pression can be shown to reduce to the form

$$\frac{1}{\sqrt{\epsilon}}=1-\frac{4\pi N}{\hbar}\sum_b \frac{\nu_{ba}|\langle a|\sum_k e_k\mathbf{e}\cdot\mathbf{r}_k|b\rangle|^2}{\nu_{ba}^2-\nu^2}. \quad (47)$$

Finally, recalling that we are dealing with a medium of low density for which $\epsilon-1$ is small, we can write

$$\epsilon=1+\frac{8\pi N}{\hbar}\sum_b \frac{\nu_{ba}|\langle a|\sum_k e_k\mathbf{e}\cdot\mathbf{r}_k|b\rangle|^2}{\nu_{ba}^2-\nu^2}. \quad (48)$$

This is precisely the expression for the dielectric constant given by the Kramers-Heisenberg dispersion formula.[12]

ACKNOWLEDGMENT

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[12] See reference 4. Also see J. H. Van Vleck, *Electric and Magnetic Susceptibilities* (Oxford University Press, London, 1932), p. 362.

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Periodic Deviations in the Schottky Effect

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The periodic deviations in the Schottky effect are recalculated in this paper. The same basic model is taken as in the work of Guth and Mullin, and of Juenker, Colladay, and Coomes. The difference between this work and the previous treatments is that WKB-type approximations are used. It is felt that these remove some of the uncertainties in the derivation. The results are essentially the same as those of Juenker, Colladay, and Coomes, except that the amplitude of the deviations found is about twice as large and has a slightly different dependence on the field.

I. INTRODUCTION

WHEN the logarithm of the current emitted from a metal is plotted against the square root of the electric field applied, the resulting curve is very nearly a straight line over a large range of the applied field. This dependence of the current on the field is known as the Schottky[1] effect. Experimentally in the Schottky region the curve has small deviations from straightness which are roughly periodic. A summary of the experimental data has been given by Juenker, Colladay, and Coomes.[2] The period of the deviations depends primarily on the field; the amplitude depends on both the field and the temperature. The theory of Guth and Mullin,[3] based on the one-dimensional potential shown in Fig. 1 and modified by Juenker, Colladay, and Coomes,[2] predicts the correct period and it also agrees closely with the observed variation of amplitude with field and temperature. However, the amplitude itself and the phase predicted by the theory disagree with the experimental results. As one possible origin of the disagreement, Brock, Houde, and Coomes[4] have suggested that the shape of the assumed potential may be incorrect in Region II (Fig. 1). The reason for making this suggestion is that the only effect of the potential in this region is to introduce a constant factor into the amplitude and a constant term into the phase of the theoretical results. Another possible origin of the disagreement, as emphasized by Juenker, Colladay, and Coomes,[2] is that some of the approximations used in developing the theory may not be applicable.

In this paper the periodic deviations in the Schottky effect are recalculated. In accordance with the sug-

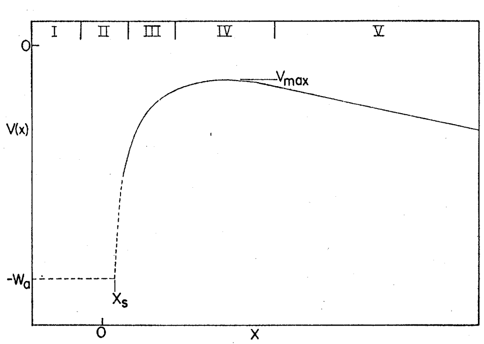

FIG. 1. The potential $V(x)$ assumed for an electron at the surface of a metal. The solid curve represents the function $-eFx-e^2(4x)^{-1}$. The dashed curve represents the same function for $x>x_s$ and the function $-W_a$ for $x<x_s$. The Roman numerals indicate special regions discussed.

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[1] W. Schottky, Physik. Z. **15**, 872 (1914).
[2] Juenker, Colladay, and Coomes, Phys. Rev. **90**, 772 (1953).
[3] E. Guth and C. J. Mullin, Phys. Rev. **59**, 575 (1941).
[4] Brock, Houde, and Coomes, Phys. Rev. **89**, 851 (1953).

gestion of Brock, Houde, and Coomes, the results are expressed in terms of a parameter μ which gives the effect of the potential in Region II; μ itself is found separately for the potential of Fig. 1. A WKB-type approximation recently proposed[5] has been used throughout; in some respects it gives improved results and it is felt that it removes some of the uncertainties in the derivation. The results obtained are essentially the same as those of Juenker, Colladay, and Coomes, except that the amplitude is about twice as large and depends in a slightly different way on the field. The details of the calculations are given in Secs. II to V; Sec. VI contains a discussion, including an estimate of the nonperiodic deviations from the Schottky effect.

