method of virtual quanta has been treated[2] using various meson theories, it is of interest to calculate the *PSPS* result. Using the meson-scattering cross sections of Ashkin *et al.*[7] and proceeding exactly as above, one obtains for charged mesons from $n-p$ collisions:

$$\sigma=0.0034g^6\gamma^{-1}(1-0.94\gamma^{-1}), \tag{7}$$

and for π^+ from $p-p$ (or π^- from $n-n$) collisions:

$$\sigma=0.0068g^6\gamma^{-1}[\ln(0.31\gamma)+4.3\gamma^{-1}]. \tag{8}$$

Comparison with the third-order field-theoretic calculation of Morette[8] shows substantial agreement in the leading term. Similar agreement between virtual quanta methods and rigorous calculations in the scalar theory has been found by Strick and ter Haar.[9]

[7] Ashkin, Simon, and Marshak, Progr. Theoret. Phys. (Japan) **5**, 634 (1950).
[8] C. Morette, Phys. Rev. **76**, 1432 (1949).
[9] E. Strick and D. ter Haar, Phys. Rev. **76**, 304 (1949).

Because of the assumptions entailed in (1), the above results are of theoretical interest only. In practice one should integrate perhaps a phenomenological $N(\omega)$ with experimental meson-scattering cross sections. This might enable one to correlate meson-induced with nucleon-induced reactions, in a manner similar to Guth and Mullin's[10] correlation of experiments on the disintegration of Be^9 by γ rays and by electrons. A comparison of a high energy $n-p$ bremsstrahlung calculation using nuclear force phenomenology[11] with the method of virtual quanta might enable one to express $N(\omega)$ in terms of nuclear force parameters. Finally it might be pointed out that the method of virtual quanta would lend itself more readily to heavier particles with weaker coupling.

[10] E. Guth and C. J. Mullin, Phys. Rev. **76**, 234 (1949).
[11] J. Ashkin and R. E. Marshak, Phys. Rev. **76**, 58 (1949) and T. Muto, Phys. Rev. **59**, 837 (1941) have done low-energy $n-p$ bremsstrahlung calculations in which the nucleon-nucleon interaction is handled by nuclear force phenomenology rather than by weak-coupling meson theory.

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The Theory of Quantized Fields. IV

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The principal development in this paper is the extension of the eigenvalue-eigenvector concept to complete sets of anticommuting operators. With the aid of this formalism we construct a transformation function for the Dirac field, as perturbed by an external source. This transformation function is enlarged to describe phase transformations and, when applied to the isolated Dirac field, yields the charge and energy-momentum eigenvalues and eigenfunctions. The transformation function describing the system in the presence of the source is then used as a generating function to construct the matrices of all ordered products of the field operators, for the isolated Dirac field. The matrices in the occupation number representation are exhibited with a classification that effectively employs a time-reversed description for negative frequency modes. The last section supplements III by constructing the matrices of all ordered products of the potential vector, for the isolated electromagnetic field.

INTRODUCTION

THIS paper and its sequel are continuations of III[1] in their concern with a single externally perturbed field. We shall discuss the Dirac field as perturbed by a second prescribed Dirac field, which appears as an external source, or by a prescribed Bose-Einstein field, as exemplified by a given electromagnetic field. The Lagrange function of this system is

$$\begin{aligned}\mathcal{L}=-\tfrac{1}{4}[\bar\psi,\gamma_\mu(-i\partial_\mu-eA_\mu)\psi+m\psi]\\-\tfrac{1}{4}[(i\partial_\mu-eA_\mu)\bar\psi\gamma_\mu+m\bar\psi,\psi]\\+\tfrac{1}{2}[\bar\psi,\eta]+\tfrac{1}{2}[\bar\eta,\psi].\end{aligned} \tag{1}$$

The resulting field equations are

$$\begin{aligned}\gamma_\mu(-i\partial_\mu-eA_\mu)\psi+m\psi=\eta,\\(i\partial_\mu-eA_\mu)\bar\psi\gamma_\mu+m\bar\psi=\bar\eta,\end{aligned} \tag{2}$$

[1] J. Schwinger, Phys. Rev. **91**, 728 (1953).

and the generators of infinitesimal changes in ψ or $\bar\psi$ on a surface σ are given by

$$G(\psi)=i\int d\sigma_\mu\bar\psi\gamma_\mu\delta\psi=i\int d\sigma\bar\psi\gamma_{(0)}\delta\psi \tag{3}$$

and

$$G(\bar\psi)=-i\int d\sigma_\mu\delta\bar\psi\gamma_\mu\psi=-i\int d\sigma\delta\bar\psi\gamma_{(0)}\psi. \tag{4}$$

It was shown in III that the vacuum state of a closed system, Ψ_0 can be characterized as the right eigenvector, with zero eigenvalues, of the positive frequency parts of the field components, and that $\Psi_0^\dagger$ is the left eigenvector, with zero eigenvalues, of the negative frequency parts of the field components. The inference that the totality of eigenvectors of these types would be of particular utility led us, in discussing a Bose-Einstein system, to introduce eigenvectors and eigenvalues for complete sets of commuting non-Hermitian operators. We shall now find it desirable to extend the eigenvalue-eigenvector concept to complete sets of *anticommuting*, non-Hermitian operators.

THE TRANSFORMATION FUNCTION

The discussion in this paper will be limited to the situation of zero electromagnetic field, as described with the elementary gauge, $A_\mu=0$. Relative to a coordinate system based on a given surface, the field equations in the absence of the sources can be written as the equations of motion,

$$\begin{aligned} i\partial_0\psi &= -i\gamma_0\gamma_k\partial_k\psi+m\gamma_0\psi=H\psi,\\ -i\partial_0\bar{\psi} &= i\partial_k\bar{\psi}\gamma_k\gamma_0+m\bar{\psi}\gamma_0=\bar{\psi}\gamma_0 H\gamma_0. \end{aligned} \tag{5}$$

We define two Hermitian coordinate operators,

$$P^{(\pm)}=\tfrac{1}{2}(1\pm(H/E)), \tag{6}$$

where E is the positive-definite quantity

$$E=(H^2)^{\frac{1}{2}}. \tag{7}$$

These operators have the projection properties

$$P^{(+)}+P^{(-)}=1,\quad P^{(+)}P^{(-)}=P^{(-)}P^{(+)}=0 \tag{8}$$

and

$$P^{(\pm)}H=\pm EP^{(\pm)}. \tag{9}$$

Hence the representation of ψ and $\bar{\psi}$ as

$$\psi=\psi^{(+)}+\psi^{(-)},\quad \bar{\psi}=\bar{\psi}^{(+)}+\bar{\psi}^{(-)}, \tag{10}$$

where

$$\psi^{(\pm)}=P^{(\pm)}\psi,\quad \bar{\psi}^{(\pm)}=\bar{\psi}\gamma_0P^{(\mp)}\gamma_0 \tag{11}$$

is a decomposition into positive and negative frequency parts, according to the resulting form of the equations of motion,

$$i\partial_0\psi^{(\pm)}=\pm E\psi^{(\pm)},\quad i\partial_0\bar{\psi}^{(\pm)}=\pm E\bar{\psi}^{(\pm)}. \tag{12}$$

It should also be noted that

$$\bar{\psi}^{(\pm)}=\overline{\psi^{(\mp)}}. \tag{13}$$

In view of the orthogonality properties expressed by

$$\int d\sigma\bar{\psi}^{(+)}\gamma_0\psi^{(+)}=\int d\sigma\bar{\psi}\gamma_0P^{(-)}P^{(+)}\psi=0, \tag{14}$$

and

$$\int d\sigma\bar{\psi}^{(-)}\gamma_0\psi^{(-)}=\int d\sigma\bar{\psi}\gamma_0P^{(+)}P^{(-)}\psi=0, \tag{15}$$

the generator $G(\psi)$ appears as

$$G(\psi)=i\int d\sigma\bar{\psi}^{(-)}\gamma_0\delta\psi^{(+)}+i\int d\sigma\bar{\psi}^{(+)}\gamma_0\delta\psi^{(-)}. \tag{16}$$

By the addition of suitable variations to this generator, we deduce the generators of infinitesimal changes in the positive frequency, or in the negative frequency parts of ψ and $\bar{\psi}$,

$$G_+=i\int d\sigma\bar{\psi}^{(-)}\gamma_0\delta\psi^{(+)}-i\int d\sigma\delta\bar{\psi}^{(+)}\gamma_0\psi^{(-)}, \tag{17}$$

$$G_-=i\int d\sigma\bar{\psi}^{(+)}\gamma_0\delta\psi^{(-)}-i\int d\sigma\delta\bar{\psi}^{(-)}\gamma_0\psi^{(+)}. \tag{18}$$

The orthogonality properties contained in (14) and (15) also enable us to write these generators as

$$G_+=i\int d\sigma\bar{\psi}\gamma_0\delta\psi^{(+)}-i\int d\sigma\delta\bar{\psi}^{(+)}\gamma_0\psi \tag{19}$$

and

$$G_-=i\int d\sigma\bar{\psi}\gamma_0\delta\psi^{(-)}-i\int d\sigma\delta\bar{\psi}^{(-)}\gamma_0\psi, \tag{20}$$

or, alternatively, as

$$G_+=i\int d\sigma\bar{\psi}^{(-)}\gamma_0\delta\psi-i\int d\sigma\delta\bar{\psi}\gamma_0\psi^{(-)} \tag{21}$$

and

$$G_-=i\int d\sigma\bar{\psi}^{(+)}\gamma_0\delta\psi-i\int d\sigma\delta\bar{\psi}\gamma_0\psi^{(+)}. \tag{22}$$

The latter forms facilitate the derivation of commutation properties on σ for the field components $\psi^{(\pm)}$, $\bar{\psi}^{(\pm)}$. These are expressed by the vanishing of all anticommutators save

$$\{\psi^{(+)}(x),\bar{\psi}^{(-)}(x')\}=P^{(+)}\gamma_0\delta_\sigma(x-x') \tag{23}$$

and

$$\{\psi^{(-)}(x),\bar{\psi}^{(+)}(x')\}=P^{(-)}\gamma_0\delta_\sigma(x-x'). \tag{24}$$

In particular, all positive frequency components are anticommutative, as are all negative frequency components.

