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The Dynamics of a Disordered Linear Chain*

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By a disordered chain we mean a chain of one-dimensional harmonic oscillators, each coupled to its nearest neighbors by harmonic forces, the inertia of each oscillator and the strength of each coupling being a random variable with a known statistical distribution law. A method is presented for calculating exactly the distribution-function of the frequencies of normal modes of vibration of such a chain, in the limit when the chain becomes infinitely long. For some special examples, in which the distribution law of the oscillator parameters is assumed to be of exponential form, the frequency spectra are calculated analytically. The theory applies equally well to a chain of masses connected by elastic springs and making mechanical vibrations, or to an electrical transmission line composed of alternating inductances and capacitances with random characteristics.

I. INTRODUCTION

CONSIDER a chain of N masses, each coupled to its nearest neighbors by elastic springs obeying Hooke's law. We shall study the longitudinal vibrations of the chain, all motions being supposed to take place in one dimension so that each mass is described by a single coordinate. Since the coupling forces are linear in the displacements, the most general vibration is a superposition of $(N-1)$ normal modes, each having a characteristic frequency. The object of this paper is to present a method for determining accurately the spectrum or distribution function of the characteristic frequencies of the chain, in the limit as the number of masses N becomes very large. As is well known, the knowledge of this distribution function enables all the thermodynamical properties of the chain to be deduced immediately.

In the case when the masses and the strengths of the springs are all equal, the calculation of the frequency spectrum is elementary. In Sec. IV of this paper we give explicit formulas for the frequency spectrum in the most general case when the masses and spring-constants are arbitrary. The case of equal masses and springs here serves as a check.

In Secs. V–VI we consider the physically interesting case in which the masses and springs are unequal but are distributed along the chain in a random way. This means that we know the probability that a given mass or spring-constant has a particular value, and that this probability is the same at every point in the chain. There are several different ways of defining precisely how the randomization of the masses and springs is to be understood; for example, the masses may be independent random variables with a known probability distribution, while the springs are all equal; or each mass may be correlated with the strengths of the two neighboring springs, and so on. In every case, given the probability distribution for masses and springs, our method leads to an exact determination of the spectrum of normal frequencies. To illustrate the method, one particular class of probability distributions is worked out in detail, and the corresponding frequency spectra are obtained explicitly.

These calculations were begun in response to a question of C. Kittel, who was concerned with the thermal properties of glass. Glass may be considered roughly to be a disordered array of coupled harmonic oscillators in 3 dimensions. The systems considered in this paper are models of a "one-dimensional glass," a disordered array of atoms in one dimension. It is not, of course, to be expected that the results of this paper have any immediate application to the 3-dimensional problem. But it seems to us remarkable that the 1-dimensional problem can be solved exactly, and we publish this analysis in the hope that the methods will be useful in

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discussing other disordered systems of a less idealized character.

II. DEFINITIONS

Let particle number j in the chain have mass m_j, and let its displacement from its equilibrium position be x_j. Let the elastic modulus of the spring between particles j and $(j+1)$ be K_j. Then the equations of motion of the system are

$$m_j\ddot{x}_j=K_j(x_{j+1}-x_j)+K_{j-1}(x_{j-1}-x_j). \tag{1}$$

It is convenient to introduce new variables

$$y_j=m_j^{\frac{1}{2}}x_j \tag{2}$$

and new constants $\lambda_1, \lambda_2, \cdots, \lambda_{2N-2}$ given by

$$\lambda_{2j-1}=K_j/m_j, \quad \lambda_{2j}=K_j/m_{j+1}. \tag{3}$$

Then the equations of motion take the form

$$\ddot{y}_j=(\lambda_{2j-1}\lambda_{2j})^{\frac{1}{2}}y_{j+1}+(\lambda_{2j-3}\lambda_{2j-2})^{\frac{1}{2}}y_{j-1} -(\lambda_{2j-1}+\lambda_{2j-2})y_j, \tag{4}$$

and the coefficient-matrix is now symmetric. Next we define variables $z_1, z_2, \cdots, z_{N-1}$, by

$$\dot{z}_j=\lambda_{2j}^{\frac{1}{2}}y_{j+1}-\lambda_{2j-1}^{\frac{1}{2}}y_j, \tag{5}$$

so that (4) becomes

$$\dot{y}_j=\lambda_{2j-1}^{\frac{1}{2}}z_j-\lambda_{2j-2}^{\frac{1}{2}}z_{j-1}. \tag{6}$$

Let variables $u_1, u_2, \cdots, u_{2N-1}$ be defined by

$$u_{2j-1}=y_j, \quad u_{2j}=z_j. \tag{7}$$

Then Eqs. (5) and (6) together may be written

$$\dot{u}_j=\lambda_j^{\frac{1}{2}}u_{j+1}-\lambda_{j-1}^{\frac{1}{2}}u_{j-1}. \tag{8}$$

The characteristic frequencies ω_j of the chain are, therefore, the characteristic roots of the $(2N-1)\times(2N-1)$ matrix Λ whose elements are given by

$$\Lambda_{j+1,\,j}=-\Lambda_{j,\,j+1}=i\lambda_j^{\frac{1}{2}}, \tag{9}$$

all other elements being zero. There is one zero root corresponding to the degenerate mode in which all the x_j are equal; the remaining roots occur in $(N-1)$ pairs, the members of a pair being $+\omega_j$ and $-\omega_j$.