II. BASIC EQUATIONS

Some of the fundamental equations needed for the study of electron emission are given in this section.[6] A model often used (in particular by Guth and Mullin, and by Juenker, Colladay, and Coomes) is that the electrons move in the one-dimensional potential, shown in Fig. 1,

$$\begin{aligned} V(x) &= -eFx-e^2(4x)^{-1}, && \text{when } x>x_s, \\ &= -eFx_s-e^2(4x_s)^{-1}=-W_a, && \text{when } x<x_s, \end{aligned} \qquad (1)$$

where $-e^2(4x)^{-1}$ is the image potential at position x normal to the metal for an electron of charge $-e$, and $-eFx$ is the potential due to the applied field F. The value of W_a is a property of the metal. Ordinarily W_a is large compared to the amount $e^{\frac{3}{2}}F^{\frac{1}{2}}$ that the peak of the potential $V(x)$ is below zero. This is the model used below, except that in most of the calculations the shape of the potential in the neighborhood of the point x_s (the dashed part of the curve in Fig. 1) is left unspecified.

The current emitted per unit area is[7]

$$j=4\pi mk^2eh^{-3}(1-\bar{r}_v)T^2\exp[(\bar{\mu}-V)(kT)^{-1}], \qquad (2)$$

where T is the temperature, k is Boltzmann's constant, and $\bar{\mu}$ is the electrochemical potential. The factor $(1-\bar{r}_v)$ is the average transmission coefficient through the surface of the metal,

$$(1-\bar{r}_v)=(kT)^{-1}\int_V^\infty[1-r_v(W)] \times\exp[-(W-V)(kT)^{-1}]dW, \qquad (3)$$

where $[1-r_v(W)]$ is the transmission through the surface at energy W. In Eqs. (2) and (3), V is the potential at some point far outside the surface of the metal. Because of the thickness of the barrier, when the applied field is in the Schottky region, only energies near and above the peak are of importance in evaluating the integral. With these formulas the problem reduces to the calculation of $[1-r_v(W)]$. Because of its dependence on the potential V, this term gives also the final influence of the field F on the current j.

Assuming that the conventional WKB approximation applies in Regions I, III, and V (Fig. 1), Herring and Nichols[8] have shown that the transmission coefficient $[1-r_v(W)]$ can be expressed in terms of a parameter λ which depends only on the potential in Region IV and a parameter μ which depends only on the potential in Region II. The following connections between the WKB approximations to the wave functions of the electron define the parameters λ and μ;

$$p^{-\frac{1}{2}}\exp\left(-i\hbar^{-1}\int_0^x p(\xi)d\xi\right) \text{ in Region III} \rightleftarrows$$

$$\lambda c_1{}^*p^{-\frac{1}{2}}\exp\left(i\hbar^{-1}\int_0^x p(\xi)d\xi\right) + c_1p^{-\frac{1}{2}}\exp\left(-i\hbar^{-1}\int_0^x p(\xi)d\xi\right) \text{ in Region V}, \qquad (4)$$

$$c_2p^{-\frac{1}{2}}\exp\left(-i\hbar^{-1}\int_0^x p(\xi)d\xi\right) \text{ in Region I} \rightleftarrows$$

$$\mu p^{-\frac{1}{2}}\exp\left(i\hbar^{-1}\int_0^x p(\xi)d\xi\right) + p^{-\frac{1}{2}}\exp\left(-i\hbar^{-1}\int_0^x p(\xi)d\xi\right) \text{ in Region III}, \qquad (5)$$

where c_1 and c_2 are complex constants and

$$p(x)=\{2m[W-V(x)]\}^{\frac{1}{2}} \qquad (6)$$

is the momentum of the electron. With these definitions the transmission coefficient, defined as the ratio of the current transmitted into Region I to the current incident in Region V, can be written as

$$[1-r_v(W)]=1-|(\lambda+\mu)(1+\lambda^*\mu)^{-1}|^2. \qquad (7)$$

In terms of the abbreviation

$$\theta(W)=\arg(\lambda^*\mu) \qquad (8)$$

the transmission coefficient becomes[9]

$$\begin{aligned} [1-r_v(W)] &= (1-|\lambda|^2)(1-|\mu|^2) \times(1+2|\lambda\mu|\cos\theta+|\lambda\mu|^2)^{-1} \\ &= (1-|\lambda|^2)(1-|\mu|^2)(1-|\lambda\mu|^2)^{-1} \times\left[1+2\sum_{n=1}^\infty(-|\lambda\mu|)^n\cos n\theta\right]. \end{aligned} \qquad (9)$$