Complete sets of anticommuting operators on σ are thus provided by $\chi^{(+)}(x)=\psi^{(+)}(x)$, $\bar{\psi}^{(+)}(x)$, and by $\chi^{(-)}(x)=\psi^{(-)}(x)$, $\bar{\psi}^{(-)}(x)$. The existence of right and left eigenvectors, respectively, with null eigenvalues, follows from the equivalence of these states with the vacuum state. Let us now extend the number system by introducing quantities $\chi^{(+)\prime}(x)=\psi^{(+)\prime}(x)$, $\bar{\psi}^{(+)\prime}(x)$ and $\chi^{(-)\prime}(x)=\psi^{(-)\prime}(x)$, $\bar{\psi}^{(-)\prime}(x)$, which anticommute among themselves and with all Dirac field operators. Then the operators

$$'\chi^{(\pm)}=\chi^{(\pm)}-\chi^{(\pm)\prime}$$

have the same commutation properties as the $\chi^{(\pm)}$, so that there exists a right eigenvector of the complete set $'\chi^{(+)}$ with zero eigenvalues,

$$(\chi^{(+)}(x)-\chi^{(+)\prime}(x))\Psi(\chi^{(+)\prime}\sigma)=0,$$

and a left eigenvector of the complete set $'\chi^{(-)}$ with zero eigenvalues,

$$\Phi(\chi^{(-)\prime}\sigma)(\chi^{(-)}(x)-\chi^{(-)\prime}(x))=0.$$

As the notation indicates, we thereby obtain a right eigenvector of the complete set $\chi^{(+)}$ with the "eigenvalues" $\chi^{(+)\prime}$, and a left eigenvector of the set $\chi^{(-)}$ with the "eigenvalues" $\chi^{(-)\prime}$. In view of the relation (13), the eigenvectors and eigenvalues are connected by

$$\Phi(\chi^{(-)\prime}\sigma)=\Psi(\chi^{(+)\prime}\sigma)^{\dagger}, \tag{25}$$

$$\bar{\psi}^{(\pm)\prime}=\overline{\psi^{(\mp)\prime}}. \tag{26}$$

The interpretation of the infinitesimal transformation equations

$$\delta\Psi(\chi^{(+)\prime}\sigma)=-iG_{+}\Psi(\chi^{(+)\prime}\sigma) \tag{27}$$

and

$$\delta\Phi(\chi^{(-)\prime}\sigma)=i\Phi(\chi^{(-)\prime}\sigma)G_{-}, \tag{28}$$

employs the identical operator properties of the field variations and of the eigenvalues. Thus $\Psi(\chi^{(+)\prime}\sigma)+\delta\Psi(\chi^{(+)\prime}\sigma)$ is the eigenvector of the $\chi^{(+)}-\delta\chi^{(+)}$ with the eigenvalues $\chi^{(+)\prime}$. But this is also the eigenvector of the $\chi^{(+)}$ with the eigenvalues $\chi^{(+)\prime}+\delta\chi^{(+)}$. Hence the alteration given by (27) is that associated with the change of the eigenvalues by $\delta\chi^{(+)}$. A similar statement applies to (28).

We shall now discuss the Dirac field under the influence of the external sources η, $\bar{\eta}$ in terms of the transformation function

$$(\chi^{(-)\prime}\sigma_1|\chi^{(+)\prime}\sigma_2)=(\Phi(\chi^{(-)\prime}\sigma_1)\Psi(\chi^{(+)\prime}\sigma_2)).$$

The dependence of this transformation function upon the eigenvalues is expressed by

$$\delta_{\chi'}(\chi^{(-)\prime}\sigma_1|\chi^{(+)\prime}\sigma_2)=i(\chi^{(-)\prime}\sigma_1|(G_{-}(\sigma_1)-G_{+}(\sigma_2))|\chi^{(+)\prime}\sigma_2), \tag{29}$$

where

$$\begin{aligned}G_{-}(\sigma_1)-G_{+}(\sigma_2)&=-i\int_{\sigma_1}d\sigma_\mu\delta\bar{\psi}^{(-)\prime}\gamma_\mu\psi+i\int_{\sigma_1}d\sigma_\mu\bar{\psi}\gamma_\mu\delta\psi^{(-)\prime}\\&\quad+i\int_{\sigma_2}d\sigma_\mu\delta\bar{\psi}^{(+)\prime}\gamma_\mu\psi-i\int_{\sigma_2}d\sigma_\mu\bar{\psi}\gamma_\mu\delta\psi^{(+)\prime}\\&=-i\oint d\sigma_\mu\delta\bar{\psi}'\gamma_\mu\psi+i\oint d\sigma_\mu\bar{\psi}\gamma_\mu\delta\psi'.\end{aligned} \tag{30}$$

In the latter form, it is understood that negative frequency eigenvalues are employed on σ_1, and positive frequency eigenvalues on σ_2.

An infinitesimal change of the external source produces the alteration

$$\delta(\chi^{(-)\prime}\sigma_1|\chi^{(+)\prime}\sigma_2)=i\left(\chi^{(-)\prime}\sigma_1\left|\int_{\sigma_2}^{\sigma_1}(dx)(\delta\bar{\eta}\psi+\bar{\psi}\delta\eta)\right|\chi^{(+)\prime}\sigma_2\right). \tag{31}$$

Accordingly, an infinitesimal change in the eigenvalues can be simulated by the surface distributed source variation

$$\begin{aligned}\delta\bar{\eta}(x)&=-i\delta\bar{\psi}^{(-)\prime}(x)\gamma_\mu\delta_\mu(x,\sigma_1)+i\delta\bar{\psi}^{(+)\prime}(x)\gamma_\mu\delta_\mu(x,\sigma_2),\\\delta\eta(x)&=i\delta_\mu(x,\sigma_1)\gamma_\mu\delta\psi^{(-)\prime}(x)-i\delta_\mu(x,\sigma_2)\gamma_\mu\delta\psi^{(+)\prime}(x),\end{aligned} \tag{32}$$

where $\delta_\mu(x,\sigma)$ is defined by

$$\int(dx)\delta_\mu(x,\sigma)f_\mu(x)=\int_\sigma d\sigma_\mu f_\mu(x).$$

We conclude that the general transformation function is obtained from the one referring to zero eigenvalues on making the followlng substitution in the latter,

$$\begin{aligned}\bar{\eta}(x)&\rightarrow\bar{\eta}(x)-i\bar{\psi}^{(-)\prime}(x)\gamma_\mu\delta_\mu(x,\sigma_1)+i\bar{\psi}^{(+)\prime}(x)\gamma_\mu\delta_\mu(x,\sigma_2),\\\eta(x)&\rightarrow\eta(x)+i\delta_\mu(x,\sigma_1)\gamma_\mu\psi^{(-)\prime}(x)-i\delta_\mu(x,\sigma_2)\gamma_\mu\psi^{(+)\prime}(x).\end{aligned} \tag{33}$$

With the notation

$$(0\sigma_1|0\sigma_2)=\exp(i\mathcal{W}_0) \tag{34}$$

and

$$(0\sigma_1|F|0\sigma_2)/(0\sigma_1|0\sigma_2)=\langle F\rangle, \tag{35}$$

the dependence of the null eigenvalue transformation function upon the source is expressed by

$$\begin{aligned}\delta_l\mathcal{W}_0/\delta\bar{\eta}(x)&=\langle\psi(x)\rangle,\\\delta_r\mathcal{W}_0/\delta\eta(x)&=\langle\bar{\psi}(x)\rangle,\end{aligned} \tag{36}$$

or

$$\delta\mathcal{W}_0=\int_{-\infty}^{\infty}(dx)[\delta\bar{\eta}\langle\psi\rangle+\langle\bar{\psi}\rangle\delta\eta], \tag{37}$$

in which it is supposed that the source vanishes externally to the volume bounded by σ_1 and σ_2. According to the field equations (2), we have

$$-i\gamma_\mu\partial_\mu\langle\psi(x)\rangle+m\langle\psi(x)\rangle=\eta(x), \tag{38}$$

and

$$i\partial_\mu\langle\bar{\psi}(x)\rangle\gamma_\mu+m\langle\bar{\psi}(x)\rangle=\bar{\eta}(x), \tag{39}$$

which are to be solved subject to the boundary conditions

$$\langle\psi^{(-)}\rangle=\langle\bar{\psi}^{(-)}\rangle=0,\quad\text{on }\sigma_1, \tag{40}$$

and

$$\langle\psi^{(+)}\rangle=\langle\bar{\psi}^{(+)}\rangle=0,\quad\text{on }\sigma_2, \tag{41}$$

that follow from the nature of the null eigenvalue states on σ_1 and σ_2. We can express these as boundary conditions in the extended domain through the requirement that the fields shall contain only positive frequencies in the region constituting the future of σ_1, and only negative frequencies in the region prior to σ_2.

The solutions meeting these conditions are

$$\langle\psi(x)\rangle=\int_{-\infty}^{\infty}(dx')G_{+}(x,x')\eta(x') \tag{42}$$

and

$$\langle\bar{\psi}(x')\rangle=\int_{-\infty}^{\infty}(dx)\bar{\eta}(x)G_+(x,x'), \tag{43}$$

where $G_+(x, x')$ is the Green's function defined by the differential equations

$$\begin{aligned}-i\gamma_\mu\partial_\mu G_+(x,x')+mG_+(x,x')&=i\partial_\mu'G_+(x,x')\gamma_\mu+mG_+(x,x')\\&=\delta(x-x'),\end{aligned} \tag{44}$$

and the boundary condition that G_+, as a function of x, shall contain only positive frequencies for $x_0>x_0'$, and only negative frequencies for $x_0<x_0'$. Since G_+ is only dependent upon $x-x'$, the same statement applies with x and x' interchanged. That the identical Green's function is encountered in (42) and (43) follows from the integrability condition deduced from (36),

$$(\delta_r/\delta\eta(x'))\langle\psi(x)\rangle=(\delta_l/\delta\bar{\eta}(x))\langle\bar{\psi}(x')\rangle=G_+(x,x').$$

We thus construct $\mathcal{W}_0$ as

$$\mathcal{W}_0=\int_{-\infty}^{\infty}(dx)(dx')\bar{\eta}(x)G_+(x,x')\eta(x'), \tag{45}$$

with a constant of integration that has the value zero since, in the absence of sources, the null eigenvalue states have the significance of the σ-independent vacuum state

$$\eta=\bar{\eta}=0:\quad (0\sigma_1|0\sigma_2)=1,\quad \mathcal{W}_0=0.$$

On performing the substitution (33), the general transformation function is obtained as

$$(\chi^{(-)\prime}\sigma_1|\chi^{(+)\prime}\sigma_2)=\exp(i\mathcal{W}), \tag{46}$$

where

$$\begin{aligned}\mathcal{W}=&\int(dx)(dx')\bar{\eta}(x)G_+(x,x')\eta(x')\\&-i\oint d\sigma_\mu\int(dx')\bar{\psi}'(x)\gamma_\mu G_+(x,x')\eta(x')\\&+i\int(dx)\oint d\sigma_\nu'\bar{\eta}(x)G_+(x,x')\gamma_\nu\psi'(x')\\&+\oint d\sigma_\mu\oint d\sigma_\nu'\bar{\psi}'(x)\gamma_\mu G_+(x,x')\gamma_\nu\psi'(x').\end{aligned} \tag{47}$$

In particular,

$$\begin{aligned}\eta=\bar{\eta}=0:\quad &(\chi^{(-)\prime}\sigma_1|\chi^{(+)\prime}\sigma_2)\\&=\exp\left[i\oint d\sigma_\mu\oint d\sigma_\nu'\bar{\psi}'(x)\gamma_\mu G_+(x,x')\gamma_\nu\psi'(x')\right].\end{aligned} \tag{48}$$

The Green's function $G_+(x, x')$ can be exhibited in the three-dimensional symbolic form

$$\begin{aligned}G_+(x,x')&=iP^{(+)}\exp[-iE(x_0-x_0')]\gamma_0\delta(\mathbf{x}-\mathbf{x}'),&& x_0>x_0'\\&=-iP^{(-)}\exp[-iE(x_0'-x_0)]\gamma_0\delta(\mathbf{x}-\mathbf{x}'),&& x_0<x_0',\end{aligned} \tag{49}$$

or, combining both situations,

$$G_+(x,x')=(i\partial_0+H)\tfrac{1}{2}iE^{-1}\times\exp(-iE|x_0-x_0'|)\gamma_0\delta(\mathbf{x}-\mathbf{x}'). \tag{50}$$

The function of E in the latter equation has the integral representation

$$\tfrac{1}{2}iE^{-1}\exp(-iE|x_0-x_0'|)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dp_0\frac{e^{-ip_0(x_0-x_0')}}{E^2-p_0^2-i\epsilon},\quad \epsilon\to+0. \tag{51}$$

The Fourier integral expression of the three-dimensional delta function in (50), combined with (51), leads to the four-dimensional integral

$$\begin{aligned}G_+(x,x')&=\frac{1}{(2\pi)^4}\int(dp)\frac{m-\gamma p}{p^2+m^2-i\epsilon}e^{ip(x-x')}\\&=\frac{1}{(2\pi)^4}\int(dp)\frac{1}{\gamma p+m-i\epsilon}e^{ip(x-x')},\quad \epsilon\to+0,\end{aligned} \tag{52}$$

which shows that the mass m must be supplemented with an infinitesimal negative imaginary constant in constructing the Green's function as the reciprocal of the differential operator in (44).

