$$\mathbf{r}_u=\mathbf{s}_u+\boldsymbol{\lambda}-(\mathbf{s}_u+\boldsymbol{\lambda}\cdot\mathbf{R})\left\{\frac{\mathbf{S}_u(m+H)+\mathbf{R}K_u}{m(m+H)H_u}\right\}+\left\{\sum_v K_v(\mathbf{s}_v+\boldsymbol{\lambda}\cdot\mathbf{R})-\sum_v\frac{([\boldsymbol{\omega}_v\times\mathbf{S}_v]\cdot\mathbf{R})}{m_v+K_v}\right\}$$
$$\times\left\{\frac{\mathbf{S}_u(m+H)+\mathbf{R}K_u}{m^2(m+H)H_u}-\frac{\mathbf{R}}{mH(m+H)}\right\}-\frac{(\boldsymbol{\omega}_u\cdot[\mathbf{S}_u\times\mathbf{R}])\mathbf{S}_u}{mH_u(m_u+H_u)(m_u+K_u)}$$
$$+\frac{(\boldsymbol{\omega}_u\cdot[\mathbf{S}_u\times\mathbf{R}])\mathbf{R}(H_u-K_u)}{m(m+H)H_u(m_u+H_u)(m_u+K_u)}-\frac{[\boldsymbol{\omega}_u\times\mathbf{R}]}{m(m_u+H_u)}$$
$$-\frac{[\boldsymbol{\omega}_u\times\mathbf{S}_u](H-m)}{m(m_u+K_u)(m_u+H_u)}-\frac{2(\boldsymbol{\omega}_u\cdot\mathbf{R})[\mathbf{S}_u\times\mathbf{R}]}{m(m+H)(m_u+K_u)(m_u+H_u)},$$

$$\mathbf{r}=\sum_u\left(\frac{K_u+H_u}{m+H}\right)(\mathbf{s}_u+\boldsymbol{\lambda})-\sum_u\frac{(\mathbf{s}_u+\boldsymbol{\lambda}\cdot\mathbf{R})}{m(m+H)}\mathbf{s}_u-\sum_u\frac{K_u(\mathbf{s}_u+\boldsymbol{\lambda}\cdot\mathbf{R})\mathbf{R}}{mH(m+H)}+\sum_u\frac{(\boldsymbol{\omega}_u\cdot[\mathbf{S}_u\times\mathbf{R}])\mathbf{R}}{m(m+H)H(m_u+K_u)}$$
$$-\frac{2\sum_u(\boldsymbol{\omega}_u\cdot[\mathbf{S}_u\times\mathbf{R}])\mathbf{S}_u}{m(m+H)(m_u+H_u)(m_u+K_u)}+\sum_u\left(\frac{K_u+H_u}{m+H}\right)\left\{-\frac{[\boldsymbol{\omega}_u\times\mathbf{R}]}{m(m_u+H_u)}\right.$$
$$\left.-\frac{[\boldsymbol{\omega}_u\times\mathbf{S}_u](H-m)}{m(m_u+K_u)(m_u+H_u)}-\frac{2(\boldsymbol{\omega}_u\cdot\mathbf{R})[\mathbf{S}_u\times\mathbf{R}]}{m(m+H)(m_u+K_u)(m_u+H_u)}\right\}+\sum_u\left\{\frac{[\boldsymbol{\omega}_u\times\mathbf{R}]}{m(m+H)}-\frac{[\boldsymbol{\omega}_u\times R_u]}{m(m_u+H_u)}\right\}, \quad (5)$$

and, as before, $\mathbf{r}_u-\mathbf{r}$ does not contain $\boldsymbol{\lambda}$.

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Reduction of Relativistic Two-Particle Wave Equations to Approximate Forms. II*

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The method of reduction of two-particle relativistic wave equations (an extension of the Foldy-Wouthuysen method), as given in Part I, was applicable only if $m_{\mathrm{I}}\neq m_{\mathrm{II}}$. Other variants of the procedure, free from this restriction, are developed now. On the basis of a discussion of properties of the matrices involved, it is found that the postulate of an "even-even" transformed Hamiltonian was too far-reaching. The less stringent requirement of a "uU separating" or an "lL separating" $\mathcal{H}_{\mathrm{tr}}$ leads to a whole class of usable transformations, which includes the transformation of Part I as a special case. Another important special case, (that of the "least change" transformation) has been calculated through in detail. Different transformations give different expressions for $\mathcal{H}_{\mathrm{tr}}$, but they coincide after (as a part of the next step of the procedure) the matrices β^{I} and β^{II} are replaced by 1 (or -1). Consequently the reduced wave equation is the same in all cases.

IN a recent paper,[1] hereafter referred to as I,† a method was developed for conversion of relativistic two-particle wave equations from the full (16-component) into an approximate (4-component) form. The procedure consists of two steps: first, a canonical transformation (strictly speaking, a sequence of canonical transformations) is performed with the help of suitable generating functions; then, twelve components of the wave equation are rejected, and only the four upper-upper or the four lower-lower components are retained, namely that quadruple which describes states with both particles possessing positive energy (the other of these two quadruples corresponds to both particles having negative energy). *The same* transformation is required to make either choice possible.

