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Relativistic Particle Dynamics. II*

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The relativistic dynamics for a system of non-interacting particles in Hamiltonian form is separated by a contact transformation into motion of their center of mass and internal motion. Interaction at a distance between them is then introduced into the expression for the rest-mass in terms of the internal variables. This gives a dynamics for which invariance over space displacements and rotations is trivial and which is rigorously invariant over Lorentz transformations. Earlier approximate treatments may be reduced by contact transformations to special cases of the general treatment.

1. INTRODUCTION

IN this paper we adopt the point of view of an earlier paper[1] and describe a system in terms of dynamical variables whose mutual Poisson brackets are given. We shall specify each particle i by three pairs of canonically conjugate dynamical variables forming vector coordinates $\mathbf{q}_i$ and momenta $\mathbf{p}_i$, three variables forming its vector intrinsic spin $\boldsymbol{\omega}_i$, and its rest mass m_i.

We shall take these to transform trivially for simultaneous rotation or displacement of the frame of reference in space, leading to an "instant" form of dynamics,[2,3] (as opposed to a "point" form in which transformation over the homogeneous Lorentz group is trivial, such as the form considered in I[1]).

The dynamics will then be specified by a Hamiltonian function H leading to the changes of the dynamical variables with time, and a vector $\mathbf{V}$, of which the components U, V, and W, give in the same way the infinitesimal transformations for change to relatively moving coordinate systems. In addition we introduce the vectors $\mathbf{R}$, components X, Y, and Z, the total linear momentum, and $\boldsymbol{\Omega}$, components L, M, and N, the total angular momentum of the system, which give in a similar way the here trivial, infinitesimal transformations for space displacement and rotation. These ten functions of the basic dynamical variables $\mathbf{q}_i$, $\mathbf{p}_i$, $\boldsymbol{\omega}_i$ (and m_i) must satisfy the conditions for all the transformations to give the inhomogeneous Lorentz group.[4]

It is, however, not implied, nor is it true, that, even in non-quantum mechanics, there exist world lines $\mathbf{r}_i=\mathbf{q}_i(t)$, for the particles, that are the same loci in space-time when transformed by the corresponding Lorentz transformations, unless there is no interaction and the particles are without spin.[5]

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1 L. H. Thomas, Phys. Rev. **85**, 868 (1952); referred to below as I.

2 See I, Eqs. (4.2) and (4.3).

3 P. M. Dirac, Revs. Modern Phys. **21**, 392 (1944).

4 See I, Sec. III.

5 M. H. L. Pryce, Proc. Roy. Soc. (London) **A195**, 621 (1948).

2. DYNAMICS FOR A SINGLE PARTICLE

For a free spinless point particle we have three coordinates q_x, q_y, q_z and canonically conjugate components of momentum p_x, p_y, p_z with Poisson brackets $(q_x, p_x)=1$, $(q_y, p_y)=1$, $(q_z, p_z)=1$, the remaining Poisson brackets vanishing. We may take the usual components of momentum, angular momentum, and energy,

$$\begin{aligned} X&=p_x, & L&=q_yp_z-q_zp_y, \\ Y&=p_y, & M&=q_zp_x-q_xp_z, & H&=(m^2c^4+p_x^2+p_y^2+p_z^2)^{\frac{1}{2}}, \\ Z&=p_z, & N&=q_xp_y-q_yp_x, \end{aligned}$$

where m is the rest mass and c the speed of light, and then, if we take

$$U=q_xH/c^2,\quad V=q_yH/c^2,\quad W=q_zH/c^2,$$

all the conditions are satisfied,

$$[\text{e.g., } (U,X)=(q_x,p_x)H/c^2=H/c^2].$$

U, V, and W do not have as immediate a physical interpretation as X, Y, Z, L, M, N, and H, because they do not commute with H and are not constants of the motion of the system. They may be called velocity operators.

If we replace q_x, q_y, q_z, by operators

$$-\frac{h}{2\pi i}\frac{\partial}{\partial p_x},\quad -\frac{h}{2\pi i}\frac{\partial}{\partial p_y},\quad -\frac{h}{2\pi i}\frac{\partial}{\partial p_z},$$

and write

$$U=\frac{1}{2c^2}(q_xH+Hq_x),\quad V=\frac{1}{2c^2}(q_yH+Hq_y),$$

$$W=\frac{1}{2c^2}(q_zH+Hq_z),$$

we obtain operators giving a relativistic quantum dynamics.

In vector notation,

$$\begin{aligned} \mathbf{R}&=\mathbf{p}, & H&=[m^2c^4+\mathbf{p}^2c^2]^{\frac{1}{2}}, \\ \boldsymbol{\Omega}&=[\mathbf{q}\times\mathbf{p}], & \mathbf{V}&=(1/2c^2)(\mathbf{q}H+H\mathbf{q}). \end{aligned} \tag{1.2}$$

For a particle with spin, we may write in like fashion:

$$\mathbf{R}=\mathbf{p},\quad H=[m^2c^4+\mathbf{p}^2c^2]^{\frac{1}{2}},\quad \mathbf{\Omega}=[\mathbf{q}\times\mathbf{p}]+\boldsymbol{\omega}, \tag{2.21}$$

where $(\omega_x,\omega_y)=\omega_z$, $(\omega_y,\omega_z)=\omega_x$, $(\omega_z,\omega_x)=\omega_y$, and ω_x, ω_y, and ω_z have zero Poisson brackets with the rest of the basic dynamical variables. We now find that we must take

$$\mathbf{V}=\frac{1}{2c^2}(\mathbf{q}H+H\mathbf{q})-\frac{[\boldsymbol{\omega}\times\mathbf{p}]}{mc^2+H}. \tag{2.22}$$

Further, the Dirac electron can be obtained from this by introducing negative energy states, putting $\boldsymbol{\omega}=(h/4\pi)\boldsymbol{\sigma}$, and making the unitary transformation given by the matrix

$$[mc^2+E-ic\rho_2(\boldsymbol{\sigma}\cdot\mathbf{p})]/[2E(mc^2+E)]^{\frac{1}{2}},$$

where $E=|H|$ and $H=-\rho_3E$, and σ_x, σ_y, and σ_z, and ρ_1, ρ_2, and ρ_3, are two independent sets of Pauli matrices.

3. DYNAMICS FOR A SYSTEM OF PARTICLES WITHOUT SPIN

In building up a dynamics for more than one particle, it is convenient to start with two non-interacting particles. We write $c=1$ for brevity and take the sums of functions of the form (2.1) for each particle,

$$\begin{aligned}\mathbf{R}&=\mathbf{p}_1+\mathbf{p}_2,\quad \mathbf{\Omega}=[\mathbf{q}_1\times\mathbf{p}_1]+[\mathbf{q}_2\times\mathbf{p}_2],\\ H&=[m_1^2+\mathbf{p}_1^2]^{\frac{1}{2}}+[m_2^2+\mathbf{p}_2^2]^{\frac{1}{2}},\\ \mathbf{V}&=\mathbf{q}_1[m_1^2+\mathbf{p}_1^2]^{\frac{1}{2}}+\mathbf{q}_2[m_2^2+\mathbf{p}_2^2]^{\frac{1}{2}}.\end{aligned} \tag{3.1}$$

This defines ten functions trivially satisfying the forty-five Poisson bracket relations[4] required for invariance.

The expression

$$m=(H^2-\mathbf{R}^2)^{\frac{1}{2}} \tag{3.2}$$

can be regarded as the effective rest mass of the system viewed as a single entity and has the important property that it commutes, or has zero Poisson bracket, with each of Eqs. (3.1).

We expect, for an instant form of dynamics, that interaction terms will enter only in H and $\mathbf{V}$ and not in $\mathbf{R}$ and $\mathbf{\Omega}$, and that they will depend in some sense only on relative or internal variables of the system, affecting the motion of the system as a whole only through m.

Suppose that we make a transformation from $\mathbf{p}_1$, $\mathbf{q}_1$, $\mathbf{p}_2$, $\mathbf{q}_2$ to variables $\mathbf{R}$, $\mathbf{r}$, total momentum and coordinates of the "center of mass," and $\mathbf{P}$, $\boldsymbol{\varrho}$, relative momentum and relative coordinates, twelve variables having similar Poisson bracket relations to $\mathbf{p}_1$, $\mathbf{q}_1$, $\mathbf{p}_2$, and $\mathbf{q}_2$, in such a way that (i) $\mathbf{\Omega}=[\mathbf{r}\times\mathbf{R}]+[\boldsymbol{\varrho}\times\mathbf{P}]$, (so that $\boldsymbol{\varrho}$ and $\mathbf{P}$ transform like vectors in a space rotation), (ii) m depends on $\mathbf{P}$ only, and (iii) $\mathbf{V}$ can be expressed in terms of m, $\mathbf{r}$, $\mathbf{R}$, and $\mathbf{\Omega}$ only. Then in the expression for H,

$$H=(m^2+\mathbf{R}^2)^{\frac{1}{2}},$$

and in the expression for $\mathbf{V}$ in terms of m, $\mathbf{r}$, $\mathbf{R}$, and $\mathbf{\Omega}$, we may introduce an interaction by replacing m by any other function of $\boldsymbol{\varrho}$ and $\mathbf{P}$ which is a scalar for space rotations. Relativistic invariance will be preserved because m still commutes with $\mathbf{r}$, $\mathbf{R}$, and $\mathbf{\Omega}$, and the commutation relations of these with each other and with H and V are not disturbed. (The transformation between $\mathbf{p}_1$, $\mathbf{q}_1$, $\mathbf{p}_2$, $\mathbf{q}_2$ and $\mathbf{R}$, $\mathbf{r}$, $\mathbf{P}$, $\boldsymbol{\varrho}$ must not be altered.)

We now obtain a transformation for many particles as similar to that to center of mass and relative coordinates in Newtonian mechanics as is consistent with the above requirements. We start with non-interacting particles having momenta $\mathbf{R}_u$ and coordinates $\mathbf{r}_u$ relative to an arbitrary observer, and write $H_u=(m_u^2+\mathbf{R}_u^2)^{\frac{1}{2}}$, $\mathbf{\Omega}_u=[\mathbf{r}_u\times\mathbf{R}_u]$, $\mathbf{V}_u=\frac{1}{2}(\mathbf{r}_uH_u+H_u\mathbf{r}_u)$, so that we can take

$$\mathbf{R}=\sum_u\mathbf{R}_u,\quad \mathbf{\Omega}=\sum_u\mathbf{\Omega}_u,\quad H=\sum_u H_u,\quad \mathbf{V}=\sum_u\mathbf{V}_u, \tag{3.3}$$

and the requirements for relativistic invariance will be met.