The three-dimensional Fourier integrals derived from (49) are advantageously presented as

$$G_+(x,x')=\tfrac{1}{2}i\int\frac{(d\mathbf{p})}{(2\pi)^3}\frac{1}{p_0}\times\begin{cases}(m-\gamma p)e^{ip(x-x')}, & x_0>x_0'\\(m+\gamma p)e^{-ip(x-x')}, & x_0<x_0',\end{cases} \tag{53}$$

where p_0 is a *positive* frequency

$$p_0=(\mathbf{p}^2+m^2)^{\frac{1}{2}}.$$

The four-rowed square matrix $-\gamma p$ has two distinct eigenvalues, $\pm m$,

$$(-\gamma p)^2=-p^2=m^2,$$

each of which is twofold degenerate. We shall designate the eigenvectors by $u_{\lambda p}$ where $\lambda=1, 2$ refers to the eigenvalue $+m$, and $\lambda=-1, -2$ indicates the eigenvalue m. Thus

$$-\gamma p u_{\lambda p}=\epsilon(\lambda)mu_{\lambda p}, \tag{54}$$

and

$$-\bar{u}_{\lambda p}\gamma p=\bar{u}_{\lambda p}\epsilon(\lambda)m,$$

where

$$\epsilon(\lambda)=\lambda/|\lambda|.$$

In view of the indefinite character of the quantities

$$(\bar{u}_{\lambda p}u_{\lambda p})=(u_{\lambda p}{}^*\gamma_0 u_{\lambda p}),$$

the orthonormality and completeness properties of these eigenvectors appear as

$$(\bar{u}_{\lambda p}u_{\lambda' p})=\delta_{\lambda\lambda'}\epsilon(\lambda),$$

and

$$\sum_\lambda \epsilon(\lambda)u_{\lambda p}\bar{u}_{\lambda p}=1.$$

The positive definite quantities

$$(\bar{u}_{\lambda p}\gamma_0 u_{\lambda p})=(u_{\lambda p}{}^*u_{\lambda p})$$

are then given by

$$(\bar{u}_{\lambda p}\gamma_0 u_{\lambda' p})=\delta_{\lambda\lambda'}(p_0/m).$$

The latter result is deduced from

$$\begin{aligned}-(\bar{u}_{\lambda p}\{\gamma_0,\gamma p\}u_{\lambda' p})&=2p_0(\bar{u}_{\lambda p}u_{\lambda' p})\\&=m(\epsilon(\lambda)+\epsilon(\lambda'))(\bar{u}_{\lambda p}\gamma_0 u_{\lambda' p}),\end{aligned}$$

which shows that $(\bar{u}_{\lambda p}u_{\lambda p})$ is indeed negative for $\lambda<0$.

When the eigenvalue equation (54), in the form

$$(1/2m)(m-\gamma p)u_{\lambda p}=\tfrac{1}{2}(1+\epsilon(\lambda))u_{\lambda p},$$

is multiplied by $\epsilon(\lambda)\bar{u}_{\lambda p}$ and summed over all λ with the aid of the completeness relation, there results

$$(1/2m)(m-\gamma p)=\sum\nolimits_+ u_{\lambda p}\bar{u}_{\lambda p},$$

where $+$ signifies the eigenvectors with $\lambda>0$. Similarly,

$$(1/2m)(m+\gamma p)=-\sum\nolimits_- u_{\lambda p}\bar{u}_{\lambda p}.$$

We employ these projection operator representations in (53), and replace the integral by a summation over cells of volume $(d\mathbf{p})$. This yields

$$\begin{aligned}G_+(x,x')&=i\sum_{+,p}\psi_{\lambda p}(x)\bar{\psi}_{\lambda p}(x'),\qquad x_0>x_0'\\&=-i\sum_{-,p}\psi_{\lambda p}(x)\bar{\psi}_{\lambda p}(x'),\quad x_0<x_0',\end{aligned}\tag{55}$$

in which

$$\psi_{\lambda p}(x)=\left(\frac{(d\mathbf{p})}{(2\pi)^3}\frac{m}{p_0}\right)^{\frac{1}{2}}u_{\lambda p}e^{i\epsilon(\lambda)px}.$$

The completeness of these functions on a given surface is implied by the discontinuity of the Green's function at $x_0=x_0'$, as derived from (44),

$$\sum\nolimits_{\lambda p}\psi_{\lambda p}(x)\bar{\psi}_{\lambda p}(x')=\gamma_0\delta_\sigma(x-x'),\tag{56}$$

The associated *ortho*-normality statement is

$$\int d\sigma\bar{\psi}_{\lambda p}\gamma_0\psi_{\lambda' p'}=\delta_{\lambda p,\lambda' p'}.\tag{57}$$

EIGENVALUES AND EIGENFUNCTIONS

We begin our applications of (46) with the isolated Dirac field, as described by (48). The eigenvalues and eigenfunctions of the energy-momentum vector P_μ are obtained from the transformation function that connects representations associated with parallel surfaces. The effect of infinitesimal translations of σ_1 and σ_2 is given by

$$\delta_x\Phi(\chi^{(-)\prime}\sigma_1)=i\Phi(\chi^{(-)\prime}\sigma_1)P_\mu\delta x_{1\mu},$$

$$\delta_x\Psi(\chi^{(+)\prime}\sigma_2)=-iP_\mu\delta x_{2\mu}\Psi(\chi^{(+)\prime}\sigma_2).$$

Accordingly, if x_1 and x_2 are the finite translations that produce σ_1 and σ_2 from a standard surface, we have

$$\Phi(\chi^{(-)\prime}\sigma_1)=\Phi(\chi^{(-)\prime})\exp(iP_\mu x_{1\mu}),$$

$$\Psi(\chi^{(+)\prime}\sigma_2)=\exp(-iP_\mu x_{2\mu})\Psi(\chi^{(+)\prime}),$$

and

$$\begin{aligned}(\chi^{(-)\prime}\sigma_1|\chi^{(+)\prime}\sigma_2)&=(\chi^{(-)\prime}|\exp(iP_\mu X_\mu)|\chi^{(+)\prime})\\&=\sum\nolimits_{\gamma'}(\chi^{(-)\prime}|\gamma')\exp(iP_\mu' X_\mu)(\gamma'|\chi^{(+)\prime}),\end{aligned}$$

where

$$X=x_1-x_2,$$

and the γ are a complete set of constants of the motion.

Before employing the transformation function (48) in this manner, we shall extend it to serve also as a generating function for the eigenvalues and eigenfunctions of the charge operator Q. An infinitesimal phase transformation on σ,

$$\begin{aligned}\delta\psi(x)&=-ie\delta\alpha\psi(x),\\\delta\bar{\psi}(x)&=ie\delta\alpha\bar{\psi}(x),\end{aligned}\tag{58}$$

induces the eigenvector transformations generated by

$$G_\alpha=Q\delta\alpha.$$

According to the orthogonality properties (14) and (15), the charge operator can be written

$$Q=e\tfrac{1}{2}\int d\sigma([\bar{\psi}^{(-)},\gamma_0\psi^{(+)}]+[\bar{\psi}^{(+)},\gamma_0\psi^{(-)}])$$

$$=e\int d\sigma(\bar{\psi}^{(-)}\gamma_0\psi^{(+)}-\psi^{(-)}(\bar{\psi}^{(+)}\gamma_0)).\tag{59}$$

In arriving at the latter form with the aid of the commutation relations (23) and (24), we have assigned the value zero to the quantity

$$\mathrm{Tr}(P^{(+)}-P^{(-)})=\mathrm{Tr}(H/E),\tag{60}$$

where the trace is applied to spatial coordinates and spinor indices. This evaluation is based upon the time reflection invariance of the theory, which indicates that a one to one correspondence can be established between positive and negative frequency modes. Thus complete cancellation occurs in summing the eigenvalues of H/E, which are ± 1.

The infinitesimal eigenvector-transformation equations are

$$\delta_\alpha\Phi(\chi^{(-)\prime}\sigma\alpha)=i\Phi(\chi^{(-)\prime}\sigma\alpha)e\delta\alpha \times\int d\sigma(\bar\psi^{(-)\prime}\gamma_0\psi^{(+)}+\bar\psi^{(+)}\gamma_0\psi^{(-)\prime})$$

and

$$\delta_\alpha\Psi(\chi^{(+)\prime}\sigma\alpha)=-ie\delta\alpha \times\int d\sigma(\bar\psi^{(-)}\gamma_0\psi^{(+)\prime}+\bar\psi^{(+)\prime}\gamma_0\psi^{(-)})\Psi(\chi^{(+)\prime}\sigma\alpha).$$

A comparison with (17) and (18) shows that the eigenvector transformations are just those produced by the eigenvalue changes

$$\delta\psi^{(\pm)\prime}=-ie\delta\alpha\psi^{(\pm)\prime},$$
$$\delta\bar\psi^{(\pm)\prime}=ie\delta\alpha\bar\psi^{(\pm)\prime},$$

as we could anticipate from (58). Finite phase transformations

$$\Phi(\chi^{(-)\prime}\sigma\alpha)=\Phi(\chi^{(-)\prime}\sigma)\exp(iQ\alpha),$$
$$\Psi(\chi^{(+)\prime}\sigma\alpha)=\exp(-iQ\alpha)\Psi(\chi^{(+)\prime}\sigma),$$

are thus described by

$$\Phi(\psi^{(-)\prime}\bar\psi^{(-)\prime}\sigma\alpha)=\Phi(e^{-ie\alpha}\psi^{(-)\prime},\,e^{ie\alpha}\bar\psi^{(-)\prime},\,\sigma)\quad(61)$$

and

$$\Psi(\psi^{(+)\prime}\bar\psi^{(+)\prime}\sigma\alpha)=\Psi(e^{-ie\alpha}\psi^{(+)\prime},\,e^{ie\alpha}\bar\psi^{(+)\prime},\,\sigma).\quad(62)$$

If different phase transformations are performed on σ_1 and σ_2, we have

$$(\chi^{(-)\prime}\sigma_1\alpha_1|\chi^{(+)\prime}\sigma_2\alpha_2)=(\chi^{(-)\prime}\sigma_1|\exp(iQ\alpha)|\chi^{(+)\prime}\sigma_2),\quad(63)$$

where

$$\alpha=\alpha_1-\alpha_2.$$

Hence

$$\begin{aligned}(\chi^{(-)\prime}\sigma_1\alpha_1|\chi^{(+)\prime}\sigma_2\alpha_2)&=(\chi^{(-)\prime}|\exp(iP_\mu X_\mu+iQ\alpha)|\chi^{(+)\prime})\\&=\sum_{\gamma'}(\chi^{(-)\prime}|\gamma')\exp(iP_\mu'X_\mu+iQ'\alpha)(\gamma'|\chi^{(+)\prime})\end{aligned}\quad(64)$$

is the generating function of the simultaneous eigenfunctions and eigenvalues of the commuting operators P_μ and Q, and this transformation function is obtained from (48) by applying the substitutions indicated in (61) and (62).