The proposed scheme was patterned after the Foldy-Wouthuysen method for one-body equations.[2] As a matter of fact, the expression for the transformed Hamiltonian in I represents a plausible, though not trivial, generalization of that obtained by Foldy and Wouthuysen. But, remarkably, our method is not

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[1] Z. V. Chraplyvy, Phys. Rev. **91**, 388 (1953). We take over the terminology and notation used there.

† *Errata to* I.—In Eq. (4) the first minus sign is to be replaced by a plus sign. In Eq. (7j) the numerical coefficient is to be 1/8 (not 3/16).

[2] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

applicable, if the two particles happen to have equal masses. Now, it is conceivable that more variants of the reduction procedure could be devised,[3] which, like that of I, would be extensions of the F-W method, but not restricted by the condition $m_{\mathrm{I}} \neq m_{\mathrm{II}}$. The investigation of such possibilities must be preceded by a discussion and classification of matrices.

REPRESENTATION OF MATRICES

The matrices of a two-particle wave equation consist of 256 elements each, labeled by two pairs of indices, Ω_{jkJK} $(j, k, J, K=1, 2, 3, 4)$. In order to represent them by means of two-dimensional arrays, we adopt the following convention: each such array will be subdivided into 16 submatrices; the second pair of subscripts (JK) will indicate the submatrix, the first pair (jk) the position of the element within it. Whenever a matrix can be represented as a direct product of two fourth-rank matrices,[4] our convention amounts to taking the "left" direct product[5] denoted by $\cdot\times$. Also, we consistently arrange the components ψ_{kK} of the ψ spinor in a column (rather than a square array), with K specifying one of its four subcolumns, and k the position within the subcolumn.

Examples:

$$\beta^{\mathrm{I}}=\beta\cdot\times\delta=\begin{bmatrix}\beta&0&0&0\\0&\beta&0&0\\0&0&\beta&0\\0&0&0&\beta\end{bmatrix},$$

$$\beta^{\mathrm{II}}=\delta\cdot\times\beta=\begin{bmatrix}\delta&0&0&0\\0&\delta&0&0\\0&0&-\delta&0\\0&0&0&-\delta\end{bmatrix},$$

$$\alpha_z{}^{\mathrm{I}}=\alpha_z\cdot\times\delta=\begin{bmatrix}\alpha_z&0&0&0\\0&\alpha_z&0&0\\0&0&\alpha_z&0\\0&0&0&\alpha_z\end{bmatrix},$$

$$\alpha_z{}^{\mathrm{II}}=\delta\cdot\times\alpha_z=\begin{bmatrix}0&0&\delta&0\\0&0&0&-\delta\\\delta&0&0&0\\0&-\delta&0&0\end{bmatrix},$$

$$\begin{aligned}\alpha_y{}^{\mathrm{I}}\alpha_z{}^{\mathrm{II}}&=(\alpha_y\cdot\times\delta)(\delta\cdot\times\alpha_z)\\&=\alpha_y\cdot\times\alpha_z=\end{aligned}\begin{bmatrix}0&0&\alpha_y&0\\0&0&0&-\alpha_y\\\alpha_y&0&0&0\\0&-\alpha_y&0&0\end{bmatrix},$$

$$\begin{aligned}\alpha_y{}^{\mathrm{II}}\alpha_x{}^{\mathrm{I}}&=(\delta\cdot\times\alpha_y)(\alpha_z\cdot\times\delta)\\&=\alpha_z\cdot\times\alpha_y=\end{aligned}\begin{bmatrix}0&0&0&-i\alpha_z\\0&0&i\alpha_z&0\\0&-i\alpha_z&0&0\\i\alpha_z&0&0&0\end{bmatrix}.$$

[3] This is quite independent of the question as to whether the original F-W method itself is unique for its purposes. A note on this subject will be published soon.

[4] This is not necessarily so in all cases, in particular following certain transformations to be introduced later.

[5] See C. C. MacDuffee, *The Theory of Matrices* (Chelsea Publishing Company, New York, 1946), p. 81.

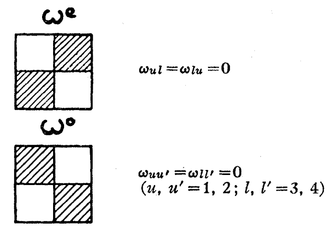

FIG. 1. The two basic types of 4×4 matrices: even and odd. At least all the shaded areas are occupied by zero elements.

($\alpha_x, \alpha_y, \alpha_z, \beta$ are the Dirac matrices and δ the fourth-rank unit matrix.)