[5] S. C. Miller, Jr., and R. H. Good, Jr., Phys. Rev. **91**, 174 (1953).
[6] C. Herring and M. H. Nichols, Revs. Modern Phys. **21**, 185 (1949).
[7] Reference 6, p. 192.
[8] Reference 6, p. 252.
[9] This expansion is given by, for example, R. Courant and D. Hilbert, *Methoden der Mathematischen Physik* (Julius Springer, Berlin, 1937), Vol. II, p. 246.

It will be shown that the constant term in the series leads to the normal Schottky effect and that each of the terms $n\geq 1$ gives a deviation oscillatory in the field F. The frequencies of these deviations are roughly proportional to n; because of this qualitative difference, one expects to be able to separate the contributions of the various terms. Also the amplitudes decrease with increasing n because of the additional $|\lambda\mu|$ factors. In this paper only the normal Schottky effect and the first oscillatory deviation will be discussed so the series only needs to be carried as far as the $n=1$ term.

III. CALCULATION OF THE PARAMETER λ

The parameter λ, defined by Eq. (4), is to be found from a discussion of the Schrödinger equation,[10]

$$(d^2\psi/dx^2)+2[W-V(x)]\psi=0, \tag{10}$$

in the region near the peak of the potential barrier where $V(x)=-Fx-(4x)^{-1}$. A WKB-type approximation for this type of potential barrier problem has been given in reference 5. From Eq. (36) of that paper one finds easily that

$$p^{-\frac{1}{2}}\exp\left(\tfrac{1}{2}\pi E+i\,\mathrm{Re}\int_x^{x_1}p(\xi)d\xi\right)\text{ in Region III}\rightleftarrows$$

$$p^{-\frac{1}{2}}\exp\left(-\tfrac{1}{2}\pi i+i\,\mathrm{Re}\int_{x_2}^{x}p(\xi)d\xi\right)$$

$$+[(2/\pi)^{\frac{1}{2}}(|E|/2e)^{-\frac{1}{2}iE}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}iE)\cosh(\tfrac{1}{2}\pi E)]$$

$$\times p^{-\frac{1}{2}}\exp\left(\tfrac{1}{4}\pi E-i\,\mathrm{Re}\int_{x_2}^{x}p(\xi)d\xi\right)\text{ in Region V,} \tag{11}$$

where x_1 and x_2 are the locations of the zeros of $p^2(x)=2[W+Fx+(4x)^{-1}]$, chosen so that $x_1<x_2$ when they are real and so that x_1 has positive imaginary part and x_2 negative imaginary part when they are complex. A branch of $p(x)$ such that $\arg p(x)$ is either 0 or $\frac{1}{2}\pi$ when x is real is to be used, and E is given by

$$E=2\pi^{-1}i\int_{x_1}^{x_2}p(\xi)d\xi. \tag{12}$$

A comparison of Connections (4) and (11) gives the parameter λ:

$$\lambda=[(2/\pi)^{\frac{1}{2}}(|E|/2e)^{\frac{1}{2}iE}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}iE)\cosh(\tfrac{1}{2}\pi E)]^{-1}$$

$$\times\exp\left(-\tfrac{1}{4}\pi E-\tfrac{1}{2}\pi i-2i\,\mathrm{Re}\int_0^{x_1}p(\xi)d\xi\right). \tag{13}$$

Using the connection between the gamma and the trigonometric functions, one finds

$$|\lambda|=(1+e^{\pi E})^{-\frac{1}{2}}. \tag{14}$$

[10] Hartree units (m, e, $\hbar=1$) are used in the rest of the paper.