It should be noted that (48) does not contain terms in which both integrals refer to a common surface. Consider, for example,

$$\int_{\sigma_1}d\sigma\int_{\sigma_1}d\sigma'\bar\psi^{(-)\prime}(x)\gamma_0G_+(x,x')\gamma_0\psi^{(-)\prime}(x') =\int_{\sigma_1}d\sigma\int_{\sigma_1}d\sigma'\bar\psi'\gamma_0P^{(+)}G_+\gamma_0P^{(-)}\psi'.\quad(65)$$

Now $G_+\gamma_0$ becomes $iP^{(+)}$ or $-iP^{(-)}$ as $x_0-x_0'\rightarrow\pm0$. Either limit results in a null value for (65) ($P^{(+)}P^{(-)}=0$). Hence,

$$\begin{aligned}(\chi^{(-)\prime}\sigma_1\alpha_1|\chi^{(+)\prime}\sigma_2\alpha_2)=\exp\Big[&-i\int_{\sigma_1}d\sigma_\mu\int_{\sigma_2}d\sigma_\nu'\bar\psi^{(-)\prime}(x)\gamma_\mu e^{ie\alpha}\\&\times G_+(x,x')\gamma_\nu\psi^{(+)\prime}(x')-i\int_{\sigma_2}d\sigma_\mu\\&\times\int_{\sigma_1}d\sigma_\nu'\bar\psi^{(+)\prime}(x)\gamma_\mu G_+(x,x')e^{-ie\alpha}\gamma_\nu\psi^{(-)\prime}(x')\Big]\end{aligned}\quad(66)$$

and we need not have indicated the positive or negative frequency parts of the field eigenvalues since these are automatically selected by the structure of the Green's function.

We insert the expressions (55) for the Green's function and introduce the following linear combinations of eigenvalues, which are not explicit functions of the surface,

$\lambda>0$:

$$\chi_{\lambda p}^{(-)\prime}=\int_{\sigma_1}d\sigma_\mu\bar\psi^{(-)\prime}(x)\gamma_\mu\psi_{\lambda p}(x)e^{-ipx_1}$$
$$\chi_{\lambda p}^{(+)\prime}=\int_{\sigma_2}d\sigma_\mu\bar\psi_{\lambda p}(x)e^{ipx_2}\gamma_\mu\psi^{(+)\prime}(x),$$

$\lambda<0$:

$$\chi_{\lambda p}^{(-)\prime}=\int_{\sigma_1}d\sigma_\mu\bar\psi_{\lambda p}(x)e^{-ipx_1}\gamma_\mu\psi^{(-)\prime}(x)$$
$$\chi_{\lambda p}^{(+)\prime}=\int_{\sigma_2}d\sigma_\mu\bar\psi^{(+)\prime}(x)\gamma_\mu\psi_{\lambda p}(x)e^{ipx_2}.$$

With these definitions, the transformation function (64) becomes[2]

$$\begin{aligned}(\chi^{(-)\prime}\sigma_1\alpha_1|\chi^{(+)\prime}\sigma_2\alpha_2)&=\exp\Big[\sum_{\lambda p}\chi_{\lambda p}^{(-)\prime}\exp(ipX+ie\epsilon(\lambda)\alpha)\chi_{\lambda p}^{(+)\prime}\Big]\\&=\prod_{\lambda p}\exp[\chi_{\lambda p}^{(-)\prime}\exp(ipX+ie\epsilon(\lambda)\alpha)\chi_{\lambda p}^{(+)\prime}]\\&=\prod_{\lambda p}\sum_{n_{\lambda p}}\frac{1}{n!}(\chi_{\lambda p}^{(-)\prime}\chi_{\lambda p}^{(+)\prime})^n\\&\qquad\times\exp[in(pX+e\epsilon(\lambda)\alpha)].\end{aligned}\quad(67)$$

In view of the anticommutative nature of the eigenvalues, the square of any $\chi_{\lambda p}^{(\pm)\prime}$ is zero, whence the expansion of the exponential terminates after the first two terms, $n_{\lambda p}=0,1$. It will be observed that the distinction between B.E. and F.D. systems is embodied primarily in the nature of the eigenvalues rather than in the formula containing those eigenvalues.

[2] A basic statement in all manipulations with eigenvalues is that the product of two eigenvalues, as a unit, behaves like an ordinary number.

On comparison with (64) we see that

$$P_\mu'=P_\mu(n)=\sum_{\lambda p} n_{\lambda p}p_\mu, \quad n_{\lambda p}=0,\,1,$$

where

$$P_0'=\sum_{\lambda p} n_{\lambda p}p_0\geq 0,$$

and that

$$Q'=\sum_{\lambda p} n_{\lambda p}e\epsilon(\lambda).$$

The occupation numbers $n_{\lambda p}$ thus form the complete set of constants of the motion. The associated eigenfunctions are

$$(n|\chi^{(+)\prime})=\prod_{\lambda p}(\chi_{\lambda p}{}^{(+)\prime})^{n_{\lambda p}}=(\chi_1{}^{(+)\prime})^{n_1}(\chi_2{}^{(+)\prime})^{n_2}\cdots,$$

and

$$(\chi^{(-)\prime}|n)={}'\prod_{\lambda p}(\chi_{\lambda p}{}^{(-)\prime})^{n_{\lambda p}}=\cdots(\chi_2{}^{(-)\prime})^{n_2}(\chi_1{}^{(-)\prime})^{n_1},$$

in which we have introduced an arbitrary standard order of the field modes. That order, when read from left to right, is symbolized by $\prod$, and in the reverse sense by $'\prod$. Thus, if only modes 1 and 2 are occupied, we have in (67)

$$\chi_1{}^{(-)\prime}\chi_1{}^{(+)\prime}\chi_2{}^{(-)\prime}\chi_2{}^{(+)\prime}=(\chi_2{}^{(-)\prime}\chi_1{}^{(-)\prime})(\chi_1{}^{(+)\prime}\chi_2{}^{(+)\prime})$$
$$={}'\prod_{\lambda p}(\chi_{\lambda p}{}^{(-)\prime})^{n_{\lambda p}}\prod_{\lambda p}(\chi_{\lambda p}{}^{(+)\prime})^{n_{\lambda p}}.$$

With the eigenvalues $\chi^{(\pm)\prime}(x)$ at corresponding points in the relation (26), we have

$$\chi_{\lambda p}{}^{(-)\prime}=\chi_{\lambda p}{}^{(+)\prime\dagger},$$

and therefore

$$(\chi^{(-)\prime}|n)=(n|\chi^{(+)\prime})^\dagger,$$

as demanded by the eigenvector connection (25).

The eigenfunction of the vacuum state, referring to an arbitrary surface, is

$$(\chi^{(-)\prime}\sigma|0)=1,$$

and therefore

$$(\chi^{(-)\prime}\sigma|n\sigma)=(\chi^{(-)\prime}\sigma|0)'\prod_{\lambda p}(\chi_{\lambda p}{}^{(-)\prime})^{n_{\lambda p}}$$
$$=(\chi^{(-)\prime}\sigma|\,'\prod_{\lambda p}(\chi_{\lambda p}{}^{(-)})^{n_{\lambda p}}|0),$$

in which we have introduced the operators on σ possessing the $\chi_{\lambda p}{}^{(-)\prime}$ as eigenvalues. Accordingly, the eigenvectors of the state with particle occupation numbers $n_{\lambda p}$ are

$$\Psi(n\sigma)=[\prod_{\lambda p}(\chi_{\lambda p}{}^{(-)})^{n_{\lambda p}}]\Psi_0,$$

and

$$\Psi(n\sigma)^\dagger=\Psi_0{}^\dagger[\prod_{\lambda p}(\chi_{\lambda p}{}^{(+)})^{n_{\lambda p}}].$$

THE MATRICES OF FIELD OPERATORS

We shall now use the transformation function (46) to obtain the matrix elements, for the isolated Dirac field, of all products of the field operators $\psi(x)$ and $\bar\psi(x)$. For this purpose, we remark that the transformation function describing the system, with sources present, is the matrix of a certain time-ordered operator for the isolated system. Indeed,

$$(\chi^{(-)\prime}\sigma_1|\chi^{(+)\prime}\sigma_2)$$
$$=\left(\chi^{(-)\prime}\sigma_1\middle|\left(\exp\left[i\int(dx)(\bar\eta(x)\psi(x)+\bar\psi(x)\eta(x))\right]\right)_+\middle|\chi^{(+)\prime}\sigma_2\right)\Big]_0, \quad (68)$$

where $]_0$ indicates that the operators and states correspond to $\eta=\bar\eta=0$. To prove this, let us replace η and $\bar\eta$ with $\lambda\eta$ and $\lambda\bar\eta$, where λ is a numerical parameter. The effect of an infinitesimal change of the latter is expressed by

$$(\partial/\partial\lambda)(\chi^{(-)\prime}\sigma_1|\chi^{(+)\prime}\sigma_2)=\left(\chi^{(-)\prime}\sigma_1\middle|\int(dx)l(x)\middle|\chi^{(+)\prime}\sigma_2\right),$$

in which we have temporarily written

$$l(x)=i(\bar\eta(x)\psi(x)+\bar\psi(x)\eta(x)).$$

On differentiating again, we find[3]

$$(\partial^2/\partial\lambda^2)(\chi^{(-)\prime}\sigma_1|\chi^{(+)\prime}\sigma_2)$$
$$=\left(\chi^{(-)\prime}\sigma_1\middle|\int(dx_1)(dx_2)(l(x_1)l(x_2))_+\middle|\chi^{(+)\prime}\sigma_2\right),$$

and, in general,

$$(\partial/\partial\lambda)^n(\chi^{(-)\prime}\sigma_1|\chi^{(+)\prime}\sigma_2)$$
$$=\left(\chi^{(-)\prime}\sigma_1\middle|\int(dx_1)\cdots(dx_n)(l(x_1)\cdots l(x_n))_+\middle|\chi^{(+)\prime}\sigma_2\right).$$

If we now construct the transformation function describing the system in the presence of sources ($\lambda=1$) in terms of that for zero sources ($\lambda=0$), as a Taylor series expansion, we obtain an infinite series which is compactly represented by (68).

