lambda}=\sum_n \beta_n T_{n;\sigma\lambda},$$

where

$$s_{n;\lambda}\equiv\varphi_n\gamma^\lambda\psi_n$$

and

$$T_{n;\sigma\lambda}\equiv(\hbar c/4i)\varphi_n[\cdot\gamma^\sigma U_\lambda+\bar{U}_\lambda\gamma^\sigma\cdot+\cdot\gamma^\lambda U_\sigma+\bar{U}_\sigma\gamma^\lambda\cdot]\psi_n$$

are just the usual expressions for the current and stress tensor for the field ψ_n.[21]

To prove these results, consider

$$\varphi D(\bar{Z}\cdot XZ)\psi$$

$$=\sum_{r,m,n}\frac{d_r}{|b_nb_m|^{\frac{1}{2}}}\sum_{k=0}^{r-1}\varphi_m\bar{Z}^k\cdot XZ^{r-1-k}\psi_n$$

$$=\sum_{r,m,n}\frac{d_r}{|b_nb_m|^{\frac{1}{2}}}\left(\sum_{k=0}^{r-1}z_m{}^kz_n{}^{r-1-k}\right)\varphi_m\cdot X\psi_n$$

$$=\sum_{r,n}\frac{rd_rz_n{}^{r-1}}{|b_n|}\varphi_n\cdot X\psi_n+\sum_{r,m\neq n}\frac{d_r}{|b_nb_m|^{\frac{1}{2}}}\frac{z_n{}^r-z_m{}^r}{z_n-z_m}\varphi_m\cdot X\psi_n$$

$$=\sum_n\frac{1}{|b_n|}D'(z_n)\varphi_n\cdot X\psi_n+\sum_{m\neq n}\frac{1}{|b_nb_m|^{\frac{1}{2}}}\frac{D(z_n)-D(z_m)}{z_n-z_m}\varphi_m\cdot X\psi_n$$

$$=\sum_n \beta_n\varphi_n\cdot X\psi_n.$$

5. THE INITIAL VALUE PROBLEM

If the order p of (2) is finite, then at the time t_0 one may take the kth derivatives with respect to x_4, $\psi^k(t_0)$, $(k=0,\cdots p-1)$ to be given functions of $\mathbf{x}$, and then by

[19] Reference 11, p. 235.

[20] M. Leichter, doctoral thesis, Ohio State University, 1952 (unpublished). This work will be submitted for publication in the near future.

[21] Reference 16, p. 189.

use of Taylor's expansion and the differential equation obtain the solution for all time. But, alternatively, we may specify the initial values of the root fields $\psi_n(t_0)$ $(n=1, \cdots p)$, and these will determine the solution $\sum \psi_n$ (we drop the factors b_n for convenience) for all later time, since the root fields satisfy first-order equations. Inasmuch as $\psi_n{}^k(t_0)$ may be expressed in terms of $\psi_n(t_0)$ through Eqs. (23), the two modes of setting the initial conditions are equivalent, since we may solve the set of p equations:

$$\psi^k(t_0)=\psi_1{}^k+\cdots+\psi_p{}^k,$$

$k=1, \cdots p-1$, for the functions $\psi_n(t_0)(n=1, \cdots p)$.

The situation is different if the order is infinite. If we give $\psi^k(t_0)$ $(k=1, \cdots\infty)$, then Taylor's theorem determines $\psi(t)$ without any use of the differential equation. If $\psi(t)$ is to be a solution of the equation, then the quantities $\psi^k(t_0)$ cannot be arbitrary but must satisfy a complicated condition determined by the infinite-order differential equation, a condition involving not simply the time t_0 but essentially all times in view of the unspecified dependence of the electromagnetic potentials on the time. This procedure is both impractical and unphysical. The physical requirements necessitate knowing what masses are initially present and their initial states; consequently, the appropriate procedure in the infinite-order case is to specify initially the root fields $\psi_n(t_0)$ $(n=1, \cdots\infty)$. It is doubtful even in the finite case whether any physical meaning could be given to the procedure of specifying the $\psi^k(t_0)$.