The expression for $\arg\lambda$ is more involved,

$$\arg\lambda=\tfrac{1}{2}E-\tfrac{1}{2}E\ln(\tfrac{1}{2}|E|)+\arg\Gamma(\tfrac{1}{2}+\tfrac{1}{2}iE)$$

$$-\tfrac{1}{2}\pi-2\,\mathrm{Re}\int_0^{x_1}p(\xi)d\xi. \tag{15}$$

The integrals in Eqs. (12) and (15) are elliptic; for $W<0$ they are

$$\int_{x_1}^{x_2}p(\xi)d\xi=\tfrac{2}{3}i|W|^{\frac{3}{2}}F^{-1}(1+a)^{\frac{1}{2}}\{\mathrm{E}[(2a)^{\frac{1}{2}}(1+a)^{-\frac{1}{2}}]$$

$$-(1-a)\mathrm{K}[(2a)^{\frac{1}{2}}(1+a)^{-\frac{1}{2}}]\}, \tag{16}$$

$$\int_0^{x_1}p(\xi)d\xi=\tfrac{2}{3}|W|^{\frac{3}{2}}F^{-1}(1+a)^{\frac{1}{2}}\{\mathrm{E}[(1-a)^{\frac{1}{2}}(1+a)^{-\frac{1}{2}}]$$

$$-a\mathrm{K}[(1-a)^{\frac{1}{2}}(1+a)^{-\frac{1}{2}}]\}, \tag{17}$$

where K and E are the complete elliptic integrals of the first and second kinds,

$$\mathrm{K}[\mathrm{k}]=\int_0^{\pi/2}(1-\mathrm{k}^2\sin^2\phi)^{-\frac{1}{2}}d\phi, \tag{18}$$

$$\mathrm{E}[\mathrm{k}]=\int_0^{\pi/2}(1-\mathrm{k}^2\sin^2\phi)^{\frac{1}{2}}d\phi, \tag{19}$$

and a is defined by

$$a=(1-FW^{-2})^{\frac{1}{2}} \tag{20}$$

(the argument of a is either 0 or $-\frac{1}{2}\pi$ and the arguments of $(1+a)^{\frac{1}{2}}$ and $(1-\mathrm{k}^2\sin^2\phi)^{\frac{1}{2}}$ are in the neighborhood of zero). A table for the evaluation of the integral in Eq. (16) when $W<-F^{\frac{1}{2}}$ has been given by Burgess, Kroemer, and Houston[11] since the same integral arose in Nordheim's discussion of field emission.[12]

IV. CALCULATION OF THE PARAMETER μ

The parameter μ, defined by Eq. (5), is to be found from a discussion of the Schrödinger equation, Eq. (10), in Regions I, II, III using the potential with the break at x_s as given in Eq. (1) and as shown in Fig. 1. Here the wave functions on the two sides of x_s are to be found separately and then are to be matched at the break. If the usual WKB method is used to approximate the wave function on the right of the break, the results obtained for $|\mu|^2$ are somewhat inaccurate, especially as W_a becomes large. This is illustrated in Fig. 27 of the review article by Herring and Nichols.[13] As W_a increases, x_s decreases and in the region just to the right of the break the $(2x)^{-1}$ term in $p^2(x)$ becomes dominant. Accordingly, one is led to use a WKB-type approximation having as basic functions the solutions

[11] Burgess, Kroemer, and Houston, Phys. Rev. **90**, 515 (1953).
[12] L. W. Nordheim, Proc. Roy. Soc. (London) **A121**, 626 (1928).
[13] Reference 6, p. 250.

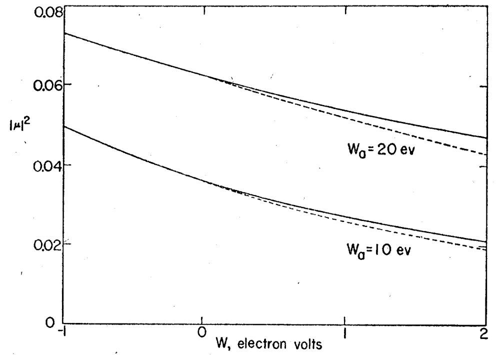

FIG. 2. The reflection coefficient $|\mu|^2$ as a function of the energy W for zero electric field. The dashed lines for $W>0$ are from the exact calculations of Nordheim and MacColl. The solid lines were found using Eq. (24).