The transformation function (46) can be expressed as

$$(\chi^{(-)\prime}\sigma_1|\chi^{(+)\prime}\sigma_2)=(\chi^{(-)\prime}\sigma_1|\chi^{(+)\prime}\sigma_2)]_0$$
$$\times\exp\left[i\int(dx)(dx')\bar\eta(x)G_+(x,x')\eta(x')+i\int(dx)(\bar\eta(x)\psi'(x)+\bar\psi'(x)\eta(x))\right], \quad (69)$$

in which we have used the symbols $\psi'(x)$, $\bar\psi'(x)$, at points in the interior of the region bounded by σ_1 and σ_2,

[3] J. Schwinger, Phys. Rev. 82, 914 (1951), Eq. (2.133).

to mean

$$\psi'(x)=i\oint d\sigma_\mu' G_+(x,x')\gamma_\mu\psi'(x')$$

$$=i\int_{\sigma_1} d\sigma_\mu' G_+(x,x')\gamma_\mu\psi^{(-)\prime}(x') - i\int_{\sigma_2} d\sigma_\mu' G_+(x,x')\gamma_\mu\psi^{(+)\prime}(x'),$$

and

$$\bar\psi'(x)=-i\oint d\sigma_\mu'\bar\psi'(x')\gamma_\mu G_+(x',x)$$

$$=-i\int_{\sigma_1} d\sigma_\mu'\bar\psi^{(-)\prime}(x')\gamma_\mu G_+(x',x) + i\int_{\sigma_2} d\sigma_\mu'\bar\psi^{(+)\prime}(x')\gamma_\mu G_+(x',x).$$

We see that

$$\psi'(x)=\sum_{+,p}\psi_{\lambda p}(x)e^{-ipx_2}\chi_{\lambda p}{}^{(+)\prime}+\sum_{-,p}\psi_{\lambda p}(x)e^{ipx_1}\chi_{\lambda p}{}^{(-)\prime}$$

is the solution of the Dirac equation with the prescribed positive frequency part $\psi^{(+)\prime}(x)$, on σ_2, and the prescribed negative frequency part $\psi^{(-)\prime}(x)$, on σ_1. Similarly,

$$\bar\psi'(x)=\sum_{-,p}\bar\psi_{\lambda p}(x)e^{-ipx_2}\chi_{\lambda p}{}^{(+)\prime}+\sum_{+,p}\bar\psi_{\lambda p}(x)e^{ipx_1}\chi_{\lambda p}{}^{(-)\prime}$$

is the solution of the adjoint Dirac equation which has the positive and negative frequency parts, $\bar\psi^{(+)\prime}(x)$ and $\bar\psi^{(-)\prime}(x)$, on σ_2 and σ_1, respectively. Note, however, that $\bar\psi'(x)$ is not the adjoint of $\psi'(x)$. Indeed,

$$\psi'(x)^\dagger\gamma_0=-i\oint d\sigma_\mu'\bar\psi'(x')\gamma_\mu G_-(x',x),$$

where

$$G_-(x,x')=\gamma_0 G_+(x',x)^\dagger\gamma_0 \qquad (70)$$

satisfies the same differential equations as $G_+(x,x')$, but obeys ingoing rather than outgoing wave temporal boundary conditions,

$$G_-(x,x')=i\sum_{-,p}\psi_{\lambda p}(x)\bar\psi_{\lambda p}(x'),\qquad x_0>x_0'$$

$$=-i\sum_{+,p}\psi_{\lambda p}(x)\bar\psi_{\lambda p}(x'),\quad x_0<x_0'. \qquad (71)$$

In terms of the notation [not to be confused with (35)]

$$(\chi^{(-)\prime}\sigma_1|F|\chi^{(+)\prime}\sigma_2)/(\chi^{(-)\prime}\sigma_1|\chi^{(+)\prime}\sigma_2)]_0=\langle F\rangle, \qquad (72)$$

we express (68) and (69) as

$$\left\langle\left(\exp\left[i\int(dx)(\bar\eta\psi+\bar\psi\eta)\right]\right)_+\right\rangle$$

$$=\exp\left[i\int(dx)(dx')\bar\eta G_+\eta+i\int(dx)(\bar\eta\psi'+\bar\psi'\eta)\right], \qquad (73)$$

or, more simply,

$$\left\langle\left(\exp\left[i\int(dx)(\bar\eta'\psi+'\bar\psi\eta)\right]\right)_+\right\rangle$$

$$=\exp\left[i\int(dx)(dx')\bar\eta(x)G_+(x,x')\eta(x')\right], \qquad (74)$$

in which we have introduced the operators

$$'\psi(x)=\psi(x)-\psi'(x),$$

$$'\bar\psi(x)=\bar\psi(x)-\bar\psi'(x).$$

An expansion of both sides in powers of η and $\bar\eta$ will yield the matrix elements of ordered operator products of the ψ and $\bar\psi$.

From the absence of terms that are linear in η and $\bar\eta$, on the right of (74), we see that

$$\langle'\psi(x)\rangle=\langle'\bar\psi(x)\rangle=0,$$

or that

$$\langle\psi(x)\rangle=\psi'(x), \qquad (75)$$

$$\langle\bar\psi(x)\rangle=\bar\psi'(x). \qquad (76)$$

The term on the left of (74) that in bilinear in η and $\bar\eta$ is

$$-\left\langle\int(dx)(dx')(\bar\eta(x)'\psi(x)'\bar\psi(x')\eta(x'))_+\right\rangle$$

$$=-\int(dx)(dx')\bar\eta(x)\langle('\psi(x)'\bar\psi(x'))_+\rangle\epsilon(x,x')\eta(x'),$$

where

$$\epsilon(x,x')=\begin{cases}+1, & x_0>x_0'\\ -1, & x_0<x_0'.\end{cases}$$

Hence

$$\langle('\psi(x)'\bar\psi(x'))_+\rangle\epsilon(x,x')=-iG_+(x,x'),$$

or

$$\langle(\psi(x)\bar\psi(x'))_+\rangle\epsilon(x,x')=\psi'(x)\bar\psi'(x')-iG_+(x,x'). \qquad (77)$$

The complete expansion of the left side in (74) is

$$\sum_{k,l}\frac{i^{k+l}}{k!l!}\int(dx_1)\cdots(dx_k)(dx_1')\cdots(dx_l')\bar\eta(x_k)\cdots$$

$$\times\bar\eta(x_1)\langle('\psi(x_1)\cdots'\psi(x_k)'\bar\psi(x_l')\cdots'\bar\psi(x_1'))_+\rangle\epsilon_{k,l}$$

$$\times\eta(x_1')\cdots\eta(x_l'), \qquad (78)$$

where $\epsilon_{k,l}$ is the alternating symbol expressed by

$$\epsilon_{k,l}=(\prod_{i<j}\epsilon(x_i,x_j))(\prod_{i,j}\epsilon(x_i,x_j'))(\prod_{i>j}\epsilon(x_i',x_j')). \qquad (79)$$

This is to be compared with

$$\sum_k\frac{i^k}{k!}\int(dx_1)\cdots(dx_k)(dx_1')\cdots(dx_k')\bar\eta(x_k)\cdots$$

$$\times\bar\eta(x_1)G_+(x_1,x_1')\cdots G_+(x_k,x_k')\eta(x_1')\cdots\eta(x_k')$$

$$=\sum_k\frac{i^k}{(k!)^2}\int(dx_1)\cdots(dx_k')\bar\eta(x_k)\cdots\bar\eta(x_1)$$

$$\times[\det{}_{(k)}G_+(x_i,x_j')]\eta(x_1')\cdots\eta(x_k'),$$

where the k-dimensional determinant constructed from the elements $G_+(x_i, x_j')$ has been introduced by subjecting the variables $x_1'\cdots x_k'$ to the set of $k!$ permutations. The anticommutativity of the η provides the algebraic signs to form the alternating combination of terms which constitutes the determinant. Therefore

$$\langle(('\psi(x_1)\cdots'\psi(x_k)'\bar\psi(x_l')\cdots'\bar\psi(x_1'))_+\rangle\epsilon_{k,l} = \delta_{k,l}(-i)^k \det{}_{(k)} G_+(x_i, x_j'), \quad (80)$$

in which both sides are completely antisymmetrical in the variables x_i, and in the variables x_j'.

Straightforward algebraic rearrangement would yield the matrix elements for successive products of the operators ψ and $\bar\psi$, as illustrated by (75) and (77). However, one can obtain an explicit formula from (73). We first consider operator products with $k=l$, and remark that such terms are isolated in (73) by substituting $\bar\eta\to t\bar\eta$, $\eta\to t^{-1}\eta$, and evaluating the integral

$$\frac{1}{2\pi i}\oint\frac{dt}{t}\left\langle\left(\exp\left[i\int(dx)(t\bar\eta\psi+t^{-1}\bar\psi\eta)\right]\right)_+\right\rangle = \frac{1}{2\pi i}\oint\frac{dt}{t}\exp\left[i\int(dx)(dx')\bar\eta G_+\eta + i\int(dx)(t\bar\eta\psi'+t^{-1}\bar\psi'\eta)\right]. \quad (81)$$

The further substitution performed on the right, $it\int(dx)\bar\eta\psi'\to t$, yields

$$\frac{1}{2\pi i}\oint\frac{dt}{t}e^t\exp\left[i\int(dx)(dx')\bar\eta(x)(G_+(x,x') + it^{-1}\psi'(x)\bar\psi'(x'))\eta(x')\right].$$

The known result of expanding the right side of (74), as expressed in (80), now shows that

$$\langle(\psi(x_1)\cdots\psi(x_k)\bar\psi(x_k')\cdots\bar\psi(x_1'))_+\rangle\epsilon_{k,k} = \frac{1}{2\pi i}\oint\frac{dt}{t}e^t\det{}_{(k)}[-iG_+(x_i,x_j')+t^{-1}\psi(x_i)\bar\psi(x_j')]$$
$$=\psi'(x_1)\cdots\psi'(x_k)\bar\psi'(x_k')\cdots\bar\psi'(x_1')+\cdots +(-i)^k\det{}_{(k)}G_+(x_i,x_j'),$$

in which the various terms will be given by the development of the determinant, combined with the theorem

$$\frac{1}{2\pi i}\oint\frac{dt}{t^{n+1}}e^t=\frac{1}{n!}.$$

The effect of the t integration is to compensate the numerical factors that appear on expanding the determinant. An example is

$$\langle(\psi(x_1)\psi(x_2)\bar\psi(x_2')\bar\psi(x_1'))_+\rangle\epsilon_{2,2}=\frac{1}{2\pi i}\oint\frac{dt}{t}e^t\begin{vmatrix}-iG_+(x_1,x_1')+t^{-1}\psi'(x_1)\bar\psi'(x_1'), & -iG_+(x_1,x_2')+t^{-1}\psi'(x_1)\bar\psi'(x_2')\\ -iG_+(x_2,x_1')+t^{-1}\psi'(x_2)\bar\psi'(x_1'), & -iG_+(x_2,x_2')+t^{-1}\psi'(x_2)\bar\psi'(x_2')\end{vmatrix}$$
$$=\psi'(x_1)\psi'(x_2)\bar\psi'(x_2')\bar\psi'(x_1')-i\psi'(x_1)\bar\psi'(x_1')G_+(x_2,x_2')-i\psi'(x_2)\bar\psi'(x_2')G_+(x_1,x_1')$$
$$+i\psi'(x_1)\bar\psi'(x_2')G_+(x_2,x_1')+i\psi'(x_2)\bar\psi'(x_1')G_+(x_1,x_2')$$
$$-G_+(x_1,x_1')G_+(x_2,x_2')+G_+(x_1,x_2')G_+(x_2,x_1').$$