Thus, we have fixed the manner in which matrix elements are to be arranged, and in the following classification of matrices we can often make use of convenient diagrams rather than the imperspicuous algebraic relationships accompanying them. In connection with this we have to mention the effect of the matrix multipliers $\frac{1}{2}(1+\beta^{\mathrm{I}})$, $\frac{1}{2}(1+\beta^{\mathrm{II}})$, $\frac{1}{2}(1+\beta^{\mathrm{I}}\beta^{\mathrm{II}})$, $\frac{1}{2}(1-\beta^{\mathrm{I}})$, $\frac{1}{2}(1-\beta^{\mathrm{II}})$, and $\frac{1}{2}(1-\beta^{\mathrm{I}}\beta^{\mathrm{II}})$. Since we have

$$\tfrac{1}{2}(1+\beta^{\mathrm{I}})=\begin{bmatrix}\kappa&0&0&0\\0&\kappa&0&0\\0&0&\kappa&0\\0&0&0&\kappa\end{bmatrix},\quad \tfrac{1}{2}(1+\beta^{\mathrm{II}})=\begin{bmatrix}\delta&0&0&0\\0&\delta&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},$$

$$\tfrac{1}{2}(1+\beta^{\mathrm{I}}\beta^{\mathrm{II}})=\begin{bmatrix}\kappa&0&0&0\\0&\kappa&0&0\\0&0&\lambda&0\\0&0&0&\lambda\end{bmatrix},$$

etc., with

$$\kappa=\begin{bmatrix}1&&&\\&1&&\\&&0&\\&&&0\end{bmatrix},\quad \lambda=\begin{bmatrix}0&&&\\&0&&\\&&1&\\&&&1\end{bmatrix},$$

it follows that each of these six factors, when pre-multiplied (post-multiplied) into a given matrix, produces zero elements arranged in double rows (columns) after a pattern characteristic of that factor.

CLASSIFICATION OF MATRICES

According to Foldy and Wouthuysen, any 4×4 matrix is either even, or odd, or a sum of an even and an odd matrix:

$$\omega=\omega^e+\omega^o=\tfrac{1}{2}(\omega+\beta\omega\beta)+\tfrac{1}{2}(\omega-\beta\omega\beta). \tag{1}$$

For a matrix to be even (odd), it has to commute (anticommute) with β; at least half of its elements have to be zeros, distributed as is shown in Fig. 1.

In order to obtain an analogous classification of the 16×16 matrices, we consider the following decomposition of the general matrix Ω:

$$\Omega=\Omega^{ee}+\Omega^{eo}+\Omega^{oe}+\Omega^{oo}, \tag{2}$$

with

$$\Omega^{ee}=\tfrac{1}{4}(\Omega+\beta^{\mathrm{I}}\Omega\beta^{\mathrm{I}}+\beta^{\mathrm{II}}\Omega\beta^{\mathrm{II}}+\beta^{\mathrm{I}}\beta^{\mathrm{II}}\Omega\beta^{\mathrm{I}}\beta^{\mathrm{II}}); \tag{2a}$$

$$\Omega^{eo}=\tfrac{1}{4}(\Omega+\beta^{\mathrm{I}}\Omega\beta^{\mathrm{I}}-\beta^{\mathrm{II}}\Omega\beta^{\mathrm{II}}-\beta^{\mathrm{I}}\beta^{\mathrm{II}}\Omega\beta^{\mathrm{I}}\beta^{\mathrm{II}}); \tag{2b}$$

$$\Omega^{oe}=\tfrac{1}{4}(\Omega-\beta^{\mathrm{I}}\Omega\beta^{\mathrm{I}}+\beta^{\mathrm{II}}\Omega\beta^{\mathrm{II}}-\beta^{\mathrm{I}}\beta^{\mathrm{II}}\Omega\beta^{\mathrm{I}}\beta^{\mathrm{II}}); \tag{2c}$$

$$\Omega^{oo}=\tfrac{1}{4}(\Omega-\beta^{\mathrm{I}}\Omega\beta^{\mathrm{I}}-\beta^{\mathrm{II}}\Omega\beta^{\mathrm{II}}+\beta^{\mathrm{I}}\beta^{\mathrm{II}}\Omega\beta^{\mathrm{I}}\beta^{\mathrm{II}}). \tag{2d}$$