of the Schrödinger equation,

$$(d^2\phi/dS^2)+S^{-1}\phi=0. \tag{21}$$

This gives the following approximate wave functions to the right of the break:

$$\psi \cong p^{-\frac{1}{2}}\left(\int_0^x p(\xi)d\xi\right)^{\frac{1}{2}} Z_1\left(\int_0^x p(\xi)d\xi\right), \tag{22}$$

where Z_1 indicates any Bessel function of the first order. It is convenient to take a specific wave function which has the asymptotic form appearing in Connection (5). This is

$$\psi_1=\left(\tfrac{1}{2}\pi p^{-1}\int_0^x p(\xi)d\xi\right)^{\frac{1}{2}}\left[\mu e^{\frac{3}{4}\pi i}H_1^{(1)}\left(\int_0^x p(\xi)d\xi\right)\right.$$
$$\left.+e^{-\frac{3}{4}\pi i}H_1^{(2)}\left(\int_0^x p(\xi)d\xi\right)\right]\rightarrow$$
$$\mu p^{-\frac{1}{2}}\exp\left(i\int_0^x p(\xi)d\xi\right)$$
$$+p^{-\frac{1}{2}}\exp\left(-i\int_0^x p(\xi)d\xi\right) \text{ in Region III,} \tag{23}$$

where the asymptotic forms of the Hankel functions $H_1^{(1)}(z)$, $H_1^{(2)}(z)$ for large z have been used.[14] The logarithmic derivative of the wave function must be continuous across the break x_s. Furthermore, the potential is constant to the left of the break so that the left-going wave given in Connection (5) for Region I is an exact solution of the Schrödinger equation up to the point x_s. Therefore the logarithmic derivative has the constant value $-ip=-i[2(W+W_a)]^{\frac{1}{2}}$ to the left of x_s and the condition

$$d\psi_1/dx=-i[2(W+W_a)]^{\frac{1}{2}}\psi_1, \quad \text{when} \quad x=x_s, \tag{24}$$

may be applied to determine μ. As a matter of convenience μ may be calculated in the form of an expansion on WW_a^{-1} and FW_a^{-2}, with the result that

$$\mu=\left[\frac{H_0^{(2)}(z)+iH_1^{(2)}(z)}{-H_1^{(1)}(z)+iH_0^{(1)}(z)}\right]_{z=(2W_a)^{-\frac{1}{2}}}$$
$$\times\{1+O(WW_a^{-1})+O(FW_a^{-2})\}. \tag{25}$$

The value of $|\mu|^2$ for zero electric field as found from Eq. (24) is compared with the exact results of Nordheim[12] and MacColl[15] in Fig. 2. In order that other models for calculating μ may be used conveniently, the value of μ is left unspecified in the next section. For many of these models the type of approximation used above will be applicable.

V. THE PERIODIC DEVIATIONS

As a first step in finding the average transmission coefficient, Eq. (14) and Eq. (9) to order $n=1$ may be substituted into Eq. (3) yielding

$$(1-\bar{r}_v)=(kT)^{-1}\int_V^\infty(1-|\mu|^2)\left[\frac{e^{\pi E}}{1-|\mu|^2+e^{\pi E}}\right.$$
$$\left.-\frac{2e^{\pi E}|\mu|\cos\theta}{(1-|\mu|^2+e^{\pi E})(1+e^{\pi E})^{\frac{1}{2}}}\right]$$
$$\times\exp\{-(W-V)(kT)^{-1}\}dW. \tag{26}$$

An exact discussion of this integral would be difficult. A series of approximations will be applied in order to express the results in a simpler form. As a first approximation, it will be assumed that $|\mu|^2$ can be neglected compared to 1 in the range of energy W which contributes to this integral. Accordingly, $1-|\mu|^2$ will be replaced simply by 1 in the integrand.

For $E<-1$[16] the quantity in the square brackets in Eq. (26) is effectively zero (it is assumed that $e^{\pi E}$ is dominant over $\exp[-W(kT)^{-1}]$ for $E<-1$) and for $E>2$ it is effectively one. Consequently, only the functional dependence of E on W in the range $-1<E<2$ is of importance in the evaluation of the integral. From Eq. (12) one sees that E varies monotonically with W and that $E=0$ when $W=V_{\max}=-F^{\frac{1}{2}}$. Therefore one is led to expand E as given by Eqs. (12) and (16) for small ϵ, where

$$\epsilon=(|V_{\max}|)^{-1}(W-V_{\max})=1+WF^{-\frac{1}{2}}. \tag{27}$$

When ϵ is small, a is small and, using the expansions of the elliptic functions[17] for small moduli, one finds that

$$E=F^{-\frac{1}{4}}\epsilon+O(\epsilon^2). \tag{28}$$

In the Schottky region the field ranges roughly from

[14] E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, New York, 1945), fourth edition, p. 138.