Next, we make a transformation to a frame in which the total momentum of the system is zero. In this new frame the particles have momenta $\mathbf{S}_u$ and energies K_u such that $\sum_u\mathbf{S}_u=0$ and $\sum_u K_u=m$.

Writing as before $H^2=m^2+\mathbf{R}^2$, and further $R^2=\mathbf{R}^2$, $R=m\sinh v$, $H=m\cosh v$, $R_u=(\mathbf{R}_u\cdot\mathbf{R})/R$, we obtain for this Lorentz transformation in the direction of R,

$$S_u=R_u\cosh v-H_u\sinh v,$$
$$K_u=-R_u\sinh v+H_u\cosh v,$$

or, returning to vectors,

$$\begin{aligned}\mathbf{S}_u&=\mathbf{R}_u-R_u\mathbf{R}/R+S_u\mathbf{R}_u/R\\ &=\mathbf{R}_u-\left\{\frac{(m+H)H_u-(\mathbf{R}\cdot\mathbf{R}_u)}{m(m+H)}\right\}\mathbf{R},\end{aligned} \tag{3.4}$$

$$K_u=-(\mathbf{R}\cdot\mathbf{R}_u)/m+HH_u/m;$$

while reversely,

$$\mathbf{R}_u=\mathbf{S}_u+\left\{\frac{(m+H)K_u+(\mathbf{R}_u\cdot\mathbf{S}_u)}{m(m+H)}\right\}\mathbf{R}, \tag{3.5}$$

$$H_u=(\mathbf{R}\cdot\mathbf{S}_u)/m+HK_u/m.$$

We verify that

(a) $\sum_u\mathbf{S}_u=0$, (b) $\sum_u K_u=m$; (c) $K_u^2=m_u^2+\mathbf{S}_u^2$,

while also

$$\mathbf{R}_u=\mathbf{S}_u+\frac{K_u+H_u}{m+H}\mathbf{R}. \tag{3.6}$$

The equation (3.4) now defines a "point transformation" in momentum space from the momenta $\mathbf{R}_u$ to the momenta $\mathbf{R}$ and $\mathbf{S}_u$, subject to the condition $\sum_u\mathbf{S}_u=0$, which satisfies the requirement (ii) above in that m

depends only on the relative momenta $\mathbf{S}_u$ in scalar fashion.

We can now find a corresponding transformation for the "coordinates" $\mathbf{r}_u$ of the particles by writing down the condition for a contact transformation which in this case is an "extended point transformation" in momentum space.

$$\sum_u(\mathbf{r}_u\cdot d\mathbf{R}_u)=(\mathbf{r}\cdot d\mathbf{R})+\sum_u(\mathbf{s}_u\cdot d\mathbf{S}_u), \quad (3.7)$$

subject to $\sum_u \mathbf{S}_u=0$. The requirement (i) above follows now necessarily from the vector form of our equations as

$$\mathbf{\Omega}=\sum_u[\mathbf{r}_u\times\mathbf{R}_u]=[\mathbf{r}\times\mathbf{R}]+\sum_u[\mathbf{s}_u\times\mathbf{S}_u].$$

Here, $\mathbf{r}$ may be chosen to be any arbitrary linear combination of the $\mathbf{r}_u$ with coefficients functions of the $\mathbf{R}_u$, and in particular to satisfy requirement (iii). We may take the Eqs. (2.21) and (2.22) in the form

$$\mathbf{\Omega}=[\mathbf{r}\times\mathbf{R}]+\boldsymbol{\omega},\quad \mathbf{V}=\tfrac{1}{2}(\mathbf{r}H+H\mathbf{r})-[\boldsymbol{\omega}\times\mathbf{R}]/(m+H),$$

and solve these to give

$$\mathbf{r}=\frac{1}{m}V+\frac{1}{m(m+H)}[\mathbf{\Omega}\times\mathbf{R}]-\frac{\mathbf{R}(\mathbf{V}\cdot\mathbf{R})}{mH(m+H)}, \quad (3.8)$$

which is of the necessary form and determines the transformation. The detailed calculations are given in Appendix I, the important results being Eqs. (1)–(5). When there are just two particles we may write

$$\mathbf{P}=\mathbf{S}_1=-\mathbf{S}_2,\quad \boldsymbol{\varrho}=\mathbf{s}_1-\mathbf{s}_2,$$

and (3.8) becomes

$$(\mathbf{r}_1\cdot d\mathbf{R}_1)+(\mathbf{r}_2\cdot d\mathbf{R}_2)=(\mathbf{r}\cdot d\mathbf{R})+(\boldsymbol{\varrho}\cdot d\mathbf{P}), \quad (3.9)$$

where $\boldsymbol{\varrho}$ and its canonically conjugate momentum $\mathbf{P}$ may be interpreted as the relative coordinate and relative momentum of the two particles. Thus, we have performed a contact transformation from the original pairs of conjugate variables describing the separate particles, $(\mathbf{r}_1, \mathbf{R}_1)$, $(\mathbf{r}_2, \mathbf{R}_2)$, to two other pairs $(\mathbf{r}, \mathbf{R})$, giving the state of the system as a whole, and $(\boldsymbol{\varrho}, \mathbf{P})$ giving its internal state.

Now any scalar combinations, ϱ^2, $(\boldsymbol{\varrho}\cdot\mathbf{P})$, $\mathbf{P}^2$, and functions of these, of the internal variables commute with the ten fundamental quantities, so an interaction can be introduced consistently by making m any function of these scalar combinations. An explicit expression for $\boldsymbol{\varrho}$ is found from Eq. (2) of Appendix I.

$$\begin{aligned}\boldsymbol{\varrho}&=\mathbf{s}_1-\mathbf{s}_2\\ &=\mathbf{r}_1-\mathbf{r}_2+\frac{(\mathbf{r}_1-\mathbf{r}_2\cdot\mathbf{R})}{m(m+H)}\bigg\{\mathbf{R}+\frac{1}{H}\left(\frac{H_1}{s+m_1^2}-\frac{H_2}{s+m_2^2}\right)\\ &\quad\times([H_2(m+H)-(\mathbf{R}_1\cdot\mathbf{R}_2)]\mathbf{R}_1\\ &\quad-[H_1(m+H)-(\mathbf{R}_1\cdot\mathbf{R}_2)]\mathbf{R}_2)\bigg\}, \end{aligned} \quad (3.10)$$

where $s=\frac{1}{2}(m^2-m_1^2-m_2^2)$. Likewise

$$\begin{aligned}\mathbf{r}=\frac{1}{m}(\mathbf{r}_1H_1+\mathbf{r}_2H_2)+\frac{1}{m+H}\bigg\{&[[\mathbf{r}_1\times\mathbf{R}_1]\times\mathbf{R}]\\ &+[[\mathbf{r}_2\times\mathbf{R}_2]\times\mathbf{R}]-\frac{H_1}{H}(\mathbf{r}_1\cdot\mathbf{R})\mathbf{R}-\frac{H_2}{H}(\mathbf{r}_2\cdot\mathbf{R})\mathbf{R}\bigg\}.\end{aligned} \quad (3.11)$$

These equations, with $H_1^2=m_1^2+\mathbf{R}_1^2$, $H_2^2=m_2^2+\mathbf{R}_2^2$, $\mathbf{R}=\mathbf{R}_1+\mathbf{R}_2$, $H=H_1+H_2$, $m^2=H^2-\mathbf{R}^2$, and

$$\mathbf{P}=\mathbf{R}_1-\left\{\frac{(m+H)H_1-(\mathbf{R}\cdot\mathbf{R}_1)}{m(m+H)}\right\}\mathbf{R},$$

give the transformation explicitly, while the fundamental quantities giving the dynamics are

$$\begin{aligned}&\mathbf{R}=\mathbf{R},\quad H=(M^2+R^2)^{\frac{1}{2}},\\ &\mathbf{\Omega}=[\mathbf{r}\times R]+[\boldsymbol{\varrho}\times\mathbf{P}],\\ &\mathbf{V}=\tfrac{1}{2}(\mathbf{r}H+H\mathbf{r})-\frac{[[\boldsymbol{\varrho}\times\mathbf{P}]\times\mathbf{R}]}{M+H},\end{aligned} \quad (3.12)$$

where M may be any function of ϱ^2, $(\boldsymbol{\varrho}\cdot\mathbf{P})$, and $\mathbf{P}^2$.

Since $\mathbf{r}$ and $\boldsymbol{\varrho}$ are linear in $\mathbf{r}_1$ and $\mathbf{r}_2$, the extension to quantum mechanics is always possible.

4. SPIN

We may introduce spin into our equations as follows: (*a*) We define the intrinsic spin $\boldsymbol{\omega}$ of a particle relative to an observer O as a four-vector which is space-like in a frame P in which the particle is at rest. Further, we assume that the components ω_x, ω_y, ω_z of $\boldsymbol{\omega}$ satisfy the Poisson bracket relations:

$$(\omega_x,\omega_y)=\omega_z;\quad (\omega_y,\omega_z)=\omega_x;\quad (\omega_z,\omega_x)=\omega_y. \quad (4.1)$$

(*b*) Viewed from any other frame of reference, the intrinsic spin defined in this manner will be given in general by a different four vector which is also space-like in the frame P. In particular, the intrinsic spin relative to the frame C in which the system has zero linear momentum will be denoted by a vector $\mathbf{n}$. Three successive Lorentz transformations from P to O, O to C, and from C back to P will now give us the relationship between $\boldsymbol{\omega}$ and $\mathbf{n}$:

$$\begin{aligned}\mathbf{n}_u&=\boldsymbol{\omega}_u+(\boldsymbol{\omega}_u\cdot\mathbf{R})\frac{\mathbf{S}_u}{m(m_u+K_u)}+\frac{(\boldsymbol{\omega}_u\cdot\mathbf{R})\mathbf{R}}{m(m+H)}\\ &\quad+\frac{(\boldsymbol{\omega}_u\cdot\mathbf{R}_u)\mathbf{S}_u(m-H)}{m(m_u+K_u)(m_u+H_u)}-\frac{(\boldsymbol{\omega}_u\cdot\mathbf{R}_u)\mathbf{R}}{m(m_u+H_u)},\\ &\text{or}\\ \boldsymbol{\omega}_u&=\mathbf{n}_u-(\mathbf{n}_u\cdot\mathbf{R})\frac{\mathbf{R}_u}{m(m_u+K_u)}+(\mathbf{n}_u\cdot\mathbf{R})\frac{\mathbf{R}}{m(m+H)}\\ &\quad+\frac{(\mathbf{n}_u\cdot\mathbf{S}_u)\mathbf{R}_u(m-H)}{m(m_u+K_u)(m_u+H_u)}+\frac{(\mathbf{n}_u\cdot\mathbf{S}_u)\mathbf{R}}{m(m_u+K_u)}.\end{aligned} \quad (4.2)$$