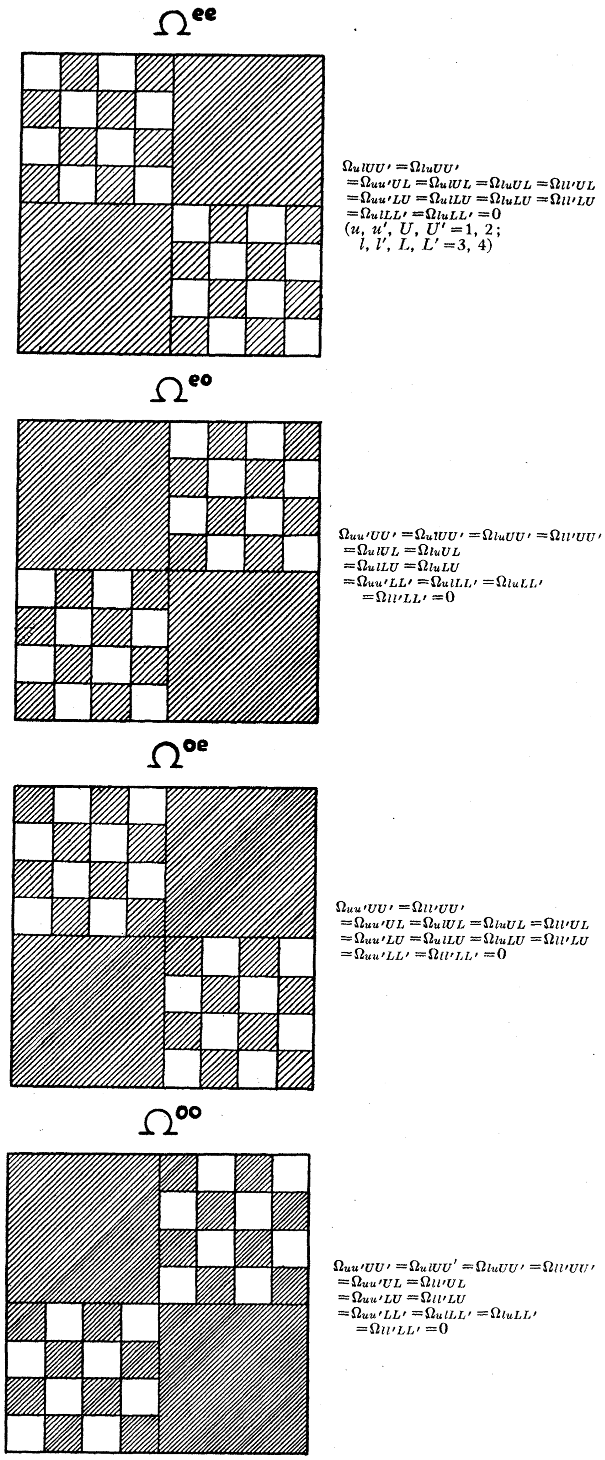

FIG. 2. The four basic types of 16×16 matrices: even-even, even-odd, odd-even, odd-odd.

Four basic types of matrices, the even-even, even-odd, odd-even, and odd-odd are defined[6] by the expressions (2a), (2b), (2c), (2d) respectively. Their commutation properties with β^{I} and β^{II} (see I, p. 389) can be deduced from the defining formulas. These formulas, when written in the form

$$\Omega^{ee}=\tfrac{1}{4}(1+\beta^{\mathrm{II}})\Omega'(1+\beta^{\mathrm{II}})+\tfrac{1}{4}(1-\beta^{\mathrm{II}})\Omega'(1-\beta^{\mathrm{II}})$$

with

$$\Omega'=\tfrac{1}{4}(1+\beta^{\mathrm{I}})\Omega(1+\beta^{\mathrm{I}})+\tfrac{1}{4}(1-\beta^{\mathrm{I}})\Omega(1-\beta^{\mathrm{I}}), \quad \text{(3a)}$$

etc., yield information as to which (at least) of the elements must be equal to zero, in order that the matrix be even-even, etc. Figure 2 shows the characteristic distribution of zeros in matrices of the four basic types.

An even-even matrix has the important property that it "keeps apart" the four kinds of components of the ψ spinor. This means that in the product $\Omega^{ee}\psi$, elements $(\Omega\psi)_{uU}$ are expressed entirely in terms of upper-upper components ψ_{uU}, and likewise elements $(\Omega\psi)_{uL}$, $(\Omega\psi)_{lU}$, $(\Omega\psi)_{lL}$ in terms of the ψ_{uL}, ψ_{lU}, ψ_{lL} respectively. Also in this regard the *ee* matrix appears to be a direct extension of Foldy's concept of an even matrix (see I, p. 388). However, there exists another matrix for which the same claim can be made. Its structure is shown by the first diagram of Fig. 3. The conditions imposed on it are weaker, it need not have as many zeros as an *ee* matrix; hence, its effect is smaller. When multiplied into ψ, it merely prevents the upper-upper components of ψ from mixing with components of other kinds, which means: $(\Omega\psi)_{uU}$ is expressed by the ψ_{uU} only, whereas the expressions for $(\Omega\psi)_{uL}$, $(\Omega\psi)_{lU}$, $(\Omega\psi)_{lL}$ do not

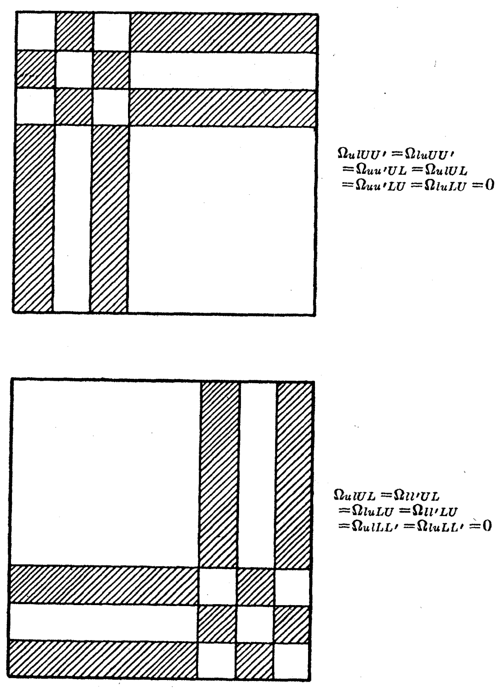

FIG. 3. Two special types of 16×16 matrices: the *uU*-separating and the *lL*-separating matrix.