Operator products with $k-l>0$ are isolated by the integral

$$\frac{1}{2\pi i}\oint\frac{dt}{t^{k-l+1}}\left\langle\left(\exp\left[i\int(dx)(t\bar\eta\psi+t^{-1}\bar\psi\eta)\right]\right)_+\right\rangle = \frac{1}{2\pi i}\oint\frac{dt}{t}e^t\left(it^{-1}\int(dx)\bar\eta\psi'\right)^{k-l} \times\exp\left[i\int(dx)(dx')\bar\eta(G_++it^{-1}\psi'\bar\psi')\eta\right].$$

On expanding the right side, it is seen that

$$\langle(\psi(x_1)\cdots\psi(x_k)\bar\psi(x_l')\cdots\bar\psi(x_1'))_+\rangle\epsilon_{k,l} = \frac{1}{2\pi i}\oint\frac{dt}{t}e^t\det{}_{(k)}[t^{-1}\psi'(x_i)^{(k-l)}, -iG_+(x_i,x_j')+t^{-1}\psi'(x_i)\bar\psi'(x_j')],$$

in which the determinant is constructed with $t^{-1}\psi'(x_i)$, $i=1\cdots k$, occupying the first $k-l$ columns, and the rectangular matrix $-iG_+(x_i, x_j')+t^{-1}\psi'(x_i)\bar\psi'(x_j')$, $i=1\cdots k$, $j=1\cdots l$, completing the k-dimensional square array. It is understood that the determinant is defined by alternating permutations of the *row* indices, applied to the product of the diagonal elements written successively from left to right. This is illustrated by

$$\langle(\psi(x_1)\psi(x_2)\psi(x_3)\bar\psi(x_1'))_+\rangle\epsilon_{3,1}=\frac{1}{2\pi i}\oint\frac{dt}{t}e^t\begin{vmatrix} t^{-1}\psi'(x_1), & t^{-1}\psi'(x_1), & -iG_+(x_1,x_1')+t^{-1}\psi'(x_1)\bar\psi'(x_1') \\ t^{-1}\psi'(x_2), & t^{-1}\psi'(x_2), & -iG_+(x_2,x_1')+t^{-1}\psi'(x_2)\bar\psi'(x_1') \\ t^{-1}\psi'(x_3), & t^{-1}\psi'(x_3), & -iG_+(x_3,x_1')+t^{-1}\psi'(x_3)\bar\psi'(x_1') \end{vmatrix}$$

$$=\psi'(x_1)\psi'(x_2)\psi'(x_3)\bar\psi'(x_1')-i\psi'(x_1)\psi'(x_2)G_+(x_3,x_1')$$

$$-i\psi'(x_2)\psi'(x_3)G_+(x_1,x_1')-i\psi'(x_3)\psi'(x_1)G_+(x_2,x_1').$$

A similar treatment for $l-k>0$ yields

$$\langle(\psi(x_1)\cdots\psi(x_k)\bar\psi(x_l')\cdots\bar\psi(x_1'))_+\rangle\epsilon_{k,l}$$
$$=\frac{1}{2\pi i}\oint\frac{dt}{t}e^t\det{}_{(l)}[-iG_+(x_i,x_j')+t^{-1}\psi'(x_i)\bar\psi'(x_j'),\ t^{-1}\bar\psi'(x_j')^{(l-k)}],$$

where this determinant contains $t^{-1}\bar\psi'(x_j')$, $j=1\cdots l$ in the first $l-k$ rows, and the l-dimensional array is filled out with the rectangular matrix, $-iG_+(x_i,x_j')+t^{-1}\psi'(x_i)\bar\psi'(x_j')$, $i=1\cdots k$, $j=1\cdots l$. Here the determinant is defined by alternating permutations of the *column* indices applied to the product of the diagonal elements, written successively from right to left. Thus, for the example $k=0$, we have

$$\langle(\bar\psi(x_l')\cdots\bar\psi(x_1'))_+\rangle\epsilon_{0l}=\langle\bar\psi(x_l')\cdots\bar\psi(x_1')\rangle$$
$$=\bar\psi'(x_l')\cdots\bar\psi'(x_1').$$

The Occupation Number Representation

Matrices in the occupation number description can be derived from these results. The simplest examples are the diagonal matrix elements referring to the vacuum state—the vacuum expectation values. Indeed, on placing all eigenvalues equal to zero in (80), we obtain

$$(0|(\psi(x_1)\cdots\psi(x_k)\bar\psi(x_l')\cdots\bar\psi(x_1'))_+|0)\epsilon_{k,l}=\delta_{k,l}(-i)^k\det{}_{(k)}G_+(x_i,x_j'),\quad(82)$$

and, in particular,

$$(0|(\psi(x)\bar\psi(x'))_+|0)\epsilon(x,x')=-iG_+(x,x').\quad(83)$$

To obtain the occupation number matrices of $\psi(x)$ and $\bar\psi(x)$, we observe that (75), for example, can be written

$$(\chi^{(-)\prime}\sigma_1|\psi(x)|\chi^{(+)\prime}\sigma_2)=\psi'(x)(\chi^{(-)\prime}\sigma_1|\chi^{(+)\prime}\sigma_2),$$

or

$$\sum_{n,n'}(\chi^{(-)\prime}|n)(n\sigma_1|\psi(x)|n'\sigma_2)(n'|\chi^{(+)\prime})$$
$$=[\sum_{+,p}\psi_{\lambda p}(x)e^{-ipx_2}\chi_{\lambda p}^{(+)\prime}+\sum_{-,p}\psi_{\lambda p}(x)e^{ipx_1}\chi_{\lambda p}^{(-)\prime}]\times\sum_n(\chi^{(-)\prime}|n)\exp[iP(n)(x_1-x_2)](n|\chi^{(+)\prime}),\quad(84)$$

which exhibits it as a generating function for $(n\sigma_1|\psi(x)|n'\sigma_2)$. We shall prefer to construct

$$(n|\psi(x)|n')=\exp(-iP(n)x_1)\times(n\sigma_1|\psi(x)|n'\sigma_2)\exp(iP(n')x_2),$$

which is independent of σ_1 and σ_2, and refers to the standard surface. On incorporating e^{-ipx_2} into $\chi_{\lambda p}^{(+)\prime}$ and e^{ipx_1} into $\chi_{\lambda p}^{(-)\prime}$, (84) becomes

$$\sum_{n,n'}(\chi^{(-)\prime}|n)(n|\psi(x)|n')(n'|\chi^{(+)\prime})$$
$$=[\sum_{+,p}\psi_{\lambda p}(x)\chi_{\lambda p}^{(+)\prime}+\sum_{-,p}\psi_{\lambda p}\chi_{\lambda p}^{(-)\prime}]\times\sum_n(\chi^{(-)\prime}|n)(n|\chi^{(+)\prime}).$$

Now

$$\chi_{\lambda p}^{(-)\prime}(\chi_{\lambda p}^{(-)\prime}|n)=\chi_{\lambda p}^{(-)\prime}\prod(\chi^{(-)\prime})^n$$
$$=0,\quad n_{\lambda p}=1$$
$$=(-1)^{n_{>\lambda p}}(\chi^{(-)\prime}|n+1_{\lambda p}),\quad n_{\lambda p}=0,$$

where $n_{>\lambda p}$ is the number of occupied states that follow λp in the standard order. Similarly,

$$(n|\chi^{(+)\prime})\chi_{\lambda p}^{(+)\prime}=\prod(\chi^{(+)\prime})^n\chi_{\lambda p}^{(+)\prime}$$
$$=0,\quad n_{\lambda p}=1$$
$$=(-1)^{n_{>\lambda p}}(n+1_{\lambda p}|\chi^{(+)\prime}),\quad n_{\lambda p}=0.$$

We see that the nonvanishing matrix elements of $\psi(x)$ are of the form

$$(n|\psi(x)|n+1_{\lambda p})=(-1)^{n_{>\lambda p}}\psi_{\lambda p}(x),\quad\lambda>0,\quad(85)$$
$$(n+1_{\lambda p}|\psi(x)|n)=(-1)^{n_{>\lambda p}}\psi_{\lambda p}(x),\quad\lambda<0,\quad(86)$$

where $n_{\lambda p}=0$ in both statements. These exhibit $\psi(x)$ as a unit charge annihilator.

In an analogous way,

$$(\chi^{(-)\prime}\sigma_1|\bar\psi(x)|\chi^{(+)\prime}\sigma_2)=\bar\psi'(x)(\chi^{(-)\prime}\sigma_1|\chi^{(+)\prime}\sigma_2)$$

yields

$$\sum_{n,n'}(\chi^{(-)\prime}|n)(n|\bar\psi(x)|n')(n'|\chi^{(+)\prime})$$
$$=[\sum_{-,p}\bar\psi_{\lambda p}(x)\chi_{\lambda p}^{(+)\prime}+\sum_{+,p}\bar\psi_{\lambda p}(x)\chi_{\lambda p}^{(-)\prime}]\times\sum_n(\chi^{(-)\prime}|n)(n|\chi^{(+)\prime}),$$

from which we obtain the nonvanishing matrix elements

$$(n+1_{\lambda p}|\bar{\psi}(x)|n)=(-1)^{n>\lambda p}\bar{\psi}_{\lambda p}(x),\quad \lambda>0,$$
$$(n|\bar{\psi}(x)|n+1_{\lambda p})=(-1)^{n>\lambda p}\bar{\psi}_{\lambda p}(x),\quad \lambda<0,$$

with $n_{\lambda p}=0$, which display $\bar{\psi}(x)$ as a unit charge creator.

The matrices of ψ and $\bar{\psi}$ suggest the utility of a classification of matrix elements that would unify a given change in occupation number for positive frequency modes with that in the reverse sense for negative frequency modes. This is accomplished by transposing the matrices with respect to the occupation numbers of modes with $\lambda<0$, which effectively introduces a time-reversed description for the negative frequency modes.