These relations then define the transformation properties of *intrinsic* spin under a Lorentz transformation from the frame O to the frame C. We may further conclude from the properties of successive Lorentz transformations that $\mathbf{n}$ and $\boldsymbol{\omega}$ will have the same magnitude but will differ in orientation. Two successive Lorentz transformations are not equivalent to a single Lorentz transformation, but rather to a Lorentz transformation plus a suitable rotation. In other words, Lorentz transformations do not have the group property. They have the group property if each Lorentz transformation is coupled with a rotation. Thus we associate with the Lorentz transformations $O-C$, $O-P$, and $C-P$, rotations of the coordinate axes defined by Eulerian angles φ, θ, ψ; α, β, γ; and ξ, η, ζ, respectively. (c) Since the components of $\boldsymbol{\omega}$ do not have the Poisson bracket relations of true momenta, we cannot introduce them in our contact transformation (3.7) directly. The angles α, β, γ and ξ, η, ζ, however, represent true coordinates and therefore have canonically conjugate true momenta A, B, C and Ξ, H, Z associated with them. If now we consider an infinitesimal rotation given by a vector $d\boldsymbol{\pi}$ with components defined in the frame O as

$$(d\pi_x)_0=\cos\alpha d\beta+\sin\beta\,\sin\alpha d\gamma,$$
$$(d\pi_y)_0=\sin\alpha d\beta-\sin\beta\,\cos\alpha d\gamma,$$
$$(d\pi_z)_0=d\alpha+\cos\beta d\gamma;$$

and we set

$$(\boldsymbol{\omega}\cdot d\boldsymbol{\pi})_0=Ad\alpha+Bd\beta+Cd\gamma, \tag{4.3}$$

then

$$\begin{aligned}(\omega_x)_0&=-\cot\beta\,\sin\alpha A+\cos\alpha B+\csc\beta\,\sin\alpha C,\\ (\omega_y)_0&=\cot\beta\,\cos\alpha A+\sin\alpha B-\csc\beta\,\cos\alpha C,\\ (\omega_z)_0&=A.\end{aligned} \tag{4.4}$$

$\omega_x, \omega_y, \omega_z$ now form a function group in terms of canonical variables A, α; B, β; C, γ such that the relations (Sec. 2) are satisfied.

Similarly, we may consider an infinitesimal rotation given by a vector $d\boldsymbol{\theta}$ with components $(d\vartheta_x)_c=\cos\xi d\eta+\sin\eta\,\sin\xi d\zeta$, $(d\vartheta_y)_c=\sin\xi d\eta-\sin\eta\,\cos\xi d\zeta$, and $(d\vartheta_z)_c=d\xi+\cos\eta d\zeta$. We can write relations similar to (4.3), (4.4) that give the components $(\eta_x)_c$, $(\eta_y)_c$, $(\eta_z)_c$ of $\mathbf{n}$ in terms of canonical variables Ξ, ξ; H, η; Z, ζ.

Now if we write

$$\sum_u \mathbf{r}_u\cdot d\mathbf{R}_u+\sum_u \boldsymbol{\omega}_u\cdot d\boldsymbol{\pi}_u =\mathbf{r}\cdot d\mathbf{R}+\sum_u \mathbf{s}_u\cdot d\mathbf{S}_u+\sum_u \mathbf{n}_u\cdot d\boldsymbol{\theta}_u, \tag{4.5}$$

where the spin terms represent their equivalent expressions in terms of true variables, this represents a contact transformation between true coordinates and momenta. Since $\mathbf{n}_u$ is related to the new variables in the same way that $\boldsymbol{\omega}_u$ is to the old, we can conclude that the components of $\mathbf{n}_u$ will have the correct Poisson bracket relations among themselves.

Further, we can show that we do not need to know either α, β, γ or ξ, η, ζ to find the transformation of $(\omega_x, \omega_y, \omega_z)$ to (η_x, η_y, η_z) or to obtain that from $\mathbf{r}_u$ to $\mathbf{s}_u$. Thus, if we consider an infinitesimal rotation $d\boldsymbol{\sigma}$ related to the Eulerian angles φ, θ, ψ in the same manner that the vectors $d\boldsymbol{\pi}$ and $d\boldsymbol{\vartheta}$ are related to α, β, and γ and ξ, η, ζ, we can write

$$d\boldsymbol{\pi}=d\boldsymbol{\vartheta}+d\boldsymbol{\sigma},$$

when all are given in the same coordinate system. Equation (4.5) can now be written as

$$\sum_u \mathbf{r}_u\cdot d\mathbf{R}_u+(\sum_u \boldsymbol{\omega}_u\cdot d\pi_u)_0 =\mathbf{r}\cdot d\mathbf{R}+\sum_u \mathbf{s}_u\cdot d\mathbf{S}_u-(\sum_u \mathbf{n}_u\cdot d\boldsymbol{\sigma}_u)_c+(\sum_u \mathbf{n}_u\cdot d\pi_u)_c.$$

Identifying coefficients,

$$(\mathbf{n}_u)_c=(\boldsymbol{\omega}_u)_0 \tag{4.6}$$

and

$$\sum_u \mathbf{r}_u\cdot d\mathbf{R}_u=\mathbf{r}\cdot d\mathbf{R}+\sum_u \mathbf{s}_u\cdot d\mathbf{S}_u-(\sum_u \mathbf{n}_u\cdot d\sigma_u)_c. \tag{4.7}$$

Equation (4.6) means that $\mathbf{n}_u$ can be obtained from $\boldsymbol{\omega}_u$ by a rotation through the Eulerian angles φ, θ, ψ. And Eq. (4.7) means that a knowledge of $d\boldsymbol{\sigma}$ alone is sufficient to give the transformation from the $\mathbf{r}_u$ to $\mathbf{r}$ and $\mathbf{s}_u$. The expression for $d\boldsymbol{\sigma}$ can be found if we observe that

$$d\mathbf{n}=d\boldsymbol{\omega}+[d\boldsymbol{\sigma}\times\boldsymbol{\omega}], \tag{4.8}$$

if $d\mathbf{n}$ and $d\boldsymbol{\omega}$ are evaluated in system C and O and the equation is in either system. The detailed calculations are given in Appendix II.

5. CONCLUSION

In discussing the many-body problem in Newtonian mechanics, one may make a contact transformation from the coordinates and momenta of the various bodies to the coordinates of the center of mass and the total momentum, and to internal coordinates and momenta. We have carried through a similar transformation in relativity mechanics, at the cost of giving up the assumption of invariant world lines.

Spin variables are treated as quasi-momenta, referred, in classical mechanics, to true coordinates like Eulerian angles. Their commutation relations are sufficient, however, to justify the transformations without this reference.

The introduction of interaction by replacing m by the function M of the internal variables in Eqs. (2.21) and (2.22), with $\boldsymbol{\omega}=\sum_u\{[\mathbf{s}_u\times\mathbf{S}_u]+\mathbf{n}_u\}$, gives a dynamics which may be modified by any further contact transformation of the variables.

Darwin's Hamiltonian[6] can be derived from our formalism if we set

$$M=m_1+m_2+\tfrac{1}{2}\mathrm{P}^2\left(\frac{1}{m_1}\times\frac{1}{m_2}\right)-\frac{1}{8}\mathrm{P}^4\left(\frac{1}{m_1{}^3}+\frac{1}{m_2{}^3}\right) +\frac{q_1q_2}{\rho}+\frac{q_1q_2}{\rho}\frac{1}{2m_1m_2}\mathrm{P}^2+\frac{q_1q_2}{\rho^3}\frac{1}{2m_1m_2}(\boldsymbol{\rho}\cdot\mathbf{P})^2,$$

[6] C. G. Darwin, Phil. Mag. 39, 537–551 (1920).

and perform a contact transformation generated by the function

$$W=\epsilon[(\boldsymbol{\rho}\cdot\mathbf{P})(\mathbf{R}\cdot\mathbf{P})-(\boldsymbol{\rho}\cdot\mathbf{R})\mathbf{P}^2],$$

where

$$\epsilon=\frac{q_1q_2}{2}\,\frac{m_2-m_1}{m_1m_2(m_1+m_2)}.$$

Similarly, two Hamiltonians[7] derived by Breit will be equivalent to our Hamiltonians if we take

$$\begin{aligned}M=2m&+\frac{\rho^2}{m}-\frac{\rho^4}{4m^3}-J(\rho)-\frac{3\hbar}{4m^2}(\boldsymbol{\rho}\times\mathbf{P})\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\\&-\frac{J}{m^2}\rho^2+\frac{\boldsymbol{\rho}}{2m^2}(\boldsymbol{\rho}\cdot\mathbf{P})+\frac{3\hbar f}{2im^2}(\boldsymbol{\rho}\cdot\mathbf{P})\\&+\frac{\hbar}{2im^2}\rho\frac{df}{d\rho}(\boldsymbol{\rho}\cdot\mathbf{P})-\frac{\hbar^2 f}{8m^2}(15+4\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\\&-\frac{\hbar^2}{4m^2}\frac{df}{\rho d\rho}[5\rho^2+\rho^2(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)-(\boldsymbol{\rho}\cdot\boldsymbol{\sigma}_1)(\boldsymbol{\rho}\cdot\boldsymbol{\sigma}_2)]\\&-\frac{\hbar^2\rho^3}{8m^2}\frac{d}{d\rho}\left(\frac{df}{\rho d\rho}\right)\end{aligned}$$

in one case, and

$$\begin{aligned}M=2m&+\frac{\mathrm{P}^2}{m}-\frac{\mathrm{P}^4}{4m^3}-J(\rho)+\frac{3\mathrm{P}^2}{2m^2}J+\frac{\hbar f}{4m^2}(\boldsymbol{\rho}\times\mathbf{P}\cdot\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\\&+\frac{f}{2m}(\boldsymbol{\rho}\cdot\mathbf{P})^2-\frac{\hbar}{2im^2}\left(5f+\rho\frac{df}{d\rho}\right)(\boldsymbol{\rho}\cdot\mathbf{P})\\&-\frac{\hbar^2}{8m^2}\left[15f+10\rho\frac{df}{d\rho}+\rho^3\frac{d}{d\rho}\left(\frac{df}{\rho d\rho}\right)\right]\end{aligned}$$

in the other case. Here $m_1=m_2=m$ and no further contact transformation is necessary. For two particles with unequal mass Breit derives another classical Hamiltonian[8] which can be derived from our equations if we set

$$\begin{aligned}M=m_1+m_2&+\frac{1}{2}\mathrm{P}^2\left(\frac{1}{m_1}+\frac{1}{m_2}\right)-\frac{1}{8}\mathrm{P}^4\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)\\&-J(\rho)+\frac{1}{2m_1m_2}\frac{1}{\rho}\frac{dJ}{d\rho}(\boldsymbol{\rho}\cdot\mathbf{P})^2-\frac{J(\rho)}{2m_1m_2}\mathrm{P}^2,\end{aligned}$$

and perform a contact transformation generated by the function

$$W=\epsilon(\boldsymbol{\rho}\cdot\mathbf{R})J(\rho),\quad\text{with}\quad \epsilon=\tfrac{1}{2}(m_2-m_1)/(m_1+m_2)^2.$$

APPENDIX I

For a system of particles defined by coordinates $\mathbf{r}_u$, canonically conjugate momenta $\mathbf{R}_u$, mass m_u, and zero spin we try to find a transformation to a new set of variables describing the system which distinguish between external and internal variables.