Apparently the correct way to specify the initial data is to specify the individual root fields. But here we have a contradiction between the mathematical specification of initial values at a time t_0 and the physical facts. For unlike other quantum observables, such as the momentum, etc., the mass (like energy) is conjugate to a time, the proper time. Consequently, the determination of mass requires observing the system over a certain period of time. To clarify the situation, suppose we know $\psi(\mathbf{x}, t_0)=f(\mathbf{x})$ at the time t_0 [for example $f(\mathbf{x})$ might be an eigenstate of angular momentum prepared by an observation of the angular momentum], then one might pick any value of n and set $\psi_n(t_0)=f(\mathbf{x})$, or one might break $f(\mathbf{x})$ up in quite arbitrary ways and take the parts for the $\psi_n(t_0)$. How are we to determine which of the various possibilities is the correct one in a given situation? In order to answer this question, we do not have to catalog all the possible ways in which we might know the mass, such as by deflections in weak static electric and magnetic fields or by the spectral or chemical data that one uses to select, say, hydrogen gas when one wants to produce a proton beam, etc.; for these methods all seem to be equivalent from the individual particle aspect of the theory[4] to the minimum requirement of making the particle interact with an electromagnetic field of some kind for a sufficient time for us to be able to classify the associated field with respect to the individual root-field equation (23) it obeys. Once the determination of the masses has been made, it will remain unchanged for all time so long as interaction takes place only through the agency of the electromagnetic field introduced with the gauge-invariant substitution $z\rightarrow Z$. This stability of the mass is a consequence of the decomposition demonstrated in the preceding section.

6. QUANTIZATION: CHARGE AND MASS RENORMALIZATION

The decomposition of the Lagrangian derived in Sec. 4 shows that the quantization scheme for the free fields,

$$\{\psi_\alpha{}^n(x),\ \varphi_\beta{}^m(x')\}=\delta_{nm}\beta_m S_{\alpha\beta}{}^m(x-x'),$$

with all other anticommutators vanishing, is consistent with the equations of motion. The appearance of the factors β_m is necessary not only because they appear in the Lagrangian but also in order that the energy-momentum four-vector represent a displacement operator and in order that the interaction representation with $H=-(1/c)j_\mu(x)A_\mu(x)$ reduce the equations of motion of ψ^n to free-field equations. If $\bar{\psi}^m$ has its usual significance, where

$$\{\psi_\alpha{}^n(x),\ \bar{\psi}_\beta{}^m(x')\}=(1/i)\delta_{nm}S_{\alpha\beta}{}^m(x-x'),$$

we see that $\varphi^m=i\beta_m\bar{\psi}^m$ and hence that φ is not simply $i\bar{\psi}$. Indeed,

$$\varphi=\sum_n\frac{1}{|b_n|^{\frac{1}{2}}}\varphi_n=i\sum_n\frac{\beta_n}{|b_n|^{\frac{1}{2}}}\bar{\psi}^n,$$

so that in terms of the ψ^n and $\bar{\psi}^n$ the factors β_n drop out entirely from all expressions. It is therefore clear that to the order of approximation in Schwinger II the charge and mass renormalization will not be affected, and that the vacuum is defined by

$$\psi^{n(+)}\Psi_0=0,\ \bar{\psi}^{n(+)}\Psi_0=0;$$

i.e., the positive frequency states are empty and the negative frequency states are full throughout.

From what has been said, it would seem that the quantization used by Pais and Uhlenbeck is inconsistent with their Lagrangian as well as the interaction representation, and that there is no interchange of the roles of particle and antiparticle for negative β_n.

Finally we wish to remark that the preceding results, based on the usual methods of quantization and conventional electromagnetics in which the β_n drop out of the theory and in which the charge and mass renormalization are unaffected (to the first order), seem to be at variance with the manifest nonlocal nature of the theory for the irreducible volume character of events. We intend to consider these questions later from a quite different approach.