[6] The definition used in I refers only to matrices representable as direct products and is, therefore, too narrow for the purposes of the present paper.

contain ψ_{uU} components. Such a matrix will be called "uU separating."

The algebraic equivalent of the diagram under consideration is found to be

$$\Omega^{us}=\Omega-\tfrac{1}{4}(1+\beta^{\mathrm{I}})(1+\beta^{\mathrm{II}})\Omega-\tfrac{1}{4}\Omega(1+\beta^{\mathrm{I}})(1+\beta^{\mathrm{II}})+\tfrac{1}{8}(1+\beta^{\mathrm{I}})(1+\beta^{\mathrm{II}})\Omega(1+\beta^{\mathrm{I}})(1+\beta^{\mathrm{II}}), \quad (4)$$

with Ω arbitrary. Every uU-separating matrix can be represented in this form. To simplify this rather unwieldy expression, we specialize it, assuming successively that Ω (and consequently also Ω^{us} itself) is even-even, even-odd, etc. If we make use of the commutation properties of β^{I} and β^{II}, we obtain the following set of rules: (1) Any even-even matrix is uU separating (but not *vice versa*). (2) An even-odd matrix is uU separating if, and only if, it can be written in the form $\frac{1}{2}(1-\beta^{\mathrm{I}})\Omega^{eo}$. Similarly, (3), (4) an odd-even matrix is rendered uU separating by the factor $\frac{1}{2}(1-\beta^{\mathrm{II}})$ in front of it, and an odd-odd matrix by the factor $\frac{1}{2}(1-\beta^{\mathrm{I}}\beta^{\mathrm{II}})$.

Rule (1) is easily verified by comparison of the corresponding diagrams. Rules (2), (3), (4) are illustrated by Fig. 4. For instance, it is seen that pre-multiplication by $\frac{1}{2}(1-\beta^{\mathrm{I}})$ provides for additional zeros in Ω^{eo}, so as to make it fit into the first scheme of Fig. 3.

The name "lL separating" is proposed for matrices possessing the property that in the elements of the product $(\Omega\psi)$ the lL components of ψ are not mixed with components of the other kinds. The most general algebraic expression for an lL-separating matrix,

$$\Omega^{ls}=\Omega-\tfrac{1}{4}(1-\beta^{\mathrm{I}})(1-\beta^{\mathrm{II}})\Omega-\tfrac{1}{4}\Omega(1-\beta^{\mathrm{I}})(1-\beta^{\mathrm{II}})+\tfrac{1}{8}(1-\beta^{\mathrm{I}})(1-\beta^{\mathrm{II}})\Omega(1-\beta^{\mathrm{I}})(1-\beta^{\mathrm{II}}) \quad (5)$$

and the corresponding distribution of zero elements (second diagram of Fig. 3) are analogous to those for uU-separating matrices. So are the four rules: (1) Any even-even matrix is lL separating. (2), (3), (4) The left multipliers, which convert a matrix into an lL-separating one, are $\frac{1}{2}(1+\beta^{\mathrm{I}})$, $\frac{1}{2}(1+\beta^{\mathrm{II}})$, and $\frac{1}{2}(1-\beta^{\mathrm{I}}\beta^{\mathrm{II}})$ for an *eo*, *oe*, *oo* matrix respectively.

Two more kinds of "separating" matrices with similar properties could be defined, but this would not lead to an exhaustive classification of matrices.

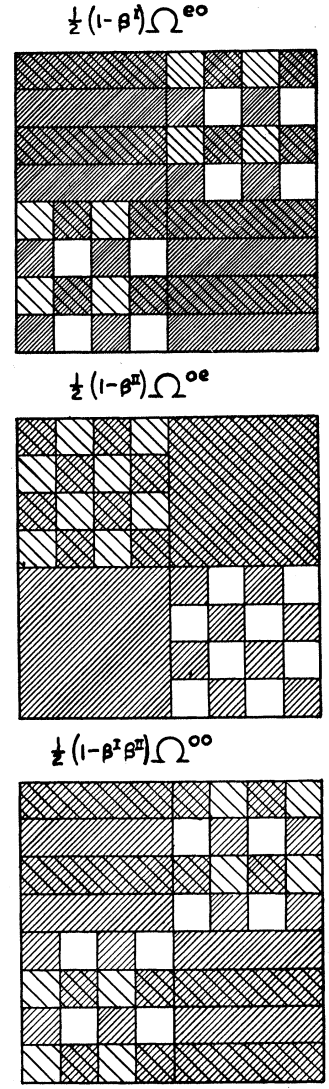

FIG. 4. The narrow-gauge shading shows the zeros due to the basic *eo*, *oe*, *oo* character of the matrix; the wide-gauge shading the zeros introduced by the respective left multiplier.