We are to have $\mathbf{R}$ and $n-1$ functions of the $\mathbf{S}_u$ subject to $\sum\mathbf{S}_u=0$ as variables and wish to find a set of variables canonically conjugate to these, including

$$\mathbf{r}=\frac{1}{m}\sum_u\mathbf{r}_uH_u+\frac{1}{m+H}\sum_u\left\{[[\mathbf{r}_u\times\mathbf{R}_u]\times\mathbf{R}]-\frac{H_u}{H}(\mathbf{r}_u\cdot\mathbf{R})\mathbf{R}\right\}$$

and $n-1$, other sets of three variables. Differentiating Eq. (3.5) in the text we can write:

$$\begin{aligned}d\mathbf{R}_u=d\mathbf{S}_u&+\left(\frac{(\mathbf{R}\cdot\mathbf{S}_u)}{m(m+H)}+\frac{K_u}{m}\right)d\mathbf{R}\\&+\mathbf{R}\left\{\frac{(\mathbf{R}\cdot d\mathbf{S}_u)}{m(m+H)}+\frac{(\mathbf{S}_u\cdot d\mathbf{R})}{m(m+H)}+\frac{(\mathbf{S}_u\cdot d\mathbf{S}_u)}{mK_u}\right.\\&\quad-\frac{(\mathbf{R}\cdot\mathbf{S}_u)}{m(m+H)^2}\frac{(\mathbf{R}\cdot d\mathbf{R})}{H}-\sum_v\frac{(\mathbf{S}_v\cdot d\mathbf{S}_v)}{K_v}\\&\quad\left.\times\left[\frac{K_u}{m^2}+(\mathbf{R}\cdot\mathbf{S}_u)\left(\frac{1}{m^2(m+H)}+\frac{1+m/H}{m(m+H)^2}\right)\right]\right\},\end{aligned}$$

$$\begin{aligned}\sum_u(\mathbf{r}_u\cdot d\mathbf{R}_u)=\sum_u\Bigg(&d\mathbf{S}_u\cdot\mathbf{r}_u+\frac{(\mathbf{r}_u\cdot\mathbf{R})}{m(m+H)}\mathbf{R}+\frac{(\mathbf{r}_u\cdot\mathbf{R})}{mK_u}\mathbf{S}_u\\&-\sum_v(\mathbf{r}_v\cdot\mathbf{R})\left\{\frac{K_v}{m^2}+\frac{(\mathbf{R}\cdot\mathbf{S}_v)}{m^2H}\right\}\frac{\mathbf{S}_u}{K_u}\Bigg)\\&+\Bigg[d\mathbf{R}\cdot\sum_u\Bigg(\mathbf{r}_u\left\{\frac{(\mathbf{R}_u\cdot\mathbf{S}_u)}{m(m+H)}+\frac{K_u}{m}\right\}\\&\quad+\frac{(\mathbf{r}_u\cdot\mathbf{R})}{m(m+H)}\mathbf{S}_u-\mathbf{R}\frac{(\mathbf{r}_u\cdot\mathbf{R})(\mathbf{R}\cdot\mathbf{S}_u)}{mH(m+H)^2}\Bigg)\Bigg].\end{aligned}$$

Comparing this with

$$\textstyle\sum_u(\mathbf{r}_u\cdot d\mathbf{R}_u)\equiv\sum_u(\mathbf{s}_u\cdot d\mathbf{S}_u)+(\mathbf{r}\cdot d\mathbf{R}),\quad \sum_u d\mathbf{S}_u\equiv 0,$$

we get:

$$\mathbf{r}=\sum_u\left\{\left(\frac{(\mathbf{R}\cdot\mathbf{S}_u)}{m(m+H)}+\frac{K_u}{m}\right)\mathbf{r}_u+\frac{1}{m(m+H)}\times\left[(\mathbf{r}_u\cdot\mathbf{R})\mathbf{S}_u-\frac{(\mathbf{r}_u\cdot\mathbf{R})(\mathbf{R}\cdot\mathbf{S}_u)}{H(m+H)}\mathbf{R}\right]\right\},$$

$$\mathbf{r}=\sum_u\left\{\left(\frac{K_u+H_u}{m+H}\right)\mathbf{r}_u+\frac{(\mathbf{r}_u\cdot\mathbf{R})}{m(m+H)}\times\left[\mathbf{S}_u+\frac{HK_u-mH_u}{H(m+H)}\mathbf{R}\right]\right\},\tag{1}$$

[7] G. Breit, Phys. Rev. **51**, 248 (1937), especially p. 259, Eq. (17.6), and p. 260, Eq. (18.2).
[8] G. Breit, reference 7, p. 253, Eq. (13.1).

$$=\sum_u\left\{\left(\frac{H_u}{m}-\frac{(\mathbf{R}\cdot\mathbf{R}_u)}{m(m+H)}\right)\mathbf{r}_u+\frac{(\mathbf{r}_u\cdot\mathbf{R})}{m(m+H)H}[H\mathbf{R}_u-\mathbf{R}H_u]\right\},$$

and, introducing a Lagrange multipler $\boldsymbol{\lambda}$:

$$\mathbf{s}_u+\boldsymbol{\lambda}=\mathbf{r}_u+(\mathbf{r}_u\cdot\mathbf{R})\left\{\frac{\mathbf{S}_u(m+H)+\mathbf{R}K_u}{m(m+H)K_u}\right\}-\sum_v(\mathbf{r}_v\cdot\mathbf{R})\frac{K_vH+(\mathbf{R}\cdot\mathbf{S}_v)}{m^2HK_v}\mathbf{S}_u$$

$$=\mathbf{r}_u+(\mathbf{r}_u\cdot R)\left\{\frac{\mathbf{R}_u(m+H)-\mathbf{R}H_u}{m(m+H)K_u}\right\}-\{\sum_v H_v(\mathbf{r}_v\cdot\mathbf{R})\}\times\left\{\frac{\mathbf{R}_u(m+H)-\mathbf{R}H_u}{mH(m+H)K_u}-\frac{\mathbf{R}}{mH(m+H)}\right\};\quad(2)$$

$$(\mathbf{s}_u+\boldsymbol{\lambda}\cdot\mathbf{R})=\frac{H_u}{K_u}(\mathbf{r}_u\cdot\mathbf{R})-\left(\frac{H_u}{HK_u}-\frac{1}{m}\right)\times\{\textstyle\sum_v H_v(\mathbf{r}_v\cdot\mathbf{R})\},$$

$$\textstyle\sum_u K_u(\mathbf{s}_u+\boldsymbol{\lambda}\cdot\mathbf{R})=\sum_v H_v(\mathbf{r}_v\cdot\mathbf{R}),$$

$$(\mathbf{r}_u\cdot\mathbf{R})=\frac{K_u}{H_u}(\mathbf{s}_u+\boldsymbol{\lambda}\cdot\mathbf{R})-\left(\frac{K}{m+H}-\frac{1}{H}\right)\times\{\textstyle\sum_v K_v(\mathbf{s}_v+\boldsymbol{\lambda}\cdot\mathbf{R})\},$$

$$\mathbf{r}_u=\mathbf{s}_u+\boldsymbol{\lambda}-(\mathbf{s}_u+\boldsymbol{\lambda}\cdot\mathbf{R})\left\{\frac{\mathbf{S}_u(m+H)+\mathbf{R}K_u}{m(m+H)H_u}\right\}+\{\textstyle\sum_v K_u(\mathbf{s}_v+\boldsymbol{\lambda}\cdot\mathbf{R})\}\times\left\{\frac{\mathbf{S}_u(m+H)+\mathbf{R}K_u}{m^2(m+H)H_u}-\frac{\mathbf{R}}{mH(m+H)}\right\},$$

$$\mathbf{r}=\sum_u\left(\frac{K_u+H_u}{m+H}\right)(\mathbf{s}_u+\boldsymbol{\lambda})-\sum_u\frac{(\mathbf{s}_u+\boldsymbol{\lambda}\cdot\mathbf{R})\mathbf{S}_u}{m(m+H)}-\sum_u\frac{K_u(\mathbf{s}_u+\boldsymbol{\lambda}\cdot\mathbf{R})\mathbf{R}}{m(m+H)}$$

$$=\boldsymbol{\lambda}-\frac{(\boldsymbol{\lambda}\cdot\mathbf{R})\mathbf{R}}{H(m+H)}+\sum_v\left(\frac{K_v+H_v}{m+H}\right)\mathbf{s}_u-\sum_v\frac{(\mathbf{s}_u\cdot\mathbf{R})\mathbf{S}_u}{m(m+H)}-\sum_u\frac{K_u(\mathbf{s}_u\cdot\mathbf{R})\mathbf{R}}{mH(m+H)},$$

$$\mathbf{r}_u=\mathbf{r}+\mathbf{s}_u-\sum_v\frac{K_v+H_v}{m+H}\mathbf{s}_v+\sum_v\frac{(\mathbf{s}_v\cdot\mathbf{R})\mathbf{S}_v}{m(m+H)}-(\mathbf{s}_u\cdot\mathbf{R})\left\{\frac{\mathbf{S}_u(m+H)+\mathbf{R}K_u}{m(m+H)H_u}\right\}+\{\textstyle\sum_v K_v(\mathbf{s}_v\cdot\mathbf{R})\}\left\{\frac{\mathbf{S}_u(m+H)+\mathbf{R}K_u}{m^2(m+H)H_u}\right\},\quad(3)$$

while

$$\textstyle\sum_u[\mathbf{r}_u\times\mathbf{R}_u]=[\mathbf{r}\times\mathbf{R}]+\sum_u[\mathbf{s}_u\times\mathbf{S}_u],\quad(4)$$

$$\sum\mathbf{r}_uH_u=\mathbf{r}H-\frac{[\sum_u[\mathbf{s}_u\times\mathbf{S}_u]\times\mathbf{R}]}{m+H}.\quad(5)$$