The generating function for the matrices of all ordered products is (73), written as

$$\sum_{n,n'}(\chi^{(-)'}|n)\left(n\left|\left(\exp\left[i\int(dx)(\bar{\eta}\psi+\bar{\psi}\eta)\right]\right)_+\right|n'\right)\times(n'|\chi^{(+)'})=(\chi^{(-)'}|\chi^{(+)'})$$
$$\times\exp\left[i\int(dx)(dx')\bar{\eta}G_+\eta+i\int(dx)(\bar{\eta}\psi'+\bar{\psi}'\eta)\right],\quad(87)$$

in which all states refer to the standard surface. The sources $\bar{\eta}$ and η are understood to be placed on the extreme left and right, respectively, as we have done in (78). We now indicate the positive and negative frequency modes separately, placing the negative frequency modes first in the standard order,

$$(\chi^{(-)'}|n)=(\chi_+^{(-)'}|n_+)(\chi_-^{(-)'}|n_-),$$
$$(n'|\chi^{(+)'})=(n_-'|\chi_-^{(+)'})(n_+'|\chi_+^{(+)'}),$$

and define the mixed eigenfunctions

$$[\bar{\psi}'|N]=(\chi_+^{(-)'}|n_+)(n_-'|\chi_-^{(+)'})(-1)^{\frac{1}{2}n_-'(n_-'-1)},$$
$$[N'|\psi']=(-1)^{\frac{1}{2}n_-(n_--1)}(\chi_-^{(-)'}|n_-)(n_+'|\chi_+^{(+)'}).\quad(88)$$

Here the integers n_- and n_-' indicate the respective number of occupied negative frequency modes, while N and N' symbolize the sets of occupation numbers $\{n_+, n_-'\}$ and $\{n_+', n_-\}$, respectively. We shall also write $N_-=n_-'$, $N_-'=n_-$. The notation $[\bar{\psi}'|N]$ refers to the fact that the $\chi_+^{(-)'}$ and $\chi_-^{(+)'}$ together comprise the quantities

$$\bar{\psi}_{\lambda p}'=\int d\sigma_\mu\bar{\psi}'(x)\gamma_\mu\psi_{\lambda p}(x)\quad(89)$$

for positive and negative λ. Thus

$$[\bar{\psi}'|N]='\prod_{\lambda p}(\bar{\psi}_{\lambda p}')^{N_{\lambda p}},\quad(90)$$

and this product is in standard order, from right to left, in virtue of the factor $(-1)^{\frac{1}{2}n_-'(n_-'-1)}$, which effectively reverses the sense of multiplication for the n_-' anticommuting eigenvalues in $(n_-'|\chi_-^{(+)'})$. Similarly, $\chi_+^{(+)'}$ and $\chi_-^{(-)'}$ are comprised in

$$\psi_{\lambda p}'=\int d\sigma_\mu\bar{\psi}_{\lambda p}(x)\gamma_\mu\psi(x)',\quad(91)$$

and

$$[N'|\psi']=\prod_{\lambda p}(\psi_{\lambda p}')^{N_{\lambda p}'}.\quad(92)$$

To carry out the transposition, we must take

$$\sum_{n,n'}(\chi_+^{(-)'}|n_+)(\chi_-^{(-)'}|n_-)(n_+n_-|F|n_+'n_-')\times(n_-'|\chi_-^{(+)'})(n_+'|\chi_+^{(+)'}),\quad(93)$$

where F is any product of the operators ψ, $\bar{\psi}$, and reverse the positions of the two negative frequency eigenfunctions. This introduces the factor $(-1)^{n_-n_-'}$, so that (93) becomes

$$\sum_{N,N'}(\pm)[\bar{\psi}'|N][N|F|N'][N'|\psi'],\quad(94)$$

in which we have written

$$[N|F|N']=(n|F|n'),$$

and

$$(\pm)=(-1)^{N_-}(-1)^{\frac{1}{2}(N_--N_-')(N_-+1-N_-')}$$
$$=(-1)^{N_-'}(-1)^{\frac{1}{2}(N_--N_-')(N_-'+1-N_-)}.\quad(95)$$

Thus, if F is the unit operator, we have

$$(\chi^{(-)'}|\chi^{(+)'})=\sum_N[\bar{\psi}'|N](-1)^{N_-}[N|\psi']$$
$$=\exp[\sum_{\lambda p}\bar{\psi}_{\lambda p}'\epsilon(\lambda)\psi_{\lambda p}'].\quad(96)$$

To complete the re-expression of the generating function (87), we remark that

$$\psi'(x)=\sum_{\lambda p}\psi_{\lambda p}(x)\psi_{\lambda p}',\quad(97)$$

and

$$\bar{\psi}'(x)=\sum_{\lambda p}\bar{\psi}_{\lambda p}(x)\bar{\psi}_{\lambda p}'.\quad(98)$$

Hence,

$$\sum_{N,N'}(\pm)[\bar{\psi}'|N]\times\left[N\left|\left(\exp\left[i\int(dx)(\bar{\eta}\psi+\bar{\psi}\eta)\right]\right)_+\right|N'\right][N'|\psi']$$
$$=\exp\left[i\int(dx)(dx')\bar{\eta}G_+\eta\right]$$
$$\times\exp[\sum_{\lambda p}\bar{\psi}_{\lambda p}'\epsilon(\lambda)\psi_{\lambda p}'+i\bar{\eta}_{\lambda p}\psi_{\lambda p}'+i\bar{\psi}_{\lambda p}'\eta_{\lambda p}],\quad(99)$$

in which we have employed the notation,

$$\bar{\eta}_{\lambda p}=\int(dx)\bar{\eta}(x)\psi_{\lambda p}(x),\quad \eta_{\lambda p}=\int(dx)\bar{\psi}_{\lambda p}(x)\eta(x).$$

It should be emphasized again that the sources $\bar{\eta}$ and η are written to the left and to the right, respectively, rather than as indicated in the left-hand member of Eq. (99), since the sign factor expressed by (95) is valid only if the matrix element in (93) is a number, rather than a quantity possessing anticommutative properties.

The second exponential factor in (99) can be written

$$\prod_{\lambda p} \exp[\bar{\psi}_{\lambda p}'\epsilon(\lambda)\psi_{\lambda p}'+i\bar{\eta}_{\lambda p}\psi_{\lambda p}'+i\bar{\psi}_{\lambda p}'\eta_{\lambda p}]$$
$$=\prod_{\lambda p}\sum_{N,N'}(\bar{\psi}_{\lambda p}')^N[\delta_{N,N'}\epsilon^N(1+N\epsilon\bar{\eta}\eta)$$
$$+i\bar{\eta}N'(1-N)+i\eta N(1-N')](\psi_{\lambda p}')^{N'}, \quad (100)$$

where the expansion of the exponential referring to a given mode yields only five nonvanishing terms, as represented on the right of (100). We shall first extract the diagonal matrix elements, which constitute the following terms of (100),

$$\prod_{\lambda p}\sum_{N_{\lambda p}}(\bar{\psi}_{\lambda p}')^{N_{\lambda p}}[\epsilon(\lambda)]^{N_{\lambda p}}[1+N_{\lambda p}\epsilon(\lambda)\bar{\eta}_{\lambda p}\eta_{\lambda p}](\psi_{\lambda p}')^{N_{\lambda p}}$$
$$=\sum_N \exp[\sum_{\lambda p} N_{\lambda p}\epsilon(\lambda)\bar{\eta}_{\lambda p}\eta_{\lambda p}]$$
$$\times[\bar{\psi}'|N](-1)^{N-}[N|\psi']. \quad (101)$$

The exponential so obtained combines with the first factor on the right of (99) to form

$$\exp\left[i\int(dx)(dx')\bar{\eta}(x)(G_+(x,x') -i\sum_{\lambda p}N_{\lambda p}\epsilon(\lambda)\psi_{\lambda p}(x)\bar{\psi}_{\lambda p}(x'))\eta(x')\right].$$

No minus signs are introduced on moving the η to the right of the eigenfunctions in (101). On comparison with (74) and (80), we see that ($N_{\lambda p}=n_{\lambda p}$):

$$(n|(\psi(x_1)\cdots\psi(x_k)\bar{\psi}(x_l')\cdots\bar{\psi}(x_1'))_+|n)\epsilon_{k,l}$$
$$=\delta_{k,l}\det{}_{(k)}[-iG_+(x_i,x_j')$$
$$-\textstyle\sum_{\lambda p}n_{\lambda p}\epsilon(\lambda)\psi_{\lambda p}(x_i)\bar{\psi}_{\lambda p}(x_j')], \quad (102)$$

of which the simplest example is

$$(n|(\psi(x)\bar{\psi}(x'))_+|n)\epsilon(x,x')$$
$$=-iG_+(x,x')-\textstyle\sum_{\lambda p}n_{\lambda p}\epsilon(\lambda)\psi_{\lambda p}(x)\bar{\psi}_{\lambda p}(x'). \quad (103)$$

The modes that occur in the general matrix element can be divided into three classes: class a, those for which $N_{\lambda p}=0$, $N_{\lambda p}'=1$; class b, modes with $N_{\lambda p}=N_{\lambda p}'$; class c, those with $N_{\lambda p}=1$, $N_{\lambda p}'=0$. A typical term in the expansion of the product (100) can then be written

$$\prod_a(i\bar{\eta}\psi')\prod_b[(\bar{\psi}')^N\epsilon^N(1+N\epsilon\bar{\eta}\eta)(\psi')^N]\prod_c(i\bar{\psi}'\eta). \quad (104)$$

The b mode product can be re-arranged as in the discussion of the diagonal matrix elements, which yields

$$\exp[\sum_b N_{\lambda p}\epsilon(\lambda)\bar{\eta}_{\lambda p}\eta_{\lambda p}](-1)^{N-b}\,{}'\prod_b(\bar{\psi}_{\lambda p}')^N\prod_b(\psi_{\lambda p}')^N.$$

If we now bring the eigenvalues $\psi_{\lambda p}'$ of the a modes and $\bar{\psi}_{\lambda p}'$ of the c modes into standard order, (100) becomes

$${}'\prod_a(i\bar{\eta}(-1)^{N>})\exp[\sum_b N\epsilon\bar{\eta}\eta](-1)^{N-b}(-1)^{N_aN_c}$$
$$\times[\bar{\psi}'|N][N'|\psi']\prod_c(i\eta(-1)^{N>}),$$

in which $N_{>\lambda p}$ represents the number of occupied b modes that follow λp in the standard order, while N_a and N_c are the total number of modes in the a and c classes.