The spectrum of characteristic frequencies is given by the function $M(\mu)$ which is defined as the proportion of the roots ω_j for which $\omega_j^2\leq\mu$. As $N\rightarrow\infty$ we expect that $M(\mu)$ will become a smooth differentiable function, and then a density of characteristic frequencies can be defined by

$$D(\mu)=(dM/d\mu). \tag{10}$$

Our first task is to determine the $M(\mu)$ and $D(\mu)$ corresponding to given λ_j.

We study the function

$$\Omega(x)=\lim_{N\rightarrow\infty}(2N-1)^{-1}\textstyle\sum_j\log(1+x\omega_j^2) =\displaystyle\int_0^\infty\log(1+x\mu)D(\mu)d\mu, \tag{11}$$

as a function of the complex variable x. The logarithm is defined as the branch of the function which is real for real positive x. Then the integral (11) is convergent and defines an analytic function of x over the whole x plane, the negative real axis from 0 to $(-\infty)$ being excluded. As x tends from above onto a point $(-z)$ on the negative real axis, the imaginary part of $\log(1+x\mu)$ tends to zero if $z\mu<1$ and to $i\pi$ if $z\mu>1$. Therefore, (11) gives

$$\mathrm{Re}[(i\pi)^{-1}\lim_{\epsilon\rightarrow0}\Omega(-z+i\epsilon)] =\int_{1/z}^\infty D(\mu)d\mu=1-M(1/z), \tag{12}$$

$$D(1/z)=-z^2\,\mathrm{Re}[(i\pi)^{-1}\lim_{\epsilon\rightarrow0}\Omega'(-z+i\epsilon)], \tag{13}$$

$$\Omega'(x)=(d\Omega/dx)=\int_0^\infty\mu(1+x\mu)^{-1}D(\mu)d\mu. \tag{14}$$

According to (12) or (13), the spectrum of characteristic frequencies is determined by the limiting values of $\Omega(x)$ on the negative real axis. We call $\Omega(x)$ the characteristic function of the chain.

III. CALCULATION OF THE CHARACTERISTIC FUNCTION[1]

Consider a chain for which all the ω_j^2 are less than a fixed bound B, and let $B|x|<1$; these restrictions will be removed later. Then the logarithm in (11) may be expanded in powers of x, giving

$$\Omega(x)=\lim_{N\rightarrow\infty}(2N-1)^{-1}S(x), \tag{15}$$

$$S(x)=\sum_{n=1}^\infty(-1)^{n-1}(x^n/n)\sum_j\omega_j^{2n} =\sum_{n=1}^\infty(-1)^{n-1}(x^n/n)\,\mathrm{Spur}(\Lambda^{2n}). \tag{16}$$

According to (9), the spur of Λ^{2n} is a sum of terms $Q(\sigma)$, one corresponding to each cycle σ, a cycle consisting of $2n$ integers

$$j_0, j_1, j_2, \cdots, j_{2n-1}, j_{2n}=j_0 \tag{17}$$

lying between 1 and $2N-1$ and satisfying for each m

$$j_{m+1}=j_m\pm1. \tag{18}$$

[1] The analysis of this section might have been shortened by using known results in the theory of Jacobi matrices. See A. Wintner, *Spektraltheorie der Unendlichen Matrizen* (Leipzig, 1929), pp. 69–73. For this remark the author is indebted to Professor Kac.

The term $Q(\sigma)$ is equal to the product

$$\prod_{j=1}^{2N-1} (\lambda_j)^{q(j)}, \tag{19}$$

where $q(j)$ is the number of times that the step $(j\rightarrow j+1)$ occurs in the cycle σ; the step $(j+1\rightarrow j)$ must also occur $q(j)$ times in σ. Hence, we may write

$$\mathrm{Spur}(\Lambda^{2n}) = \sum_q R(q) \prod_j (\lambda_j)^{q(j)}, \tag{20}$$

where the summation is over all possible sets of integers $q(j)$ whose sum is n, and $R(q)$ is the number of cycles σ which exist corresponding to a given set of $q(j)$. The evaluation of $R(q)$ is carried out in Appendix I. The result is as follows. Let $q(a)$ be the first nonvanishing $q(j)$, and let $q(b)$ be the last. Then

$$R(q) = 2n[q(a)]^{-1}\left[\prod_{j=a+1}^{b} L(j)\right], \tag{21}$$

$$L(j) = (q(j)+q(j-1)-1)!/[(q(j))!(q(j-1)-1)!]. \tag{22}$$

If we substitute from (20) and (21) into (16), we find

$$S(x) = \sum_{a,b} S(a,b), \tag{23}$$

$$S(a,b) = -2\sum_q [q(a)]^{-1}(-x\lambda_a)^{q(a)} \times\left[\prod_{j=a+1}^{b} L(j)(-x\lambda_j)^{q(j)}\right]. \tag{24}$$

In Eq. (23) the summation is over all integers a, b satisfying $1\leq a\leq b\leq 2N-1$. In Eq. (24) the summation is over all sets of integers $q(j)$ which are nonzero for $a\leq j\leq b$. That is to say, each $q(j)$ for $a\leq j\leq b$ is summed over all positive integral values.