[15] L. A. MacColl, Phys. Rev. **56**, 699 (1939).

[16] In what follows, $x=-\pi$ will be used uniformly as the cut-off point for e^x.

[17] Reference 14, p. 73.

10^3 to 10^6 volts/cm; therefore F ranges roughly from 2×10^{-7} to 2×10^{-4} and $F^{-\frac{1}{4}}$ from 50 to 10. Consequently, ϵ is at most 0.2 in the region $-1<E<2$ and, for the purposes of evaluating the integral, only the linear term in ϵ need be retained in Eq. (28). Then, combining Eqs. (27) and (28), one has

$$E=F^{-\frac{1}{4}}(1+WF^{-\frac{1}{2}}), \tag{29}$$

and the integration variable in Eq. (26) can conveniently be changed from W to E. Furthermore, the lower limit can be taken to be $-\infty$.

The same ideas will be used for $\arg\lambda$, given by Eqs. (15) and (17). Using the expansions of the elliptic integrals[17] about unit modulus, one finds that

$$\mathrm{Re}\int_0^{x_1} p(\xi)d\xi=F^{-\frac{1}{4}}[\tfrac{2}{3}-\tfrac{1}{4}\epsilon\ln|\epsilon| +((5/4)\ln2-\tfrac{3}{4})\epsilon+O(\epsilon^2\ln|\epsilon|)]. \tag{30}$$

Also the gamma function may be expanded about $E=0$,

$$\arg\Gamma(\tfrac{1}{2}+\tfrac{1}{2}iE)=-(\tfrac{1}{2}\gamma+\ln2)E+O(E^3), \tag{31}$$

where $\gamma=0.577$ is Euler's constant. As a consequence of the arguments about the important range of E, the higher-order terms in Eq. (30) will be disregarded. The terms of order E^3 and higher in Eq. (31) will be neglected also, although this is a coarser approximation [numerical calculations indicate that, as a consequence, the amplitude of the periodic deviations in Eq. (41) is low by about 10 percent and that the phase is off by less than 3°]. Collecting Eqs. (8), (15), (30), (31), one finds

$$\theta=b+\pi cE, \tag{32}$$

where, as abbreviations,

$$b=\tfrac{1}{2}\pi+\arg\mu+(4/3)F^{-\frac{1}{4}}, \tag{33}$$

$$c=\pi^{-1}(-2+\tfrac{1}{2}\gamma+3\ln2-\tfrac{1}{8}\ln F). \tag{34}$$

With the further abbreviation

$$d=(\pi kT)^{-1}F^{\frac{3}{4}}, \tag{35}$$

Eq. (26) in the small $|\mu|^2$ approximation becomes

$$(1-\bar{r}_v)=d\exp[(V+F^{\frac{1}{2}})(kT)^{-1}]\int_{-\infty}^{\infty}\left[\frac{e^{-\pi Ed}}{1+e^{\pi E}} -\frac{2|\mu|\ \mathrm{Re}(e^{-\pi Ed+ib+\pi iEc})}{(1+e^{\pi E})^{\frac{3}{2}}}\right]e^{\pi E}\pi dE, \tag{36}$$

where the integration variable has been changed from W to E and Eq. (32) has been used for θ. Next it will be assumed that μ does not vary appreciably with the energy W in the range of W which is important so that $|\mu|$ and b are constants for the integration. The symbol μ_p will be used for this value of μ. Then, on changing the integration variable from E to $e^{\pi E}$ it is seen that the integrals are simply beta functions, so that

$$(1-\bar{r}_v)=d\exp[(V+F^{\frac{1}{2}})(kT)^{-1}]\{\Gamma(1-d)\Gamma(d) -4\pi^{-\frac{1}{2}}|\mu_p|\ \mathrm{Re}[e^{ib}\Gamma(1-d+ic)\Gamma(\tfrac{1}{2}+d-ic)]\}. \tag{37}$$