CHOICE OF GENERATING FUNCTIONS FOR CANONICAL TRANSFORMATIONS

The original wave equation $\mathcal{H}\psi=E\psi$ (or, alternatively, $\mathcal{H}\psi=-E\psi$) has to be transformed into $\mathcal{H}_{\mathrm{tr}}\psi_{\mathrm{tr}}=E\psi_{\mathrm{tr}}$ (or $\mathcal{H}_{\mathrm{tr}}\psi_{\mathrm{tr}}=-E\psi_{\mathrm{tr}}$), because we intend to separate out and to retain the only four component-equations which contain only the uU components (or the lL components) of the spinor ψ. That is, the transformed Hamiltonian $\mathcal{H}_{\mathrm{tr}}$ is expected to be uU separating in the one case, and lL separating in the other. Any other effect of the transformation is either superfluous or irrelevant. Now the transformation proposed in I led to an even-even $\mathcal{H}_{\mathrm{tr}}$; therefore, while satisfactory, it overshot the mark: instead of providing us with one suitable quadruple of equations, disentangled from the rest, it yielded four quadruples, not interlinked with each other.

In accordance with the present more limited objective of the transformation, its generating function will now be chosen according to less stringent rules; iS will consist of expressions of the following form (rather than that given in Eq. (5a, b, c) of I]:

$$\mathfrak{s}_{eo}=\frac{\beta^{\mathrm{II}}}{2m_{\mathrm{II}}c^2}\left\{t_{eo}+\frac{1\mp\beta^{\mathrm{I}}}{2}t_{eo}\right\}, \quad (6a)$$

$$\mathfrak{s}_{oe}=\frac{\beta^{\mathrm{I}}}{2m_{\mathrm{I}}c^2}\left\{t_{oe}+\frac{1\mp\beta^{\mathrm{II}}}{2}t_{oe}\right\}; \quad (6b)$$

$$\mathfrak{S}_{oo}=\frac{1}{4c^2}\left\{\frac{\beta^{\mathrm{I}}+\beta^{\mathrm{II}}}{m_{\mathrm{I}}+m_{\mathrm{II}}}\mathrm{t}_{oo}+\frac{\beta^{\mathrm{I}}-\beta^{\mathrm{II}}}{m_{\mathrm{I}}-m_{\mathrm{II}}}\mathrm{t}_{oe}\right\},\quad \text{if } m_{\mathrm{I}}\neq m_{\mathrm{II}}; \tag{6c}$$

$$\mathfrak{S}_{oo}'=\frac{1}{8mc^2}\{(\beta^{\mathrm{I}}+\beta^{\mathrm{II}})t_{oo}+(1-\beta^{\mathrm{I}}\beta^{\mathrm{II}})\mathrm{t}_{oo}'\}\quad \text{if } m_{\mathrm{I}}=m_{\mathrm{II}}. \tag{6d}$$

Here t_{eo}, t_{oe}, t_{oo} are terms appearing in the Hamiltonian; t_{eo}, etc., is a term obtained from t_{eo}, etc., by replacing its matrix part by another matrix which is of the same (*eo*, etc.) type, but otherwise arbitrary. Wherever double signs occur, the upper one refers to a transformation leading to a *uU*-separating $\mathcal{H}_{\mathrm{tr}}$, whereas the lower one is to be taken, if an *lL*-separating $\mathcal{H}_{\mathrm{tr}}$ is required.

It is easily verified that

$$t_{eo}+[\mathfrak{S}_{eo},\ (\beta^{\mathrm{I}}m_{\mathrm{I}}+\beta^{\mathrm{II}}m_{\mathrm{II}})c^2]=-\tfrac{1}{2}(1\mp\beta^{\mathrm{I}})\mathrm{t}_{eo};$$
$$t_{oe}+[\mathfrak{S}_{oe},\ (\beta^{\mathrm{I}}m_{\mathrm{I}}+\beta^{\mathrm{II}}m_{\mathrm{II}})c^2]=-\tfrac{1}{2}(1\mp\beta^{\mathrm{II}})\mathrm{t}_{oe};$$
$$t_{oo}+[\mathfrak{S}_{oo},\ (\beta^{\mathrm{I}}m_{\mathrm{I}}+\beta^{\mathrm{II}}m_{\mathrm{II}})c^2]=\tfrac{1}{2}(1-\beta^{\mathrm{I}}\beta^{\mathrm{II}})(t_{oo}-\mathrm{t}_{oo});$$
$$t_{oo}+[\mathfrak{S}_{oo}',\ (\beta^{\mathrm{I}}+\beta^{\mathrm{II}})mc^2]=\tfrac{1}{2}(1-\beta^{\mathrm{I}}\beta^{\mathrm{II}})t_{oo},$$

which means: when used in Eq. (2) of I, the terms $\mathfrak{S}_{eo}$ of *iS* changes (by its highest-order contribution) the even-odd (undesirable) term t_{eo} of the Hamiltonian into another one, again even-odd, but acceptable, since it is *uU* separating (or *lL* separating). The undesirable terms t_{oe}, t_{oo} are affected likewise by $\mathfrak{S}_{oe}$, $\mathfrak{S}_{oo}$, (or $\mathfrak{S}_{oo}'$), respectively. In addition, each of the $\mathfrak{S}$ expressions will contribute some terms of lower order, of which the even-even and the acceptable ones will appear in the final $\mathcal{H}_{\mathrm{tr}}$, whereas those of undesired types must be treated in a similar manner by means of subsequent transformations.