The summation over $q(b)$ in Eq. (24) can immediately be performed, because by Eq. (22) the sum is an elementary binomial expansion with exponent $(-q(b-1))$. Therefore,

$$S(a,b) = -2\sum_q{}' (q(a))^{-1}(-x\lambda_a)^{q(a)} \times\left[\prod_{j=a+1}^{b-1} L(j)(-x\lambda_j)^{q(j)}\right][(1+x\lambda_b)^{-q(b-1)}-1], \tag{25}$$

where the summation is now over all sets of integers $q(j)$ which are nonzero on $a\leq j\leq b-1$. That is to say,

$$S(a,b)+S(a,b-1) = S_1(a,b-1), \tag{26}$$

where $S_1(a,b-1)$ is the sum $S(a,b-1)$ with the variable $x\lambda_{b-1}$ replaced by $[x\lambda_{b-1}/(1+x\lambda_b)]$. If we apply the same reduction to $S_1(a,b-1)$, we obtain

$$S(a,b)+S(a,b-1)+S(a,b-2) = S_2(a,b-2), \tag{27}$$

where $S_2(a,b-2)$ is $S(a,b-2)$ with $x\lambda_{b-2}$ replaced by $[x\lambda_{b-2}/(1+x\lambda_{b-1}/(1+x\lambda_b))]$. Continuing in this way, we find

$$\sum_{j=a}^{b} S(a,j) = S_{b-a}(a,a), \tag{28}$$

where $S_{b-a}(a,a)$ is $S(a,a)$ with $x\lambda_a$ replaced by the continued fraction

$$\xi(a,b) = x\lambda_a/(1+x\lambda_{a+1}/[\cdots(1+x\lambda_{b-1}/(1+x\lambda_b))\cdots]). \tag{29}$$

Now from (24),

$$S(a,a) = -2\sum_{q(a)}[q(a)]^{-1}(-x\lambda_a)^{q(a)} = 2\log(1+x\lambda_a). \tag{30}$$

Therefore,

$$\sum_{j=a}^{b} S(a,j) = 2\log(1+\xi(a,b)). \tag{31}$$

We may now write $b=2N-1$ and make $N\rightarrow\infty$ in (29) and (31). This gives

$$\sum_{j=a}^{\infty} S(a,j) = 2\log(1+\xi(a)), \tag{32}$$

where $\xi(a)$ is the infinite continued fraction

$$\xi(a) = x\lambda_a/(1+x\lambda_{a+1}/(1+x\lambda_{a+2}/(\cdots. \tag{33}$$

If we sum over all a and use (15) and (23),

$$\Omega(x) = \lim_{N\rightarrow\infty} N^{-1}\sum_{a=1}^{2N-1}\log[1+\xi(a)]. \tag{34}$$

Equations (33) and (34) give the explicit representation of $\Omega(x)$ for an arbitrary chain with given coefficients λ_j. The result was proved only under the assumptions that all the ω_j^2 were less than a fixed bound B and that $B|x|<1$. However, the integral (11) defining $\Omega(x)$, and the continued fractions (33), are convergent for all real positive x and for any set of positive coefficients λ_j. Since these expressions are analytic functions of x, it is easy to prove by an analytic continuation argument that Eq. (34) holds for all real positive x and without restriction on the ω_j.

IV. THE FREQUENCY SPECTRUM FOR AN ARBITRARY CHAIN

To determine the frequency spectrum for a chain with given λ_j, it is only necessary to calculate $\Omega(x)$ from Eqs. (33) and (34), and then to use (13) or (14) to find the spectrum. The second step, however, requires the analytic continuation of $\Omega(x)$ from positive to negative real values. If $\Omega(x)$ can be obtained in closed analytic form for positive x, the continuation process is usually easy. For example, consider the case of an infinite chain of equal masses m linked by springs of equal modulus K. In this case all λ_j are equal to $\lambda=(K/m)$, and all $\xi(a)$ are equal to the infinite continued fraction

$$\xi = x\lambda/(1+x\lambda/(\cdots. \tag{35}$$

Since ξ satisfies

$$\xi = x\lambda/(1+\xi) \tag{36}$$

and is small for small x, we find

$$\xi = \tfrac{1}{2}[(1+4x\lambda)^{\frac{1}{2}}-1], \tag{37}$$

and by (34),

$$\Omega(x) = 2\log\{\tfrac{1}{2}[(1+4x\lambda)^{\frac{1}{2}}+1]\}. \tag{38}$$

If we differentiate (38) we obtain

$$\Omega'(x) = x^{-1}[1-(1+4x\lambda)^{-\frac{1}{2}}]. \tag{39}$$

This function is analytic and real for $x > -(4\lambda)^{-1}$. It can be continued analytically through the upper half plane to values $x < -(4\lambda)^{-1}$, where it becomes

$$\Omega'(x) = x^{-1}[1+i(-1-4x\lambda)^{-\frac{1}{2}}]. \tag{40}$$

Hence, Eq. (13) gives the spectrum of characteristic frequencies

$$D(\mu) = (1/\pi)(4\lambda\mu-\mu^2)^{-\frac{1}{2}}, \quad \mu < 4\lambda,$$
$$D(\mu) = 0, \quad \mu > 4\lambda, \tag{41}$$

a result which in this case can easily be checked by an elementary calculation.