APPENDIX II

In this Appendix we take the differential of the expression for $\mathbf{n}_u$ as given in Eq. (4.2) of the text and try to put it in the form of Eq. (4.8). Rewriting Eq. (4.2),

$$\mathbf{n}_0=\boldsymbol{\omega}_0+(\boldsymbol{\omega}_0\cdot\mathbf{R})\frac{\mathbf{S}_0}{m(m_0+K_0)}+\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R})}{m(m+H)}\mathbf{R}+\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R}_0)\mathbf{S}_0(m-H)}{m(m_0+K_0)(m_0+H_0)}-\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R}_0)}{m(m_0+H_0)}\mathbf{R}.$$

We notice that it contains $\boldsymbol{\omega}_0$, $\mathbf{R}$, $\mathbf{S}_0$, m, H, R_0, H_0, K_0 as variables. But we can express the differentials of all these variables in terms of the differentials of the first four variables only. From (3.4), (3.5), (3.6) in the text

$$dH=(m/H)dm+(\mathbf{R}\cdot d\mathbf{R})/H,$$

$$d\mathbf{R}_0=-\frac{H_0}{mH}dm\mathbf{R}+\frac{K_0+H_0}{m+H}d\mathbf{R}+\frac{\mathbf{R}(\mathbf{S}_0\cdot d\mathbf{R})}{m(m+H)}-\frac{\mathbf{R}(mH_0-HK_0)}{mH(m+H)^2}(\mathbf{R}\cdot d\mathbf{R})+d\mathbf{S}_0+\mathbf{R}\left\{\frac{(\mathbf{S}_0\cdot d\mathbf{S}_0}{mK_0}+\frac{\mathbf{R}\cdot d\mathbf{S}_0}{m(m+H)}\right\},$$

$$dH_0=-\frac{dm}{mH}(HH_0-mK_0)+\frac{(\mathbf{S}_0\cdot d\mathbf{R})}{m}+\frac{K_0(\mathbf{R}\cdot d\mathbf{R})}{mH}+\frac{(\mathbf{R}\cdot d\mathbf{S}_0)}{m}+\frac{H(\mathbf{S}_0\cdot d\mathbf{S}_0)}{mK_0},$$

with

$$dm=\textstyle\sum_v dK_v=\sum_v(\mathbf{S}_v\cdot d\mathbf{S}_v)/K_v.$$

Using these relationships we can now get $d\mathbf{n}_0$ as the sum of four expressions: (*a*) one containing terms in $d\boldsymbol{\omega}_0$; (*b*) one containing terms in $d\mathbf{R}$; (*c*) one containing terms in $d\mathbf{S}_0$; (*d*) and one containing terms in dm. For (*a*) we have

$$d\mathbf{n}_0=d\boldsymbol{\omega}_0+(d\boldsymbol{\omega}_0\cdot\mathbf{R})\frac{\mathbf{S}_0}{m(m_0+K_0)}+\frac{(d\boldsymbol{\omega}_0\cdot\mathbf{R})}{m(m+H)}\mathbf{R}+\frac{(d\boldsymbol{\omega}_0\cdot\mathbf{R}_0)\mathbf{S}_0(m-H)}{m(m_0+K_0)(m_0+H_0)}-\frac{(d\boldsymbol{\omega}_0\cdot\mathbf{R}_0)}{m(m_0+H_0)}\mathbf{R}.$$

For (*d*) we have

$$dH=(m/H)dm;\quad d\mathbf{R}_0=-(H_0/mH)dm\mathbf{R};\quad dH_0=-(dm/mH)(HH_0-mK_0),$$

and

$$d\mathbf{n}_0=\left\{\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R})\mathbf{S}_0}{m(m_0+K_0)}+\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R})\mathbf{R}}{m(m+H)}+\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R}_0)\mathbf{S}_0(m-H)}{m(m_0+K_0)(m_0+H_0)}-\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R}_0)\mathbf{R}}{m(m_0+H_0)}\right\}\left(-\frac{dm}{m}\right)+\left\{\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R})\mathbf{S}_0(m-H)}{m(m_0+K_0)(m_0+H_0)}\right.$$

$$\left.-\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R})\mathbf{R}}{m(m_0+H_0)}\right\}\left(-\frac{H_0dm}{mH}\right)-\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R})\mathbf{R}}{mH(m+H)}dm+\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R}_0)\mathbf{S}_0\left(\frac{H-m}{m}\right)dm}{m(m_0+K_0)(m_0+H_0)}$$

$$+\left\{\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R}_0)\mathbf{S}_0(m-H)}{m(m_0+K_0)(m_0+H_0)}-\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R}_0)}{m(m_0+H_0)}\right\}\frac{dm(HH_0-mK_0)}{mH(m_0+H_0)},$$

$$d\mathbf{n}_0\equiv\frac{dm}{m}\frac{1}{H(m_0+H_0)}[\mathbf{n}_0\times[\mathbf{R}\times\mathbf{S}_0]].$$

For (*c*) we have:

$$d\mathbf{R}_0=d\mathbf{S}_0+\mathbf{R}\left\{\frac{(\mathbf{S}_0\cdot d\mathbf{S}_0)}{mK_0}+\frac{(\mathbf{R}\cdot d\mathbf{S}_0}{m(m+H)}\right\},\quad dH_0=\frac{(\mathbf{R}\cdot d\mathbf{S}_0)}{m}+\frac{H(\mathbf{S}_0\cdot d\mathbf{S}_0)}{mK_0},$$

and

$$d\mathbf{n}_0=\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R})d\mathbf{S}_0}{m(m_0+K_0)}-\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R})\mathbf{S}_0(\mathbf{S}_0\cdot d\mathbf{S}_0)}{m(m_0+K_0)^2K_0}+\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R})d\mathbf{S}_0(m-H)}{m(m_0+K_0)(m_0+H_0)}-\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R}_0)\mathbf{S}_0(\mathbf{S}_0\cdot d\mathbf{S}_0)(m-H)}{m(m_0+K_0)^2K_0(m_0+H_0)}$$

$$+\left\{\frac{\mathbf{S}_0(m-H)}{m(m_0+K_0)}-\frac{\mathbf{R}}{m}\right\}\left\{\frac{(\boldsymbol{\omega}_0\cdot d\mathbf{S}_0)+(\boldsymbol{\omega}_0\cdot\mathbf{R})}{(m_0+H_0)}-\frac{\left[\frac{(\mathbf{S}_0\cdot d\mathbf{S}_0)}{mK_0}+\frac{(\mathbf{R}\cdot d\mathbf{S}_0)}{m(m+H)}\right]}{m_0+H_0}-\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R}_0)}{(m_0+H_0)^2}\left[\frac{(\mathbf{R}\cdot d\mathbf{S}_0)}{m}+\frac{H(\mathbf{S}_0\cdot d\mathbf{S}_0)}{mK_0}\right]\right\},$$

$$=\frac{d\mathbf{S}_0}{m(m_0+K_0)}\left\{\frac{(H-m)}{(m_0+H_0)}(\mathbf{n}_0\cdot\mathbf{S}_0)+\frac{(m_0+K_0)}{(m_0+H_0)}(\mathbf{n}_0\cdot\mathbf{R})\right\}+\left\{\frac{\mathbf{S}_0(m-H)}{m(m_0+K_0)}-\frac{\mathbf{R}}{m_0}\right\}\left\{\frac{(\mathbf{n}_0\cdot d\mathbf{S}_0)}{(m_0+H_0)}+\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R})(\mathbf{S}_0\cdot d\mathbf{S}_0)m_0}{mK_0(m_0+H_0)(m_0+K_0)}\right.$$

$$\left.-\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R}_0)(\mathbf{S}_0\cdot d\mathbf{S}_0)(Hm_0+mK_0)}{m(m_0+K_0)K_0(m_0+H_0)^2}\right\}-\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R})\mathbf{S}_0(\mathbf{S}_0\cdot d\mathbf{S}_0)}{m(m_0+K_0)^2K_0}-\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R}_0)(\mathbf{S}_0\cdot d\mathbf{S}_0)}{m(m_0+K_0)^2K_0(m_0+H_0)},$$

$$=\frac{d\mathbf{S}_0}{m(m_0+K_0)}\left\{\frac{(H-m)}{(m_0+H_0)}(\mathbf{n}_0\cdot\mathbf{S}_0)+\left(\frac{m_0+K_0}{m_0+H_0}\right)(\mathbf{n}_0\cdot\mathbf{R})\right\}+\left\{\frac{\mathbf{S}_0(m-H)}{m(m_0+K_0)}-\frac{\mathbf{R}}{m}\right\}\left\{\frac{(\mathbf{n}_0\cdot d\mathbf{S}_0)}{(m_0+H_0)}-\frac{(\mathbf{n}_0\cdot\mathbf{S}_0)(\mathbf{S}_0\cdot d\mathbf{S}_0)}{K_0(m_0+H_0)(m_0+K_0)}\right\}$$

$$-\frac{\mathbf{S}_0(\mathbf{S}_0\cdot d\mathbf{S}_0)}{mK_0(m_0+K_0)^2}\left\{\frac{(H-m)}{(m_0+H_0)}(\mathbf{n}_0\cdot\mathbf{S}_0)+\frac{(m_0+K_0)}{(m_0+H_0)}(\mathbf{n}_0\cdot\mathbf{R})\right\},$$

and

$$d\mathbf{n}_0=[\mathbf{n}_0\times[d\mathbf{S}_0\times\mathbf{R}]]\frac{1}{m(m_0+H_0)}+[\mathbf{n}_0\times[d\mathbf{S}_0\times\mathbf{S}_0]]\frac{(H-m)}{m(m_0+K_0)(m_0+H_0)}+[\mathbf{n}_0\times[\mathbf{R}\times\mathbf{S}_0]]\frac{(\mathbf{S}_0\cdot d\mathbf{S}_0)}{mK_0(m_0+H_0)(m_0+K_0)}.$$