Here d is small; at 1500°K, d ranges from 6×10^{-4} to 10^{-1} as the field varies from 10^3 to 10^6 volts/cm. Only a first approximation for small d to each of the terms will be calculated. The dependence on d then disappears from the second term in the braces, and the connection between the gamma and trigonometric functions can be used to express the absolute values of the gamma functions simply in terms of hyperbolic functions. Then, using Eq. (2), one finds for the emitted current,

$$j=\tfrac{1}{2}\pi^{-2}(kT)^2\exp[(\bar{\mu}+F^{\frac{1}{2}})(kT)^{-1}] \times\{1-4(2\pi c)^{\frac{1}{2}}(\sinh2\pi c)^{-\frac{1}{2}}d|\mu_p| \times\cos[b+\arg\Gamma(1+ic)+\arg\Gamma(\tfrac{1}{2}-ic)]\}. \tag{38}$$

It is seen from Eq. (34) that $2\pi c$ is greater than 3 in the Schottky region, so $2^{-\frac{1}{2}}e^{\pi c}$ can be written instead of $(\sinh2\pi c)^{\frac{1}{2}}$. Also the Stirling approximation,

$$\Gamma(z)\cong(2\pi)^{\frac{1}{2}}e^{-z}z^{z-\frac{1}{2}}, \tag{39}$$

can be used to obtain simple expressions for the arguments of the gamma functions. Finally, the common logarithm of the current is needed for comparison with the experimental results; in taking the logarithm of the quantity in braces, the approximation $\ln(1+z)\cong z$ may be used since the second term is small compared to one. When these three operations are performed, the result is

$$\log_{10}j=\log_{10}[\tfrac{1}{2}\pi^{-2}(kT)^2] +(\log_{10}e)(kT)^{-1}(\bar{\mu}+F^{\frac{1}{2}})+F_2, \tag{40}$$

where the first terms give the normal Schottky effect and the primary contribution to the periodic deviations is given by

$$F_2=(\log_{10}e)\pi^{-\frac{1}{2}}e^{2-\frac{1}{2}\gamma}(kT)^{-1}c^{\frac{1}{2}}F^{7/8}|\mu_p| \times\sin\{(4/3)F^{-\frac{1}{4}}+\arg\mu_p+\tfrac{1}{2}\arctan c +\tfrac{1}{2}c\ln[(4+4c^2)(1+4c^2)^{-1}]\}. \tag{41}$$

In these expressions b and d have been written out explicitly to show the dependence on F and μ_p. The parameter c, given by Eq. (34), still depends on the field F but varies slowly compared to the dependences explicitly shown.

This result applies for arbitrary shapes of the potential curve in Regions I and II (Fig. 1) provided that, in agreement with the assumption made following Eq. (26), $|\mu|^2$ is small compared to one in the range $-F^{\frac{1}{2}}(1+F^{\frac{1}{4}})<W<\pi kT$ and provided that, in agreement with the assumption made following Eq. (36), μ does not vary appreciably in the range $-F^{\frac{1}{2}}(1+F^{\frac{1}{4}}) <W<-F^{\frac{1}{2}}(1-2F^{\frac{1}{4}})$. The F-dependent limits here are found by translating the important range of E into terms of W according to Eq. (29).

The μ discussed in Sec. IV satisfies both these conditions, as is seen from the following argument. A

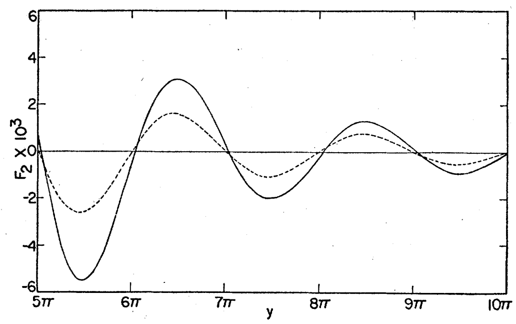

FIG. 3. The periodic deviation from the Schottky effect F_2 as a function of $y=(4/3)F^{-\frac{1}{4}}$, based on the potential shown in Fig. 1 and for $W_a=10$ electron volts and $T=1500°$K. The solid line represents the results of the present calculations; the dotted line is a plot of the results of Juenker, Colladay, and Coomes.