In general we will not be able to calculate $\Omega'(x)$ in closed analytic form, and so we require an explicit formula for $D(\mu)$ in terms of the values of $\Omega'(x)$ for positive x. This formula will enable $D(\mu)$ to be calculated if $\Omega'(x)$ is only given numerically or approximately, so that a direct use of analytic continuation is impossible. The formula is

$$D(\mu) = (2\pi^2\mu)^{-1}\int_{-\infty}^{\infty}(\cosh\pi\alpha)d\alpha \times \int_0^{\infty}(x\mu)^{-\frac{1}{2}}\cos[\alpha\log(x\mu)]\Omega'(x)dx. \tag{42}$$

Its derivation is given in Appendix II. The x integration is to be carried out first and the α integration second. Taken in this order, the double integration will always be convergent for values of μ at which $D(\mu)$ exists and is continuous. With Eqs. (33), (34), and (42), we have in principle an exact determination of the frequency spectrum of an arbitrary chain.

V. DISORDERED CHAINS

We now apply the preceding theory to the case of a disordered chain, i.e., an infinite chain whose elements are distributed in a random way according to some known probability law. We consider two types of disordered chain, differing in the way in which the randomization of the elements is defined. Type I is mathematically the simpler, whereas Type II provides the closer approximation to a real chain of randomly arranged atoms.

Type I. Each of the parameters λ_j *defined by Eq. (3) is an independent random variable with the probability distribution function* $G(\lambda)$.

In this case each of the quantities $\xi(a)$ defined by Eq. (33) will have a probability distribution $F(\xi)$, the same for all a. Now

$$\xi(a) = x\lambda_a/(1+\xi(a+1)) \tag{43}$$

and the variables λ_a and $\xi(a+1)$ are uncorrelated, since $\xi(a+1)$ depends only on $\lambda_{a+1}, \lambda_{a+2}, \cdots$. If we equate the probability distributions of the left and right sides of Eq. (43), we find an integral equation for $F(\xi)$,

$$F(\xi) = \int_0^{\infty} F(\xi')G[\xi(1+\xi')/x]((1+\xi')/x)d\xi'. \tag{44}$$

The solution of (44) for given $G(\lambda)$ can in some cases be obtained in closed form (see Sec. VI). In all cases the solution can be obtained numerically by inserting an arbitrary trial function $F(\xi')$ on the right of (44) and iterating the equation repeatedly. The series of successive iterates will converge rapidly to the true $F(\xi)$, in consequence of the good convergence of the continued fractions (33). When we have found the solution $F(\xi)$ of (44), and have normalized it by

$$\int_0^{\infty} F(\xi)d\xi = 1, \tag{45}$$

the characteristic function of the chain is given by Eq. (34) and is

$$\Omega(x) = 2\int_0^{\infty} F(\xi)\log(1+\xi)d\xi. \tag{46}$$

From this the frequency spectrum may be found as described in Sec. IV.

The frequency spectrum determined in this way is strictly an average over a statistical ensemble of chains, each chain in the ensemble having definite values of the λ_j. But by an argument familiar in the statistical mechanics of systems containing many particles, the same frequency spectrum will be found for an arbitrary chain chosen out of the ensemble, except for an exceptional class of chains whose total probability tends to zero as the number of atoms N tends to infinity. Therefore, we may say that this frequency spectrum is the correct spectrum for a single disordered chain of infinite length.

Type II. Each mass m_j *is an independent random variable with distribution function* $G(m)$, *the spring-constants* K_j *being fixed and equal.*

In this case the λ_j are given by

$$\lambda_{2j-1} = \lambda_{2j-2} = Km_j^{-1}. \tag{47}$$

We introduce the variables

$$\eta_j = [\xi(2j)]^{-1}. \tag{48}$$

Then, from Eq. (33) we derive the recurrence formula

$$\eta_j = (m_{j+1}/xK) + [\eta_{j+1}/(1+\eta_{j+1})]. \tag{49}$$

The variables η_{j+1}, m_{j+1} now are uncorrelated. Hence, Eq. (49) leads to an integral equation

$$F(\eta)=xK\int_0^\infty G[xK(\eta-(\eta'/(1+\eta')))]F(\eta')d\eta', \quad (50)$$

defining the distribution function $F(\eta)$ of each of the η_j. The characteristic function (34) is given now by

$$\Omega(x)=\lim N^{-1}\sum_{j=1}^{N}\log[(1+\xi(2j))(1+\xi(2j-1))], \quad (51)$$

since $\xi(2j)$ and $\xi(2j-1)$ will have different distribution functions. From Eq. (33) we find

$$(1+\xi(2j))(1+\xi(2j-1))=1+\xi(2j)+(xK/m_j), \quad (52)$$

and here the variables $\xi(2j)$ and m_j are uncorrelated. If we insert Eq. (52) into (51), the characteristic function becomes

$$\Omega(x)=\int_0^\infty F(\eta)d\eta\int_0^\infty G(m)dm\log[1+\eta^{-1}+xKm^{-1}], \quad (53)$$

where $F(\eta)$ is defined as the solution of (50).