Finally for (*b*) we have:

$$dH=\frac{(\mathbf{R}\cdot d\mathbf{R})}{H},\quad dH_0=\frac{(\mathbf{S}_0\cdot d\mathbf{R})}{m}+\frac{K_0(\mathbf{R}\cdot d\mathbf{R})}{mH},\quad d\mathbf{R}_0=\frac{K_0+H_0}{m+H}d\mathbf{R}+\frac{\mathbf{R}(\mathbf{S}_0\cdot d\mathbf{R})}{m(m+H)}-\frac{\mathbf{R}(mH_0-HK_0)}{mH(m+H)^2}(\mathbf{R}\cdot d\mathbf{R}),$$

and

$$d\mathbf{n}_0=\frac{(\omega_0\cdot d\mathbf{R})\mathbf{S}_0}{m(m_0+K_0)}+\frac{(\omega_0\cdot d\mathbf{R})\mathbf{R}}{m(m+H)}+\frac{(\omega_0\cdot\mathbf{R})d\mathbf{R}}{m(m+H)}-\frac{(\omega_0\cdot\mathbf{R})\mathbf{R}}{m(m+H)^2}\frac{(\mathbf{R}\cdot d\mathbf{R})}{H}+\frac{(\omega_0\cdot d\mathbf{R})\mathbf{S}_0(m-H)(K_0+H_0)}{m(m_0+K_0)(m_0+H_0)(m+H)}$$

$$+\frac{(\omega_0\cdot\mathbf{R})(\mathbf{S}_0\cdot d\mathbf{R})(m-H)\mathbf{S}_0}{m^2(m_0+K_0)(m_0+H_0)(m+H)}-\frac{(\omega_0\cdot\mathbf{R})(\mathbf{R}\cdot d\mathbf{R})(mH_0-HK_0)\mathbf{S}_0(m-H)}{m^2H(m+H)^2(m_0+K_0)(m_0+H_0)}-\frac{(\omega_0\cdot d\mathbf{R})(\mathbf{R})(K_0+H_0)}{m(m_0+H_0)(m+H)}$$

$$-\frac{(\omega_0\cdot\mathbf{R})(\mathbf{S}_0\cdot d\mathbf{R})\mathbf{R}}{m^2(m_0+H_0)(m+H)}+\frac{(\omega_0\cdot\mathbf{R})(mH_0-HK_0)(\mathbf{R}\cdot d\mathbf{R})\mathbf{R}}{m^2H(m+H)^2(m_0+H_0)}-\frac{(\omega_0\cdot\mathbf{R}_0)\mathbf{S}_0(\mathbf{R}\cdot d\mathbf{R})}{m(m_0+K_0)(m_0+H_0)H}-\frac{(\omega_0\cdot\mathbf{R}_0)\mathbf{S}_0(m-H)}{m(m_0+K_0)(m_0+H_0)^2}$$

$$\times\left\{\frac{(\mathbf{S}_0\cdot d\mathbf{R})}{m}+\frac{K_0(\mathbf{R}\cdot d\mathbf{R})}{mH}\right\}-\frac{(\omega_0\cdot\mathbf{R}_0)d\mathbf{R}}{m(m_0+H_0)}+\frac{(\omega_0\cdot\mathbf{R}_0)\mathbf{R}}{m(m_0+H_0)^2}\left\{\frac{(\mathbf{S}_0\cdot d\mathbf{R})}{m}+\frac{K_0(\mathbf{R}\cdot d\mathbf{R})}{mH}\right\}.$$

In this expression we recognize terms in $\mathbf{S}_0$(i), terms in $d\mathbf{R}$(ii), and terms in $\mathbf{R}$(iii). For (i) we have

$$\frac{(\omega_0\cdot d\mathbf{R})}{m(m_0+K_0)}+\frac{(\omega_0\cdot d\mathbf{R})(m-H)(K_0+H_0)}{m(m_0+K_0)(m_0+H_0)(m+H)}+\frac{(\omega_0\cdot\mathbf{R})(\mathbf{S}_0\cdot d\mathbf{R})(m-H)}{m^2(m_0+K_0)(m_0+H_0)(m+H)}-\frac{(\omega_0\cdot\mathbf{R})(\mathbf{R}\cdot d\mathbf{R})(mH_0-HK_0)(m-H)}{m^2H(m+H)^2(m_0+K_0)(m_0+H_0)}$$

$$-\frac{(\omega_0\cdot\mathbf{R}_0)(\mathbf{R}\cdot d\mathbf{R})}{m(m_0+K_0)(m_0+H_0)H}-\frac{(\omega_0\cdot\mathbf{R}_0)(m-H)}{m(m_0+K_0)(m_0+H_0)^2}\frac{(\mathbf{S}_0\cdot d\mathbf{R})}{m}-\frac{(\omega_0\cdot\mathbf{R}_0)(m-H)K_0(\mathbf{R}\cdot d\mathbf{R})}{m(m_0+K_0)(m_0+H_0)^2mH}$$

$$=\frac{(\omega_0\cdot d\mathbf{R})(mm_0+2mH_0+m_0H+mK_0-HK_0)}{m(m+H)(m_0+K_0)(m_0+H_0)}+(\mathbf{S}_0\cdot d\mathbf{R})\left\{\frac{(\omega_0\cdot\mathbf{R})(m-H)}{m^2(m_0+K_0)(m_0+H_0)(m+H)}-\frac{(\omega_0\cdot\mathbf{R}_0)(m-H)}{m^2(m_0+K_0)(m_0+H_0)^2}\right\}$$

$$-(\mathbf{R}\cdot d\mathbf{R})\left\{\frac{(\omega_0\cdot\mathbf{R})(mH_0-HK_0)(m-H)}{m^2H(m+H)^2(m_0+K_0)(m_0+H_0)}+\frac{(\omega_0\cdot\mathbf{R}_0)(mm_0+mH_0+mK_0-HK_0)}{m^2(m_0+K_0)(m_0+H_0)^2H}\right\}$$

$$\equiv\frac{(\mathbf{n}_0\cdot d\mathbf{R})}{m(m_0+H_0)}-\frac{(\mathbf{n}_0\cdot\mathbf{R})(\mathbf{R}\cdot d\mathbf{R})}{mH(m+H)(m_0+H_0)}+\frac{2(\mathbf{n}_0\cdot d\mathbf{R})(mH_0-HK_0)}{m(m+H)(m_0+K_0)(m_0+H_0)}-\frac{2(\mathbf{n}_0\cdot\mathbf{R})(\mathbf{S}_0\cdot d\mathbf{R})}{m(m+H)(m_0+K_0)(m_0+H_0)}.$$

For (ii) we have

$$\frac{(\omega_0\cdot\mathbf{R})}{m(m+H)}-\frac{(\omega_0\cdot\mathbf{R}_0)}{m(m_0+H_0)}\equiv-\frac{(\mathbf{n}_0\cdot\mathbf{R}_0)}{m(m_0+H_0)}+\frac{(\mathbf{n}_0\cdot\mathbf{R})}{m(m+H)}-\frac{2(\mathbf{n}_0\cdot\mathbf{S}_0)(mH_0-HK_0)}{m(m+H)(m_0+K_0)(m_0+H_0)}+\frac{2(\mathbf{n}_0\cdot\mathbf{R})(K_0-m_0)}{m(m+H)(m_0+H_0)}.$$

For (iii) we have

$$\frac{(\omega_0\cdot d\mathbf{R})}{m(m+H)}-\frac{(\omega_0\cdot d\mathbf{R})(K_0+H_0)}{m(m_0+H_0)(m+H)}-\frac{(\omega_0\cdot\mathbf{R})(\mathbf{R}\cdot d\mathbf{R})}{mH(m+H)^2}-\frac{(\omega_0\cdot\mathbf{R})(\mathbf{S}_0\cdot d\mathbf{R})}{m^2(m_0+H_0)(m+H)}+\frac{(\omega_0\cdot\mathbf{R})(mH_0-HK_0)(\mathbf{R}\cdot d\mathbf{R})}{m^2H(m+H)^2(m_0+H_0)}$$

$$+\frac{(\omega_0\cdot R_0)}{m(m_0+H_0)^2}\left\{\frac{(\mathbf{S}_0\cdot d\mathbf{R})}{m}+\frac{K_0(\mathbf{R}\cdot d\mathbf{R})}{mH}\right\}$$

$$\equiv\frac{(\mathbf{n}_0\cdot d\mathbf{R})(K_0+H_0)}{m(m_0+H_0)(m+H)}-\frac{(\mathbf{n}_0\cdot d\mathbf{R})}{m(m+H)}-\frac{(\mathbf{n}_0\cdot\mathbf{R})(\mathbf{R}\cdot d\mathbf{R})(K_0+H_0)}{mH(m+H)^2(m_0+H_0)}+\frac{(\mathbf{n}_0\cdot\mathbf{R}_0)(\mathbf{R}\cdot d\mathbf{R})}{mH(m+H)(m_0+H_0)}-\frac{2(\mathbf{n}_0\cdot d\mathbf{R})(K_0-m_0)}{m(m+H)(m_0+H_0)}$$

$$-\frac{2(\mathbf{n}_0\cdot\mathbf{R})(K_0+H_0)(\mathbf{S}_0\cdot d\mathbf{R})}{m(m+H)^2(m_0+K_0)(m_0+H_0)}+\frac{2(\mathbf{n}_0\cdot\mathbf{R}_0)(\mathbf{S}_0\cdot d\mathbf{R})}{m(m+H)(m_0+K_0)(m_0+H_0)}.$$

Gathering all these expressions, we get

$$d\mathbf{n}_0=-\frac{(\mathbf{n}_0\cdot\mathbf{R}_0)d\mathbf{R}}{m(m_0+H_0)}+\frac{(\mathbf{n}_0\cdot d\mathbf{R})\mathbf{R}_0}{m(m_0+H_0)}-\frac{(\mathbf{n}_0\cdot d\mathbf{R})\mathbf{R}}{m(m+H)}+\frac{(\mathbf{n}_0\cdot\mathbf{R})d\mathbf{R}}{m(m+H)}-\frac{(\mathbf{n}_0\cdot\mathbf{R})(\mathbf{R}\cdot d\mathbf{R})\mathbf{R}_0}{mH(m+H)(m_0+H_0)}+\frac{(\mathbf{n}_0\cdot\mathbf{R}_0)(\mathbf{R}\cdot d\mathbf{R})\mathbf{R}}{mH(m+H)(m_0+H_0)}$$
$$-\frac{2(\mathbf{n}_0\cdot\mathbf{S}_0)(mH_0-HK_0)d\mathbf{R}}{m(m+H)(m_0+K_0)(m_0+H_0)}+\frac{2(\mathbf{n}_0\cdot d\mathbf{R})\mathbf{S}_0(mH_0-HK_0)}{m(m+H)(m_0+K_0)(m_0+H_0)}-\frac{2(\mathbf{n}_0\cdot d\mathbf{R})\mathbf{R}(K_0-m_0)}{m(m+H)(m_0+H_0)}$$
$$+\frac{2(\mathbf{n}_0\cdot\mathbf{R})d\mathbf{R}(K_0-m_0)}{m(m+H)(m_0+H_0)}-\frac{2(\mathbf{n}_0\cdot\mathbf{R})\mathbf{R}_0(\mathbf{S}_0\cdot d\mathbf{R})}{m(m+H)(m_0+K_0)(m_0+H_0)}+\frac{2(\mathbf{n}_0\cdot\mathbf{R}_0)\mathbf{R}(\mathbf{S}_0\cdot d\mathbf{R})}{m(m+H)(m_0+K_0)(m_0+H_0)}.$$