Because of the mentioned arbitrariness in t_{eo}, etc., the expressions (6) are rather general;[7] in fact, they yield a whole class of transformations. Special cases include: for $\mathrm{t}_{eo}=\mathrm{t}_{oe}=0$, $\mathrm{t}_{oo}=t_{oo}$ $(m_{\mathrm{I}}\neq m_{\mathrm{II}})$, the "radical" transformation of I, which not only modifies, but even destroys undesirable terms; and, as the other extreme, for $\mathrm{t}_{eo}=-t_{eo}$, $\mathrm{t}_{oe}=-t_{oe}$, $\mathrm{t}_{oo}=\mathrm{t}_{oo}'=0$, the "least change" transformation, which secures the indispensable amount of modification, putting the necessary zeros in the matrices so as to make them acceptable, without causing any further "damage." It should be remarked that this is not the simplest transformation, which we rather should expect for $\mathrm{t}_{eo}=\mathrm{t}_{oe}=\mathrm{t}_{oo}=\mathrm{t}_{oo}'=0$.

THE TRANSFORMED HAMILTONIANS

By repeated application of the "least change" transformation, the following expression for the transformed Hamiltonian, approximate out to the order $(1/c)^2$, has been obtained:

$$\mathcal{H}_{\mathrm{tr}}=\beta^{\mathrm{I}}m_{\mathrm{I}}c^2+\beta^{\mathrm{II}}m_{\mathrm{II}}c^2+(\mathcal{E}\mathcal{E}) \tag{7a}$$

$$+\frac{\beta^{\mathrm{I}}(1\pm\beta^{\mathrm{II}})}{4m_{\mathrm{I}}c^2}(\mathcal{O}\mathcal{E})^2+\frac{\beta^{\mathrm{II}}(1\pm\beta^{\mathrm{I}})}{4m_{\mathrm{II}}c^2}(\mathcal{E}\mathcal{O})^2 \tag{7b}$$

$$+\frac{1\pm\beta^{\mathrm{II}}}{16m_{\mathrm{I}}^2c^4}[[(\mathcal{O}\mathcal{E}),(\mathcal{E}\mathcal{E})],(\mathcal{O}\mathcal{E})]+\frac{1\pm\beta^{\mathrm{I}}}{16m_{\mathrm{II}}^2c^4}[[(\mathcal{E}\mathcal{O}),(\mathcal{E}\mathcal{E})],(\mathcal{E}\mathcal{O})] \tag{7c}$$

$$-\frac{\beta^{\mathrm{I}}(1\pm\beta^{\mathrm{II}})}{16m_{\mathrm{I}}^3c^6}(\mathcal{O}\mathcal{E})^4-\frac{\beta^{\mathrm{II}}(1\pm\beta^{\mathrm{I}})}{16m_{\mathrm{II}}^3c^6}(\mathcal{E}\mathcal{O})^4 \tag{7d}$$

$$+\frac{\beta^{\mathrm{I}}\beta^{\mathrm{II}}\pm\beta^{\mathrm{I}}\pm2\beta^{\mathrm{II}}}{32m_{\mathrm{I}}m_{\mathrm{II}}c^4}[[(\mathcal{O}\mathcal{E}),(\mathcal{O}\mathcal{O})]_+,(\mathcal{E}\mathcal{O})]_+ +\frac{\beta^{\mathrm{I}}\beta^{\mathrm{II}}\pm\beta^{\mathrm{II}}\pm2\beta^{\mathrm{I}}}{32m_{\mathrm{I}}m_{\mathrm{II}}c^4}[[(\mathcal{E}\mathcal{O}),(\mathcal{O}\mathcal{O})]_+,(\mathcal{O}\mathcal{E})]_+$$
$$+\frac{1\mp\beta^{\mathrm{II}}}{32m_{\mathrm{I}}m_{\mathrm{II}}c^4}[[(\mathcal{O}\mathcal{E}),(\mathcal{O}\mathcal{O})],(\mathcal{E}\mathcal{O})]+\frac{1\mp\beta^{\mathrm{I}}}{32m_{\mathrm{I}}m_{\mathrm{II}}c^4}[[(\mathcal{E}\mathcal{O}),(\mathcal{O}\mathcal{O})],(\mathcal{O}\mathcal{E})] \tag{7e}$$

$$+\frac{\beta^{\mathrm{I}}+\beta^{\mathrm{II}}}{4(m_{\mathrm{I}}+m_{\mathrm{II}})c^2}(\mathcal{O}\mathcal{O})^2 \tag{7f}$$