An interesting special case of a Type II chain is a chain composed of two kinds of atoms with masses m, M, distributed at random in the proportion $p:(1-p)$. This corresponds to the distribution function

$$G(m')=p\delta(m'-m)+(1-p)\delta(m'-M). \quad (54)$$

The equation (50) for $F(\eta)$ then reduces to a difference equation,

$$\begin{aligned}F(\eta)=p[1-\eta+(m/xK)]^{-2}F[(1-\eta+(m/xK))^{-1}-1]\\+(1-p)[1-\eta+(M/xK)]^{-2}\\\times F[(1-\eta+(M/xK))^{-1}-1], \quad (55)\end{aligned}$$

which can be solved very rapidly by iteration since no integration is involved. The characteristic function (53) becomes

$$\begin{aligned}\Omega(x)=\int_0^\infty F(\eta)d\eta[p\log(1+\eta^{-1}+xKm^{-1})\\+(1-p)\log(1+\eta^{-1}+xKM^{-1})]. \quad (56)\end{aligned}$$

Only a moderate amount of numerical work would be needed to calculate the frequency spectrum for any given values of m, M and p.

A chain in which the masses are all equal while the spring strengths are independent random variables can be treated by exactly the same method as a Type II chain. It is only necessary to interchange the roles of $\xi(2j)$ and $\xi(2j-1)$. If we go beyond Types I and II, we could consider a more general type of chain, in which the masses m_j are independent random variables, while the spring strength K_j is a known function of the two adjoining masses m_j and m_{j+1}. This would be a model for a chain composed of different kinds of atoms arranged at random, the strength of the bond between two atoms depending on the chemical nature of the two atoms. The frequency spectrum of such a chain can also be calculated by the methods of this paper, only the formulas become rather more complicated.

VI. A SPECIAL FAMILY OF DISORDERED CHAINS

In this section we calculate explicitly the spectrum of normal frequencies for a special family of disordered chains. This will serve as an illustration, to show quantitatively the effect which a given degree of disorder has upon the spectrum. We consider a chain C_n of Type I, in which each λ_j is an independent random variable with the probability distribution

$$G_n(\lambda)=[n^n/(n-1)!]\lambda^{n-1}e^{-n\lambda}. \quad (57)$$

The integer n takes the values 1, 2, 3, $\cdots$. The distribution G_n has mean value 1 and standard deviation $n^{-\frac{1}{2}}$. Thus, C_1 is a highly disordered chain, C_2 is less disordered, and in the limit as $n\rightarrow\infty$, C_n becomes the uniform chain with all $\lambda_j=1$. For large n Eq. (57) takes asymptotically the Gaussian form

$$G_n(\lambda)\sim(n/2\pi)^{\frac{1}{2}}\exp[-\tfrac{1}{2}n(\lambda-1)^2]. \quad (58)$$

We choose these C_n for the illustration for reasons of mathematical convenience only. They happen to have frequency spectra which can be calculated to the end analytically. And although they do not correspond closely to any known physical situation, they illustrate clearly enough the behavior of disordered chains in general.

If we substitute Eq. (57) into Eq. (44), we find an integral equation for $F(\xi)$ which has the exact solution

$$F_n(\xi)=K_n^{-1}\xi^{n-1}(1+\xi)^{-n}e^{-n\xi/x}, \quad (59)$$

where K_n is a function of x determined by the normalization condition (45). Hence, Eq. (46) gives

$$\Omega_n(x)=2L_n(x)/K_n(x), \quad (60)$$

$$L_n(x)=\int_0^\infty \xi^{n-1}(1+\xi)^{-n}\log(1+\xi)e^{-n\xi/x}d\xi, \quad (61)$$

$$K_n(x)=\int_0^\infty \xi^{n-1}(1+\xi)^{-n}e^{-n\xi/x}d\xi. \quad (62)$$

We next have to carry out the analytic continuation of L_n and K_n to negative x. This is done in Appendix III. Hence, if we use Eq. (12), we find the following analytic expression for the frequency spectrum of C_n,

$$M_n(z)=\frac{G_1^2-F_0G_2e^{-nz}+[F_0F_2-F_1^2+((\pi^2/6)-t_{n-1})F_0^2]e^{-2nz}}{[G_1+(F_1-(\log nz+s_{n-1}+\gamma)F_0)e^{-nz}]^2+\pi^2F_0^2e^{-2nz}}. \quad (63)$$

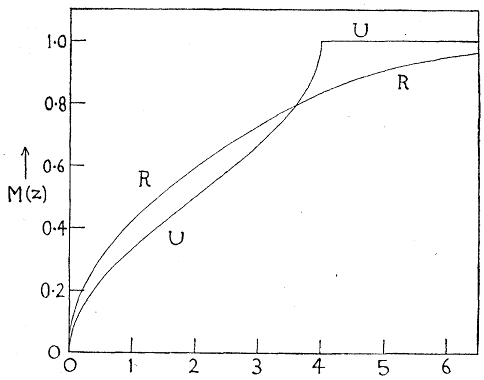

FIG. 1. Integral spectrum of characteristic frequencies for a uniform chain (curve UU) and for a random chain in which each spring-parameter λ has the distribution-function $\exp(-\lambda)$ (curve RR). $M(z)$ gives the proportion of frequencies ω for which $\omega^2\leq z$.