representative value of the temperature T is 1500°K, of W_a is 10 electron volts, and the fields are ordinarily less than 10^6 volts/cm. Therefore $FW_a^{-2}<0.001$ and also, since $|W|<0.015$ within the ranges discussed in the above paragraph, $|WW_a^{-1}|<0.04$. Accordingly the higher order terms in Eq. (25) may be disregarded. The value of $|\mu|^2$ thus never departs appreciably from its value when $W=0$ and $F=0$; from Fig. 2 it is seen that this is much less than one as required. Furthermore, μ is slowly varying with W in the required region and evidently

$$\mu_p=\left[\frac{H_0^{(2)}(z)+iH_1^{(2)}(z)}{-H_1^{(1)}(z)+iH_0^{(1)}(z)}\right]_{z=(2W_a)^{-\frac{1}{2}}}. \qquad (42)$$

VI. DISCUSSION

On the basis of the model in Fig. 1, the periodic deviation predicted is given in Eqs. (41) and (42). The phase prediction agrees with that of Juenker, Colladay, and Coomes[2] as is seen from Fig. 3, in which the periodic deviations are plotted for a particular case. The only temperature dependence is the T^{-1} factor in the amplitude, and the primary field dependence of the phase is the $(4/3)F^{-\frac{1}{4}}$ term; these features agree with the results of Juenker, Colladay, and Coomes as well as with the original work of Guth and Mullin.[3] The field dependence of the amplitude is contained in the factor $c^{\frac{1}{2}}F^{7/8}$, where c is defined by Eq. (34). This differs slightly from the previous calculations which gave an $F^{\frac{3}{4}}$ dependence.

Thus, except for the difference in the amplitude, the present results are the same as those of Juenker, Colladay, and Coomes and their analysis of the agreement with experiment still applies. Especially they have re-emphasized that the predicted phase seems to differ from the experimental phase by about 90° for all metals and for all values of the field studied so far and that the difference may be due to a wrong choice of the model for calculating μ. The amplitude found here appears to be in better agreement with the experimental amplitudes. However, this improvement may not be significant because if a different model should correct the phase through the $\arg\mu_p$ term, it might also affect the amplitude through the $|\mu_p|$ factor. Also, Brock, Houde, and Coomes[4] have pointed out that, because of patch effects, measurements of the amplitude are less certain than measurements of the phase.

An estimate of the nonperiodic deviation from the Schottky effect may be obtained from a more detailed discussion of the first term in the square brackets in Eq. (26). If, as indicated by the discussion preceding Eq. (42), it may be assumed that $|\mu|^2$ may be taken constant in the range $-F^{\frac{1}{2}}(1+F^{\frac{1}{4}})<W<\pi kT$, then the nonperiodic part of the average transmission coefficient is

$$(1-\bar{r}_v)_{NP}=\frac{1-|\mu_p|^2}{kT}\int_V^\infty\frac{e^{\pi E}\exp\{-(W-V)(kT)^{-1}\}}{1-|\mu_p|^2+e^{\pi E}}dW. \qquad (43)$$

This integral may be evaluated by the same methods as used in Sec. V; using Eq. (29) to change the integration variable from W to E, extending the lower limit to $-\infty$, and integrating on $(1-|\mu_p|^2)^{-1}e^{\pi E}$, one finds

$$(1-\bar{r}_v)_{NP}=\frac{1-|\mu_p|^2}{(1-|\mu_p|^2)^d}\frac{\pi d}{\sin\pi d}\exp\left(\frac{V+F^{\frac{1}{2}}}{kT}\right)$$
$$\cong[1-|\mu_p|^2(1-d)+\tfrac{1}{6}\pi^2d^2]\times\exp\{(V+F^{\frac{1}{2}})(kT)^{-1}\}. \qquad (44)$$

In the last step an expansion for small $|\mu_p|^2$ and d has been made and enough terms have been retained to show the primary dependence on the field regardless of whether $|\mu_p|^2$ is large or small compared to d. The final result is that the expression for $\log_{10}j$ in Eq. (40) is to be increased by the term

$$F_1=-(\log_{10}e)|\mu_p|^2[1-(\pi kT)^{-1}F^{\frac{3}{4}}]+\tfrac{1}{6}(\log_{10}e)(kT)^{-2}F^{\frac{3}{2}}, \qquad (45)$$

if the nonperiodic deviation also is required. This is smaller than the first periodic deviation and is ordinarily not observed.