Now combining the terms two by two, we can write this last expression as:

$$d\mathbf{n}_0=\frac{[\mathbf{n}_0\times[\mathbf{R}_0\times d\mathbf{R}]]}{m(m_0+H_0)}-\frac{[\mathbf{n}_0\times[\mathbf{R}\times d\mathbf{R}]]}{m(m+H)}-\frac{[\mathbf{n}_0\times[\mathbf{R}_0\times\mathbf{R}]](\mathbf{R}\cdot d\mathbf{R})}{mH(m+H)(m_0+H_0)}-\frac{2[\mathbf{n}_0\times[d\mathbf{R}\times\mathbf{S}_0]](mH_0-HK_0)}{m(m+H)(m_0+K_0)(m_0+H_0)}$$
$$-\frac{2[\mathbf{n}_0\times[\mathbf{R}\times d\mathbf{R}]](K_0-m_0)}{m(m+H)(m_0+H_0)}-\frac{2[\mathbf{n}_0\times[\mathbf{R}_0\times\mathbf{R}]](\mathbf{S}_0\cdot d\mathbf{R})}{m(m+H)(m_0+K_0)(m_0+H_0)}$$
$$\equiv\frac{[\mathbf{n}_0\times[\mathbf{R}_0\times d\mathbf{R}]]}{m(m_0+H_0)}-\frac{[\mathbf{n}_0\times[\mathbf{R}\times d\mathbf{R}]]}{m(m+H)}-\frac{[\mathbf{n}_0\times[\mathbf{R}_0\times\mathbf{R}]](\mathbf{R}\cdot d\mathbf{R})}{mH(m+H)(m_0+H_0)}-\frac{2[\mathbf{n}_0\times\mathbf{S}_0](\mathbf{R}_0\times[\mathbf{R}\times d\mathbf{R}])}{m(m+H)(m_0+K_0)(m_0+H_0)}.$$

Now collecting the (a), (b) and (c) expressions for $d\mathbf{n}_0$ we finally get:

$$d\mathbf{n}_0=d\boldsymbol{\omega}_0+\frac{(d\boldsymbol{\omega}_0\cdot\mathbf{R})\mathbf{S}_0}{m(m_0+K_0)}+\frac{(d\boldsymbol{\omega}_0\cdot\mathbf{R})\mathbf{R}}{m(m+H)}+\frac{(d\boldsymbol{\omega}_0\cdot\mathbf{R}_0)\mathbf{S}_0(m-H)}{m(m_0+K_0)(m_0+H_0)}-\frac{(d\boldsymbol{\omega}_0\cdot\mathbf{R}_0)}{m(m_0+H_0)}\mathbf{R}+\Bigg[\frac{[\mathbf{S}_0\times\mathbf{R}]dm}{mH(m_0+H_0)}+\frac{[\mathbf{R}\times d\mathbf{S}_0]}{m(m_0+H_0)}$$
$$+\frac{[\mathbf{S}_0\times d\mathbf{S}_0](H-m)}{m(m_0+K_0)(m_0+H_0)}+\frac{[\mathbf{S}_0\times\mathbf{R}](\mathbf{S}_0\cdot d\mathbf{S}_0)}{mK_0(m_0+H_0)(m_0+K_0)}-\frac{[\mathbf{R}_0\times d\mathbf{R}]}{m(m_0+H_0)}+\frac{[\mathbf{R}\times d\mathbf{R}]}{m(m+H)}+\frac{[\mathbf{R}_0\times\mathbf{R}](\mathbf{R}\cdot d\mathbf{R})}{mH(m+H)(m_0+H_0)}$$
$$+\frac{2\mathbf{S}_0(\mathbf{R}_0\cdot[\mathbf{R}\times d\mathbf{R}])}{m(m+H)(m_0+K_0)(m_0+H_0)}\times\mathbf{n}_0\Bigg].$$

Also,

$$\Bigg(\frac{[\mathbf{S}_0\times\mathbf{R}]dm}{mH(m_0+H_0)}+\frac{[\mathbf{R}\times d\mathbf{S}_0]}{m(m_0+H_0)}+\frac{[\mathbf{S}_0\times d\mathbf{S}_0](H-m)}{m(m_0+K_0)(m_0+H_0)}+\frac{[\mathbf{S}_0\times\mathbf{R}](\mathbf{S}_0\cdot d\mathbf{S}_0)}{mK_0(m_0+H_0)(m_0+K_0)}-\frac{(\mathbf{R}_0\times d\mathbf{R})}{m(m_0+H_0)}+\frac{[\mathbf{R}\times d\mathbf{R}]}{m(m+H)}$$
$$+\frac{[\mathbf{R}_0\times\mathbf{R}](\mathbf{R}\cdot d\mathbf{R})}{mH(m+H)(m_0+H_0)}+\frac{2\mathbf{S}_0(\mathbf{R}_0\cdot[\mathbf{R}\times d\mathbf{R}])}{m(m+H)(m_0+K_0)(m_0+H_0)}\cdot\mathbf{n}_0\Bigg)$$
$$=\Bigg(\frac{[\mathbf{S}_0\times\mathbf{R}]dm}{mH(m_0+H_0)}+\frac{[\mathbf{R}\times d\mathbf{S}_0]}{m(m_0+H_0)}+\frac{[\mathbf{S}_0\times d\mathbf{S}_0](H-m)}{m(m_0+K_0)(m_0+H_0)}+\frac{[\mathbf{S}_0\times\mathbf{R}](\mathbf{S}_0\cdot dS_0)}{mK_0(m_0+H_0)(m_0+K_0)}+\frac{2\mathbf{R}(\mathbf{S}_0\cdot[\mathbf{R}\times d\mathbf{S}_0])}{m(m+H)(m_0+K_0)(m_0+H_0)}$$
$$-\frac{[\mathbf{R}_0\times d\mathbf{R}]}{m(m_0+H_0)}+\frac{[\mathbf{R}\times d\mathbf{R}]}{m(m+H)}+\frac{[\mathbf{R}_0\times\mathbf{R}](\mathbf{R}\cdot d\mathbf{R})}{mH(m+H)(m_0+H_0)}\cdot\boldsymbol{\omega}_0\Bigg).$$

Using this result we can now write the contact transformation:

$$\sum_u(\mathbf{r}_u\cdot d\mathbf{R}_u)+\sum_u(\boldsymbol{\omega}_u\cdot d\boldsymbol{\pi}_u)$$
$$=\sum_u\Bigg(d\mathbf{S}_u\cdot\mathbf{r}_u+\frac{(\mathbf{r}_u\cdot\mathbf{R})}{m(m+H)}\mathbf{R}+\frac{(\mathbf{r}_u\cdot\mathbf{R})}{mK_u}\mathbf{S}_u-\sum_v\frac{(\mathbf{r}_v\cdot\mathbf{R})H_v\mathbf{S}_v}{mHK_u}+\frac{[\boldsymbol{\omega}_u\times\mathbf{R}]}{m(m_u+H_u)}+\frac{[\boldsymbol{\omega}_u\times\mathbf{S}_u](H-m)}{m(m_u+K_u)(m_u+H_u)}$$

$$+\frac{(\boldsymbol{\omega}_u\cdot[\mathbf{S}_u+R])\mathbf{S}_u}{mK_u(m_u+H_u)(m_u+K_u)}+\frac{2(\boldsymbol{\omega}_u\cdot\mathbf{R})[\mathbf{S}_u\times\mathbf{R}]}{m(m+H)(m_u+K_u)(m_u+H_u)}+\sum_v\frac{(\boldsymbol{\omega}_v\cdot[\mathbf{S}_v\times\mathbf{R}])}{(m_v+H_v)}\frac{\mathbf{S}_u}{mHK_u}\Bigg)$$

$$+\Bigg(d\mathbf{R}\cdot\sum_u\Bigg\{\mathbf{r}_u\Bigg(\frac{H_u}{m}-\frac{(\mathbf{R}\cdot\mathbf{R}_u)}{m(m+H)}+\frac{(\mathbf{r}_u\cdot\mathbf{R})}{mH(m+H)}(H\mathbf{R}_u-\mathbf{R}H_u)-\frac{[\boldsymbol{\omega}_u\times\mathbf{R}_u]}{m(m_u+H_u)}+\frac{[\boldsymbol{\omega}_u\times\mathbf{R}]}{m(m+H)}$$

$$+\frac{([\boldsymbol{\omega}_u\times\mathbf{R}]\cdot\mathbf{R})\mathbf{R}}{mH(m+H)(m_u+H_u)}\Bigg\}\Bigg)+\sum_u\Bigg(d\boldsymbol{\vartheta}_u\cdot\boldsymbol{\omega}_u+(\boldsymbol{\omega}_u\cdot\mathbf{R})\frac{\mathbf{S}_u}{m(m_u+K_u)}+\frac{(\boldsymbol{\omega}_u\cdot\mathbf{R})\mathbf{R}}{m(m+H)}$$

$$+\frac{(\boldsymbol{\omega}_u\cdot\mathbf{R}_u)\mathbf{S}_u(m-H)}{m(m_u+K_u)(m_u+H_u)}-\frac{(\boldsymbol{\omega}_u\cdot\mathbf{R}_u)\mathbf{R}}{m(m_u+H_u)}\Bigg),\qquad(1)$$

which is of the form

$$\textstyle\sum_u(\mathbf{r}_u\cdot d\mathbf{R}_u)+\sum_u(\boldsymbol{\omega}_u\cdot d\boldsymbol{\pi}_u)\equiv\sum_u(\mathbf{S}_u\cdot d\mathbf{S}_u)+(\mathbf{r}\cdot d\mathbf{R})+\sum_u(\mathbf{n}_u\cdot d\boldsymbol{\theta}_u),$$

where $\sum d\mathbf{S}_u=0$. We can also write for the total angular momentum:

$$\textstyle\sum_u[\mathbf{s}_u\times\mathbf{S}_u]+[\mathbf{r}\times\mathbf{R}]+\sum_u\mathbf{n}_u$$

$$=\sum_u[\mathbf{r}_u\times\mathbf{R}_u]+\sum_u\frac{[[\boldsymbol{\omega}_u\times\mathbf{R}]\times\mathbf{S}_u]}{m(m_u+H_u)}+\sum_u\frac{[[\boldsymbol{\omega}_u\times\mathbf{S}_u]\times\mathbf{S}_u](H-m)}{m(m_u+K_u)(m_u+H_u)}+\sum_u\frac{2(\boldsymbol{\omega}_u\cdot\mathbf{R})[[\mathbf{S}_u\times\mathbf{R}]\times\mathbf{S}_u]}{m(m+H)(m_u+K_u)(m_u+H_u)}$$

$$-\sum_u\frac{[[\boldsymbol{\omega}_u\times\mathbf{R}_u]\times\mathbf{R}]}{m(m_u+H_u)}+\sum_u\boldsymbol{\omega}_u+\sum_u\frac{(\boldsymbol{\omega}_u\cdot\mathbf{R})\mathbf{S}_u}{m(m_u+K_u)}+\sum_u\frac{(\boldsymbol{\omega}_u\cdot\mathbf{R})\mathbf{R}}{m(m+H)}+\sum_u\frac{(\boldsymbol{\omega}_u\cdot\mathbf{R}_u)\mathbf{S}_u(m-H)}{m(m_u+K_u)(m_u+H_u)}$$

$$-\sum_u\frac{(\boldsymbol{\omega}_0\cdot\mathbf{R}_u)\mathbf{R}}{m(m_u+H_u)}=\sum_u[\mathbf{r}_u\times\mathbf{R}_u]+\sum_u\boldsymbol{\omega}_u.\qquad(2)$$