Here, F_0, F_1, and F_2 are finite polynomials defined by

$$F_0=\sum_{j=0}^{n-1}\binom{n-1}{j}(-nz)^j/j!, \tag{64}$$

$$F_1=\sum_{j=0}^{n-1}\binom{n-1}{j}[(-nz)^j/j!]s_j, \tag{65}$$

$$F_2=\sum_{j=0}^{n-1}\binom{n-1}{j}[(-nz)^j/j!](s_j^2+t_j), \tag{66}$$

with

$$s_j=\sum_{l=1}^{j}l^{-1},\quad t_j=\sum_{l=1}^{j}l^{-2}. \tag{67}$$

The G_1 and G_2 are integral functions of z with power-series expansions

$$G_1=\sum_{j=0}^{\infty}\binom{n-1+j}{j}[(-nz)^j/j!]s_j, \tag{68}$$

$$G_2=\sum_{j=0}^{\infty}\binom{n-1+j}{j}[(-nz)^j/j!](s_j^2+t_j), \tag{69}$$

and γ is Euler's constant. Asymptotically for large z, G_1 and G_2 have semiconvergent series expansions

$$G_1=[(-1)^n/(n-1)!]\sum_{j=n}^{\infty}(nz)^{-j}[(j-1)!]^2/(j-n)!, \tag{70}$$

$$G_2=2[(-1)^n/(n-1)!]\sum_{j=n}^{\infty}(nz)^{-j}\times[(j-1)!]^2 s_{j-1}/(j-n)!. \tag{71}$$

From Eq. (63) we can find at once the limiting behavior of M_n for small z,

$$M_n(z)\sim[(\pi^2/6)-t_{n-1}][\pi^2+(\log nz+s_{n-1}+\gamma)^2]^{-1}. \tag{72}$$

This rises very rapidly as z increases from zero, showing that a disordered chain has a much greater proportion of very low characteristic frequencies than a uniform chain. The limiting behavior of M_n for large z is found from Eqs. (70) and (71) to be

$$M_n(z)\sim 1-2(\log nz-s_{n-1}+\gamma)e^{-nz}\times(nz)^{2n-1}[(n-1)!]^{-2}. \tag{73}$$

In Fig. 1 the numerical value of $M_1(z)$ is plotted together with the corresponding function for the uniform chain which is by Eq. (41)

$$M_\infty(z)=\pi^{-1}\arccos[1-\tfrac{1}{2}z],\quad z<4,$$
$$=1,\quad z>4. \tag{74}$$

From Eq. (63) we can also calculate the form of the frequency spectrum for large n, i.e., for a chain composed of only slightly fluctuating elements. In this case we represent the spectrum by a complex contour integral which is evaluated by the method of steepest descent; details are given in Appendix IV. The results are the following. For any fixed z in the range $0<z<4$, we have for sufficiently large n,

$$M_n(z)\sim\pi^{-1}\arccos[1-\tfrac{1}{2}z]+(2\pi n)^{-1}[(4/z)-1]^{-\frac{1}{2}}. \tag{75}$$

For fixed $z>4$ and large n,

$$M_n(z)\sim 1-\pi^{-1}\alpha\exp[-\alpha-2n(\sinh\alpha-\alpha)], \tag{76}$$

$$\alpha=\arg\cosh[\tfrac{1}{2}z-1]. \tag{77}$$

At the critical point $z=4$ and for large n,

$$M_n(z)\sim 1-[\Gamma(\tfrac{1}{3})]^{-2}[12/n]^{\frac{1}{3}}=1-(0.32\cdots)n^{-\frac{1}{3}}. \tag{78}$$

The errors in Eqs. (75), (76), (78) are in each case of higher order in $(1/n)$ than the last term given. From these results it is seen how the spectrum approaches the limiting form (74) as $n\to\infty$.

VII. APPLICATION TO TRANSMISSION LINES

Consider an ideal loss-free transmission line composed of a series of inductances L_1, L_2, $\cdots$ with a capacitance C_j between each pair L_j and L_{j+1}. The equation of motion for the current I_j in the inductance L_j is

$$L_j d^2I_j/dt^2=C_j^{-1}(I_{j+1}-I_j)+C_{j-1}^{-1}(I_{j-1}-I_j). \tag{79}$$

This is identical with Eq. (1), only with L_j replacing m_j and C_j^{-1} replacing K_j. Therefore, the whole of the theory of this paper applies without alteration to a transmission line whose elements L_j and C_j are statistically random variables. The function $M(z)$ gives the integrated frequency distribution of the normal modes of propagation of current in the line.

In conclusion, the author wishes to thank Professor Kittel for suggesting this problem to him, Professor Luttinger for some useful discussions, and the University of California for its hospitality during the summer of 1953 when the work was done.