$$+\frac{\beta^{\mathrm{I}}+\beta^{\mathrm{II}}}{16(m_{\mathrm{I}}+m_{\mathrm{II}})m_{\mathrm{I}}m_{\mathrm{II}}c^6}[(\mathcal{O}\mathcal{E}),(\mathcal{E}\mathcal{O})]^2 \tag{7g}$$

$$-\frac{(1\pm\beta^2)(1\pm\beta^{\mathrm{II}})(\beta^{\mathrm{I}}m_{\mathrm{I}}+\beta^{\mathrm{II}}m_{\mathrm{II}})}{64m_{\mathrm{I}}^2m_{\mathrm{II}}^2c^6}[(\mathcal{O}\mathcal{E})^2,(\mathcal{E}\mathcal{O})^2]_+ \tag{7h}$$

[7] However, they may be not the most general possible. In particular, under certain conditions t_{eo}, etc. could differ from t_{eo}, etc. not only as regards the matrix contained, but also in its other parts as well. Yet it seemed of little importance to pursue such possibilities.

$$+\frac{(1\pm\beta^{\mathrm{I}})(\beta^{\mathrm{II}}\pm 7)m_{\mathrm{I}}+(1\mp\beta^{\mathrm{II}})(5\beta_{\mathrm{I}}\mp 3)m_{\mathrm{II}}}{128m_{\mathrm{I}}^2m_{\mathrm{II}}^2c^6}(\mathcal{E}\mathcal{O})(\mathcal{O}\mathcal{E})^2(\mathcal{E}\mathcal{O})$$

$$+\frac{(1\pm\beta^{\mathrm{II}})(\beta^{\mathrm{I}}\pm 7)m_{\mathrm{II}}+(1\mp\beta^{\mathrm{I}})(5\beta^{\mathrm{II}}\mp 3)m_{\mathrm{I}}}{128m_{\mathrm{I}}^2m_{\mathrm{II}}^2c^6}(\mathcal{O}\mathcal{E})(\mathcal{E}\mathcal{O})^2(\mathcal{O}\mathcal{E}) \quad (7\mathrm{i})$$

$$+\frac{(1+\beta^{\mathrm{I}}\beta^{\mathrm{II}})(m_{\mathrm{I}}-m_{\mathrm{II}})}{16(m_{\mathrm{I}}+m_{\mathrm{II}})m_{\mathrm{I}}m_{\mathrm{II}}c^4}[[(\mathcal{E}\mathcal{O}),(\mathcal{O}\mathcal{E})],(\mathcal{O}\mathcal{O})] \quad (7\mathrm{j})$$

$$+\tfrac{1}{2}(1\mp\beta^{\mathrm{I}})\{(\mathcal{E}\mathcal{O})+\sum\mathfrak{N}_{eo}\} \quad (7\mathrm{k})$$

$$+\tfrac{1}{2}(1\mp\beta^{\mathrm{II}})\{(\mathcal{O}\mathcal{E})+\sum\mathfrak{N}_{oe}\} \quad (7\mathrm{l})$$

$$+\tfrac{1}{2}(1-\beta^{\mathrm{I}}\beta^{\mathrm{II}})\{(\mathcal{O}\mathcal{O})+\sum\mathfrak{N}_{oo}\}. \quad (7\mathrm{m})$$

It contains in (7a) the (unchanged) even-even part of the original Hamiltonian; in (7k, l, m) the original even-odd, odd-even, and odd-odd part (denoted by $(\mathcal{E}\mathcal{O})$, $(\mathcal{O}\mathcal{E})$, $(\mathcal{O}\mathcal{O})$), modified in the desired manner; in (7b$\cdots$j) new even-even terms produced by the transformation, our main point of interest; and in (7k, l, m) new terms of the *eo*, *oe*, and *oo* type, numerous, but irrelevant, and for this reason not specified, but only indicated.

A comparison of this Eq. (7) with the Eq. (7) of I shows considerable differences, which is not surprising. However, it should be remembered that the transformed Hamiltonian is merely an intermediate stage. As was pointed out in I (p. 390), the next, and final step, is the replacement, now made possible, of 16×16 matrices by suitable 4×4 matrices; in particular, both β^{I} and β^{II} are to be replaced by the (fourth rank) unit or the minus unit matrix, according to whether the uU or the lL quadruple is to be retained. Now it is seen that putting $\beta^{\mathrm{I}}=\beta^{\mathrm{II}}=1$ and taking the upper signs, (or $\beta^{\mathrm{I}}=\beta^{\mathrm{II}}=-1$ with the lower signs) makes the Hamiltonians (7) and I(7) coincide. Thus, different transformations of our class finally lead to the same reduced wave equation. This fact is essential, in order that the proposed reduction procedure be sound.

Although only two special transformations have been considered in full detail, there are strong indications that the same result would ensue for all other transformations generated by (6). In order to prove this in general, it would be necessary to calculate $\mathcal{H}_{\mathrm{tr}}$ using the full, nonspecialized expressions (6). However, in this general case the calculations, while still straightforward, become prohibitively lengthy and tedious.

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