Also,

$$H\mathbf{r}-\frac{[\sum_u[\mathbf{S}_u\times\mathbf{S}_u]+\sum_u\mathbf{n}_u\times\mathbf{R}]}{H+m}=\sum_u H_u\mathbf{r}_u-\sum_u\frac{[\boldsymbol{\omega}_u\times\mathbf{R}_u]}{H_u+m_u}.\qquad(3)$$

Following the same method as above, Appendix I, now:

$$\mathbf{S}_u+\boldsymbol{\lambda}=\mathbf{r}_u+(\mathbf{r}_u\cdot\mathbf{R})\left\{\frac{\mathbf{S}_u(m+H)+\mathbf{R}K_u}{m(m+H)K_u}\right\}-\sum_v(\mathbf{r}_v\cdot\mathbf{R})H_v\frac{\mathbf{S}_u}{mHK_u}+\frac{[\boldsymbol{\omega}_u\times\mathbf{R}]}{m(m_u+H_u)}+\frac{[\boldsymbol{\omega}_u\times\mathbf{S}_u](H-m)}{m(m_u+K_u)(m_u+H_u)}$$

$$+\frac{(\boldsymbol{\omega}_u\cdot[\mathbf{S}_u\times\mathbf{R}])\mathbf{S}_u}{mK_u(m_u+H_u)(m_u+K_u)}+\frac{2(\boldsymbol{\omega}_u\cdot\mathbf{R})[\mathbf{S}_u\times\mathbf{R}]}{m(m+H)(m_u+K_u)(m_u+H_u)}+\sum_v\frac{(\boldsymbol{\omega}_v\cdot[\mathbf{S}_v\times\mathbf{R}])}{m_v+H_v}\frac{\mathbf{S}_u}{mHK_u}.\qquad(4)$$

$$(\mathbf{S}_u+\boldsymbol{\lambda}\cdot\mathbf{R})=\frac{H_u}{K_u}(\mathbf{r}_u\cdot\mathbf{R})-\frac{(mH_u-HK_u)}{mHK_u}\{\textstyle\sum_v H_v(\mathbf{r}_v\cdot\mathbf{R})\}$$

$$+\frac{\{[\boldsymbol{\omega}_u\times\mathbf{S}_u]\cdot\mathbf{R}\}(H_u-K_u)}{K_u(m_u+H_u)(m_u+K_u)}+\sum_v\frac{(\boldsymbol{\omega}_v\cdot[\mathbf{S}_u\times\mathbf{R}])}{(m_v+H_v)}\frac{(mH_u-HK_u)}{mHK_u},$$

$$\sum_u K_u(\mathbf{S}_u+\boldsymbol{\lambda}\cdot\mathbf{R})=\sum_u H_u(\mathbf{r}_u\cdot\mathbf{R})+\sum_u\frac{([\boldsymbol{\omega}_u\times\mathbf{S}_u]\cdot\mathbf{R})(H_u-K_u)}{(m_u+H_u)(m_u+K_u)},$$

$$(\mathbf{r}_u\cdot\mathbf{R})=\frac{K_u}{H_u}(\mathbf{s}_u+\boldsymbol{\lambda}\cdot\mathbf{R})+\frac{(mH_u-HK_u)}{mHH_u}\sum_v K_v(\mathbf{s}_v+\boldsymbol{\lambda}\cdot\mathbf{R})-\frac{\{[\boldsymbol{\omega}_u\times\mathbf{S}_u]\cdot\mathbf{R}\}(H_u-K_u)}{H_u(m_u+H_u)(m_u+K_u)}$$

$$-\frac{(mH_u-HK_u)}{mHH_u}\sum_v\frac{([\boldsymbol{\omega}_v\times\mathbf{S}_v]\cdot\mathbf{R})}{m_v+K_v},$$

$$\mathbf{r}_u=\mathbf{s}_u+\boldsymbol{\lambda}-(\mathbf{s}_u+\boldsymbol{\lambda}\cdot\mathbf{R})\left\{\frac{\mathbf{S}_u(m+H)+\mathbf{R}K_u}{m(m+H)H_u}\right\}+\left\{\sum_v K_v(\mathbf{s}_v+\boldsymbol{\lambda}\cdot\mathbf{R})-\sum_v\frac{([\boldsymbol{\omega}_v\times\mathbf{S}_v]\cdot\mathbf{R})}{m_v+K_v}\right\}$$
$$\times\left\{\frac{\mathbf{S}_u(m+H)+\mathbf{R}K_u}{m^2(m+H)H_u}-\frac{\mathbf{R}}{mH(m+H)}\right\}-\frac{(\boldsymbol{\omega}_u\cdot[\mathbf{S}_u\times\mathbf{R}])\mathbf{S}_u}{mH_u(m_u+H_u)(m_u+K_u)}$$
$$+\frac{(\boldsymbol{\omega}_u\cdot[\mathbf{S}_u\times\mathbf{R}])\mathbf{R}(H_u-K_u)}{m(m+H)H_u(m_u+H_u)(m_u+K_u)}-\frac{[\boldsymbol{\omega}_u\times\mathbf{R}]}{m(m_u+H_u)}$$
$$-\frac{[\boldsymbol{\omega}_u\times\mathbf{S}_u](H-m)}{m(m_u+K_u)(m_u+H_u)}-\frac{2(\boldsymbol{\omega}_u\cdot\mathbf{R})[\mathbf{S}_u\times\mathbf{R}]}{m(m+H)(m_u+K_u)(m_u+H_u)},$$

$$\mathbf{r}=\sum_u\left(\frac{K_u+H_u}{m+H}\right)(\mathbf{s}_u+\boldsymbol{\lambda})-\sum_u\frac{(\mathbf{s}_u+\boldsymbol{\lambda}\cdot\mathbf{R})}{m(m+H)}\mathbf{s}_u-\sum_u\frac{K_u(\mathbf{s}_u+\boldsymbol{\lambda}\cdot\mathbf{R})\mathbf{R}}{mH(m+H)}+\sum_u\frac{(\boldsymbol{\omega}_u\cdot[\mathbf{S}_u\times\mathbf{R}])\mathbf{R}}{m(m+H)H(m_u+K_u)}$$
$$-\frac{2\sum_u(\boldsymbol{\omega}_u\cdot[\mathbf{S}_u\times\mathbf{R}])\mathbf{S}_u}{m(m+H)(m_u+H_u)(m_u+K_u)}+\sum_u\left(\frac{K_u+H_u}{m+H}\right)\left\{-\frac{[\boldsymbol{\omega}_u\times\mathbf{R}]}{m(m_u+H_u)}\right.$$
$$\left.-\frac{[\boldsymbol{\omega}_u\times\mathbf{S}_u](H-m)}{m(m_u+K_u)(m_u+H_u)}-\frac{2(\boldsymbol{\omega}_u\cdot\mathbf{R})[\mathbf{S}_u\times\mathbf{R}]}{m(m+H)(m_u+K_u)(m_u+H_u)}\right\}+\sum_u\left\{\frac{[\boldsymbol{\omega}_u\times\mathbf{R}]}{m(m+H)}-\frac{[\boldsymbol{\omega}_u\times R_u]}{m(m_u+H_u)}\right\},\quad(5)$$

and, as before, $\mathbf{r}_u-\mathbf{r}$ does not contain $\boldsymbol{\lambda}$.

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Reduction of Relativistic Two-Particle Wave Equations to Approximate Forms. II*

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The method of reduction of two-particle relativistic wave equations (an extension of the Foldy-Wouthuysen method), as given in Part I, was applicable only if $m_{\rm I}\neq m_{\rm II}$. Other variants of the procedure, free from this restriction, are developed now. On the basis of a discussion of properties of the matrices involved, it is found that the postulate of an "even-even" transformed Hamiltonian was too far-reaching. The less stringent requirement of a "uU separating" or an "lL separating" $\mathcal{H}_{\rm tr}$ leads to a whole class of usable transformations, which includes the transformation of Part I as a special case. Another important special case, (that of the "least change" transformation) has been calculated through in detail. Different transformations give different expressions for $\mathcal{H}_{\rm tr}$, but they coincide after (as a part of the next step of the procedure) the matrices $\beta^{\rm I}$ and $\beta^{\rm II}$ are replaced by 1 (or -1). Consequently the reduced wave equation is the same in all cases.

IN a recent paper,[1] hereafter referred to as I,† a method was developed for conversion of relativistic two-particle wave equations from the full (16-component) into an approximate (4-component) form. The procedure consists of two steps: first, a canonical transformation (strictly speaking, a sequence of canonical transformations) is performed with the help of suitable generating functions; then, twelve components of the wave equation are rejected, and only the four upper-upper or the four lower-lower components are retained, namely that quadruple which describes states with both particles possessing positive energy (the other of these two quadruples corresponds to both particles having negative energy). *The same* transformation is required to make either choice possible.

The proposed scheme was patterned after the Foldy-Wouthuysen method for one-body equations.[2] As a matter of fact, the expression for the transformed Hamiltonian in I represents a plausible, though not trivial, generalization of that obtained by Foldy and Wouthuysen. But, remarkably, our method is not

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[1] Z. V. Chraplyvy, Phys. Rev. **91**, 388 (1953). We take over the terminology and notation used there.

† *Errata to* I.—In Eq. (4) the first minus sign is to be replaced by a plus sign. In Eq. (7j) the numerical coefficient is to be 1/8 (not 3/16).

[2] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).