APPENDIX I. EVALUATION OF $R(q)$

Define $R(q,c)$ to be the number of cycles σ corresponding to a given set of $q(j)$, and beginning and ending with a given $j_0=j_{2n}=c$. Suppose first $c<b$. From each of these σ we can derive a new cycle σ' by simply omitting from σ the steps $(b\to b+1)$ and $(b+1\to b)$, since by hypothesis the step $(b+1\to b+2)$ does not occur in σ. These σ' will all belong to the set of integers q', where $q'(b)=0$ and $q'(j)=q(j)$ for $j<b$. The number of the σ' is $R(q',c)$. From a given σ' we derive the parent σ by inserting $q(b)$ pairs of steps $(b\to b+1)$, $(b+1\to b)$ distributed in any way among the $q(b-1)$ places where the integer b occurs in σ'. So the number of σ corresponding to a given σ' is the number of ways of distributing $q(b)$ identical objects among $q(b-1)$ boxes, which is $L(b)$ defined by Eq. (22). Hence,

$$R(q,c)=L(b)R(q',c). \qquad \text{(A.1)}$$

Proceeding repeatedly in the same way, we shall reduce q to the set of integers q'' defined by $q''(j)=0$ for $j>c$, $q''(j)=q(j)$ for $j\leq c$. We find

$$R(q,c)=\left[\prod_{j=c+1}^{b} L(j)\right]R(q'',c). \qquad \text{(A.2)}$$

We now start reducing q'' from the other end by omitting steps $(a\to a+1)$ and $(a+1\to a)$. In this way we eventually reach q''' defined by $q'''(j)=0$ for $j>c$ or $j<c-1$, $q'''(j)=q(j)$ for $j=c$ or $j=c-1$. We find

$$R(q'',c)=\left[\prod_{j=a+1}^{c-1} L'(j)\right]R(q''',c), \qquad \text{(A.3)}$$

$$L'(j)=(q(j)+q(j-1)-1)!/[(q(j)-1)!(q(j-1))!]. \qquad \text{(A.4)}$$

But $R(q''',c)$ is just the number of ways of arranging $q(c)$ pairs of steps $(c\to c+1)(c+1\to c)$ and $q(c-1)$ pairs $(c\to c-1)(c-1\to c)$, beginning and ending at c. This number is

$$R(q''',c)=(q(c)+q(c-1))!/[(q(c))!(q(c-1))!]. \qquad \text{(A.5)}$$

If we put together (A.2), (A.3), and (A.5), we obtain

$$R(q,c)=(q(c)+q(c-1))(q(a))^{-1}\left[\prod_{j=a+1}^{b} L(j)\right]. \qquad \text{(A.6)}$$

We now sum (A.6) over all values of c from a to $b+1$. Then $\sum q(c)=\sum q(c-1)=n$, and so Eq. (21) of the text is verified.

APPENDIX II. DERIVATION OF EQUATION (42)

Consider the Fourier transform of $x^{\frac{1}{2}}\Omega'(x)$ considered as a function of the variable $\log x$, namely

$$r(\alpha)=\int_{-\infty}^{\infty} x^{\frac{1}{2}}\Omega'(x)\exp[-i\alpha\log x]d(\log x). \qquad \text{(A.7)}$$

If we use Eq. (14), this becomes

$$\begin{aligned} r(\alpha)&=\int_0^{\infty} D(\mu)d\mu\int_{-\infty}^{\infty} dy\,2\mu e^{y-2i\alpha y}[1+\mu e^{2y}]^{-1}\\ &=\int_0^{\infty}\mu^{\frac{1}{2}}D(\mu)d\mu e^{i\alpha\log\mu}\int_{-\infty}^{\infty}e^{-2i\alpha y}\operatorname{sech}y\,dy\\ &=(\pi\operatorname{sech}\pi\alpha)\int_{-\infty}^{\infty}\mu^{\frac{1}{2}}D(\mu)e^{i\alpha\log\mu}d(\log\mu). \qquad \text{(A.8)}\end{aligned}$$

Thus, $[r(\alpha)\cosh\pi\alpha]$ is also the Fourier transform of $\mu^{\frac{1}{2}}D(\mu)$ as a function of $(\log\mu)$. If we invert the transform (A.8), we obtain

$$\mu^{\frac{1}{2}}D(\mu)=(2\pi^2)^{-1}\int_{-\infty}^{\infty}\exp(-i\alpha\log\mu)r(\alpha)\cosh\pi\alpha d\alpha, \qquad \text{(A.9)}$$

and substitution for $r(\alpha)$ from (A.7) gives Eq. (42).

APPENDIX III. ANALYTIC CONTINUATION OF K_n AND L_n

We write

$$I(q)=I(q,n,y)=\int_0^{\infty}\xi^{n-1}(1+\xi)^{q-n}e^{-\xi y}d\xi, \qquad \text{(A.10)}$$

where $y=n/x$. For positive y, this is an analytic function of q for every q, and in particular

$$K_n(x)=I(0),\quad L_n(x)=(dI/dq)_{q=0}. \qquad \text{(A.11)}$$

We evaluate $I(q)$ assuming $q>n-1$, in which case

$$I(q)=J(q)+R(q), \qquad \text{(A.12)}$$

where

$$\begin{aligned} J(q)&=\int_{-1}^{\infty}\xi^{n-1}(1+\xi)^{q-n}e^{-\xi y}d\xi\\ &=e^y\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\int_0^{\infty}z^{q-1-j}e^{-zy}dz\\ &=e^y y^{-q}\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\Gamma(q-j)y^j, \qquad \text{(A.13)}\end{aligned}$$

$$\begin{aligned} R(q)&=-\int_0^1(-\xi)^{n-1}(1-\xi)^{q-n}e^{\xi y}d\xi\\ &=(-1)^n\sum_{j=0}^{\infty}(y^j/j!)\\ &\quad\times[\Gamma(n+j)\Gamma(q-n+1)/\Gamma(q+j+1)]. \qquad \text{(A.14)}\end{aligned}$$

The series in (A.13) and (A.14) are convergent and represent functions of y which are analytic over the whole y plane except for the simple branch point at $y=0$ produced by the factor y^{-q}. As functions of q, $J(q)$ and $R(q)$ are analytic, but each has a simple pole at $q=0$ which only cancels in the sum $I(q)$. Hence, we may evaluate K_n and L_n using (A.13) and (A.14), expanding these expressions in powers of q and retaining terms of order -1, 0 and 1. The terms in q^{-1} cancel as they should. The terms in q^0 and q^1 give, respectively,

$$K_n(x)=e^y\sum_{j=0}^{n-1}\binom{n-1}{j}(y^j/j!)(s_j-\gamma-\log y)+\sum_{j=0}^{\infty}\binom{n+j-1}{j}(y^j/j!)(s_j-s_{n-1}), \quad (A.15)$$

$$2L_n(x)=e^y\sum_{j=0}^{n-1}\binom{n-1}{j}(y^j/j!)[(s_j-\gamma-\log y)^2+t_j+\pi^2/6]-\sum_{j=0}^{\infty}\binom{n+j-1}{j}(y^j/j!)\times[(s_j-s_{n-1})^2+t_j+t_{n-1}], \quad (A.16)$$

in which γ is Euler's constant, and s_j, t_j are defined by Eq. (67). These formulas show K_n and L_n explicitly as analytic functions of $y=n/x$, with a logarithmic branch point at $y=0$. In order to obtain the analytic continuation of K_n and L_n to a point $(-x)$ on the negative real axis, going through the upper half of the x plane, we have only to replace, in Eqs. (A.15) and (A.16), y by $(-y)$ and $\log y$ by $(\log y-i\pi)$. This leads at once, by Eq. (12), to the result stated in Eq. (63).

APPENDIX IV. METHOD OF STEEPEST DESCENT

We consider the analytic continuation of the function $K_n(x)$ given by Eq. (62) through the upper half-plane to a point $x=-z^{-1}$ on the negative real axis. This continuation may be written as a contour integral

$$K_n(-z^{-1})=\int_0^{-\infty}[F(\xi)]^n\xi^{-1}d\xi, \quad (A.17)$$

$$F(\xi)=\xi(1+\xi)^{-1}e^{z\xi}, \quad (A.18)$$

and the path of integration passes from 0 to $(-\infty)$ above the singularity at $\xi=-1$. Similarly, by Eq. (61)

$$L_n(-z^{-1})=\int_0^{-\infty}[F(\xi)]^n\log(1+\xi)\xi^{-1}d\xi. \quad (A.19)$$

We choose the path of integration to pass over the lowest saddle point of $F(\xi)$ between the minima at 0 and $(-\infty)$. The saddle points η are given by the quadratic equation

$$\eta^2+\eta+z^{-1}=0. \quad (A.20)$$

Suppose first $0<z<4$. Then, there is one saddle point

$$\eta=\tfrac{1}{2}[-1+i((4/z)-1)^{\frac{1}{2}}] \quad (A.21)$$

in the upper half-plane. The path of integration crosses this saddle at an angle of 135° to the positive real axis. For large n we calculate K_n and L_n by considering contributions to the integrals from the neighborhood of the saddle only. If we use Eq. (12), this gives the result (75). Next suppose $z>4$. Then there are two saddle points on the real axis

$$\eta_+=\tfrac{1}{2}[-1+(1-(4/z))^{\frac{1}{2}}],\quad \eta_-=\tfrac{1}{2}[-1-(1-(4/z))^{\frac{1}{2}}]. \quad (A.22)$$

The path of integration goes along the real axis over η_+ from 0 to η_-, then breaks off at 90° to the real axis and goes from η_- to $(-\infty)$ above $\xi=-1$. For large n the real parts of K_n and L_n come from the integral near η_+, but the imaginary parts come from contributions in the neighborhood of η_-. In this way we find the result (76), which tends to 1 for large n because $|F(\eta_-)|<|F(\eta_+)|$. Finally, consider the case $z=4$. Then there is one saddle point at $\eta=-\frac{1}{2}$, at which the first two derivatives of $F(\xi)$ vanish. The path of integration goes from 0 to η along the real axis, then breaks off at an angle of 120° and goes to $(-\infty)$. The integral in the neighborhood of η now gives Eq. (78).