The right side of (99) is thus expressed as

$$\sum_{N,N'}{}'\prod_a(i\bar{\eta}(-1)^{N>})\exp[\ ](-1)^{N-b+N_aN_c}$$
$$\times[\bar{\psi}'|N][N'|\psi']\prod_c(i\eta(-1)^{N>})$$

where

$$\exp[\ ]=\exp\left[i\int(dx)(dx')\bar{\eta}(x)\times(G_+(x,x')-i\sum_b N_{\lambda p}\epsilon(\lambda)\psi_{\lambda p}(x)\bar{\psi}_{\lambda p}(x'))\eta(x')\right].$$

We must obey the injunction that all $\bar{\eta}$ appear on the left, and all η on the right. The effect of moving an η in the above exponential past the product of the eigenfunctions is to introduce a factor of $(-1)^{N-N'}$, where N and N' represent the total number of occupied modes in the respective eigenfunctions. Accordingly,

$$(\pm)\left[N\left|\left(\exp\left[i\int(dx)(\bar{\eta}\psi+\bar{\psi}\eta)\right]\right)_+\right|N'\right]$$
$$=(-1)^{N-b+N_aN_c}\,{}'\prod_a(i\bar{\eta}(-1)^{N>})\exp\left[i(-1)^{N-N'}\times\int(dx)(dx')\bar{\eta}(G_+-i\sum_b N\epsilon\psi\bar{\psi})\eta\right]\prod_c(i\eta(-1)^{N>}).$$

On expansion, we obtain the followlng result for the general nonvanishing matrix element,

$$[N|(\psi(x_1)\cdots\psi(x_k)\bar{\psi}(x_l')\cdots\bar{\psi}(x_1'))_+|N']\epsilon_{k,l}$$
$$=(-1)^{\frac{1}{2}(N-a-N-c)^2-\frac{1}{2}(N-a+N-c)}\det{}_{(k+l-r)}\begin{bmatrix}0, & (-1)^{N>}\bar{\psi}_c(x_j') \\ (-1)^{N>}\psi_a(x_i), & -iG_+(x_i,x_j')-\sum_b N_{\lambda p}\epsilon(\lambda)\psi_{\lambda p}(x_i)\bar{\psi}_{\lambda p}(x_j')\end{bmatrix} \quad (105)$$

in which

$$k-N_a=l-N_c=r \tag{106}$$

must be a non-negative integer. Also,

$$N_a=N'-N_b, \quad N_c=N-N_b.$$

In this determinant, 0 stands for the null matrix of N_c rows and N_a columns, and $(-1)^{N>}\psi_a(x_i)$ represents a matrix of k rows and N_a columns in which the eigenfunctions of the various a modes are standardly arrayed in the successive columns. The matrix $(-1)^{N>}\bar{\psi}_c(x_j')$ is one of N_c rows and l columns, with the various c mode adjoint eigenfunctions, in standard order, occupying the successive rows. Finally, we have the matrix $-iG_+(x_i,x_j')-\sum_b N_{\lambda p}\epsilon(\lambda)\psi_{\lambda p}(x_i)\bar{\psi}_{\lambda p}(x_j')$ of k rows and l columns. Thus the dimension of this determinant is

$$k+N_c=l+N_a=k+l-r.$$

Since this can also be written as N_a+N_c+r, we see that the integer r is also the maximum number of Green's function factors that appear in the development of the determinant.

For the elementary example $k=1$, $l=0$, we have $N_a=1$, $N_c=r=0$, and the nonvanishing matrix element

$$[N|\psi(x)|N+1_{\lambda p}]=(-1)^{N>\lambda p}\psi_{\lambda p}(x)$$

which unifies (85) and (86) as intended. Similarly,

$$[N+1_{\lambda p}|\bar{\psi}(x)|N]=(-1)^{N>\lambda p}\bar{\psi}_{\lambda p}(x).$$

With this classification, $\psi(x)$ and $\bar{\psi}(x)$ appear as single "particle" annihilators and creators, respectively. The selection rules (106) for the general matrix element can then be described as follows. The operator contains k annihilators and l creators. If r of these operators combine in pairs to produce a null net effect, the remaining $k-r$ annihilators and $l-r$ creators will empty $N_a=k-r$ occupied modes, and fill $N_c=l-r$ occupied modes. Of course $N=N'+l-k$.

The diagonal matrix elements (102) represent the extreme situation in which

$$r=k=l, \quad N_a=N_c=0.$$

At the opposite limit is

$$r=0, \quad N_a=k, \quad N_c=l,$$

where the matrix element (105) becomes

$$\pm\det{}_{(k)}[(-1)^{N>}\psi_a(x_i)]\,\det{}_{(l)}[(-1)^{N>}\bar{\psi}_c(x_j')]$$

and

$$\pm=(-1)^{\frac{1}{2}N_{-a}(N_{-a}-1)}(-1)^{\frac{1}{2}N_{-c}(N_{-c}-1)}(-1)^{N_aN_c-N_{-a}N_{-c}}.$$

MAXWELL FIELD MATRIX ELEMENTS

This section is a supplement to paper III, in which the transformation function describing the Maxwell field with an external current is used to construct the matrices of all products of the potential vector for the isolated electromagnetic field. The transformation function (III, 20) can be expressed as

$$(F^{(-)'}\sigma_1|F^{(+)'}\sigma_2)=(F^{(-)'}\sigma_1|F^{(+)'}\sigma_2)]_0 \times\exp\left[i\mathcal{W}_0+i\int(dx)J_\nu(x)A_\nu'(x)\right], \tag{107}$$

where $]_0$ indicates zero external current, and

$$\begin{aligned}A_\nu'(x')&=2\int_{\sigma_1}d\sigma_\mu F_{\mu\nu}{}^{(-)'}(x)D_+(x-x')\\&\quad-2\int_{\sigma_2}d\sigma_\mu F_{\mu\nu}{}^{(+)'}(x)D_+(x-x')\\&=2\oint d\sigma_\mu F_{\mu\nu}'(x)D_+(x-x').\end{aligned} \tag{108}$$

We also recall that

$$\begin{aligned}\mathcal{W}_0&=\tfrac{1}{2}\int(dx)(dx')J_\mu(x)D_+(x-x')J_\mu(x')\\&=\tfrac{1}{2}\int(dx)(dx')[J_k(x)(\delta_{kl}D_+(x-x'))^{(T)}J_l(x')\\&\qquad-J_0(x)\mathcal{D}(x-x')J_0(x')],\end{aligned}$$

where the latter form is appropriate to the radiation gauge.

The dependence of the transformation function upon the external current is expressed by

$$\delta_J(F^{(-)'}\sigma_1|F^{(+)'}\sigma_2)=i\left(F^{(-)'}\sigma_1\left|\int(dx)\delta J_\mu(x)A_\mu(x)\right|F^{(+)'}\sigma_2\right). \tag{109}$$

In the radiation gauge, $A_0(x)$ is the numerical quantity

$$A_0(x)=\int(dx')\mathcal{D}(x-x')J_0(x').$$

Hence

$$\delta_{J_0}(F^{(-)'}\sigma_1|F^{(+)'}\sigma_2)=-i\int(dx)(dx')\delta J_0(x)\mathcal{D}(x-x')J_0(x')(F^{(-)'}\sigma_1|F^{(+)'}\sigma_2),$$

and

$$(F^{(-)'}\sigma_1|F^{(+)'}\sigma_2)=(F^{(-)'}\sigma_1|F^{(+)'}\sigma_2)]_{J_0=0}\times\exp\left[-i\tfrac{1}{2}\int(dx)(dx')J_0(x)\mathcal{D}(x-x')J_0(x')\right],$$

which gives the simple dependence of the transformation function upon J_0, in the radiation gauge. This factor is evident in the radiation gauge version of $\exp(i\mathcal{W}_0)$. Accordingly, we restrict ourselves to transverse currents, for which $J_0=0$.

On introducing the scale factor λ, $J_k \to \lambda J_k$ we infer from (109) that

$$(\partial/\partial\lambda)(F^{(-)\prime}\sigma_1|F^{(+)\prime}\sigma_2) = i\left(F^{(-)\prime}\sigma_1\left|\int (dx)J_k(x)A_k(x)\right|F^{(+)\prime}\sigma_2\right).$$

Repeated differentiation yields

$$(\partial/\partial\lambda)^n(F^{(-)\prime}\sigma_1|F^{(+)\prime}\sigma_2) = i^n\left(F^{(-)\prime}\sigma_1\left|\int (dx_1)\cdots(dx_n)J_{k1}(x_1)\cdots J_{kn}(x_n) \times (A_{k1}(x_1)\cdots A_{kn}(x_n))_+\right|F^{(+)\prime}\sigma_2\right),$$

and the transformation function appropriate to the external current ($\lambda=1$), is obtained from that of the isolated electromagnetic field ($\lambda=0$) as

$$(F^{(-)\prime}\sigma_1|F^{(+)\prime}\sigma_2) = \left(F^{(-)\prime}\sigma_1\left|\left(\exp\left[i\int (dx)J_k(x)A_k(x)\right]\right)_+\right|F^{(+)\prime}\sigma_2\right)\Bigg]_0.$$

If we employ the notation

$$(F^{(-)\prime}\sigma_1| \quad |F^{(+)\prime}\sigma_2)/(F^{(-)\prime}\sigma_1|F^{(+)\prime}\sigma_2)]_0 = \langle\ \rangle,$$

the transformation function (107) can be expressed by

$$\left\langle\left(\exp\left[i\int (dx)J_k(x)A_k(x)\right]\right)_+\right\rangle = \exp\left[i\tfrac{1}{2}\int (dx)(dx')J_k(x)(\delta_{kl}D_+(x-x'))^{(T)} \times J_l(x')+i\int (dx)J_k(x)A_k'(x)\right]. \quad (110)$$

It will be advantageous to suppress the vector indices. Accordingly, we rewrite (110) as

$$\left\langle\left(\exp\left[i\int (dx)J(x)A(x)\right]\right)_+\right\rangle = \exp\left[i\tfrac{1}{2}\int (dx)(dx')J(x)D_+(x-x')J(x') +i\int (dx)J(x)A'(x)\right]. \quad (111)$$

An alternative version is

$$\left\langle\left(\exp\left[i\int (dx)J(x)'A(x)\right]\right)_+\right\rangle = \exp\left[i\tfrac{1}{2}\int (dx)(dx')J(x)D_+(x-x')J(x')\right], \quad (112)$$

where

$$'A(x) = A(x) - A'(x).$$

An expansion of both sides in (111) or (112) will supply the matrix elements of ordered A products.

The right side of (112) is an even function of J. Accordingly,

$$\langle('A(x_1)\cdots 'A(x_{2n-1}))_+\rangle = 0.$$

In particular,

$$\langle 'A(x)\rangle = 0,$$

or

$$\langle A(x)\rangle = A'(x). \quad (113)$$

The general even term in the expansion of the left side in (112) is

$$\frac{i^{2n}}{(2n)!}\int (dx_1)\cdots(dx_{2n})J(x_1)\cdots J(x_{2n}) \times\langle('A(x_1)\cdots 'A(x_{2n}))_+\rangle.$$

This is to be compared with

$$\frac{i^n}{2^n n!}\int (dx_1)\cdots(dx_{2n})J(x_1)\cdots J(x_{2n}) \times D_+(x_1-x_2)\cdots D_+(x_{2n-1}-x_{2n}) = \frac{i^n}{(2n)!}\int (dx_1)\cdots(dx_{2n})J(x_1)\cdots J(x_{2n}) \times \mathrm{sym}_{(n)}D_+(x_i-x_j),$$

in which has been introduced what we shall call the nth symmetrant of D_+. This is defined by

$$\mathrm{sym}_{(n)}D_+(x_i-x_j) = \sum_{\text{perm}} D_+(x_{i1}-x_{i2})\cdots D_+(x_{i2n-1}-x_{i2n}),$$

and the summation is extended over all distinct permutations of the indices $i_1\cdots i_{2n}$, which are some rearrangement of the integers $1\cdots 2n$. Since D_+ is an even function of its argument, and the order of the n factors is irrelevant, the number of such permutations is $(2n)!/2^n n! = (2n-1)(2n-3)\cdots$.

The matrix of an even product of the $'A$ is thus expressed by

$$\langle('A(x_1)\cdots 'A(x_{2n}))_+\rangle = (-i)^n\,\mathrm{sym}_{(n)}D_+(x_i-x_j). \quad (114)$$

The first two such products are

$$\langle('A(x)'A(x'))_+\rangle = -iD_+(x-x'),$$

or

$$\langle(A(x)A(x'))_+\rangle = A'(x)A'(x') - iD_+(x-x') \quad (115)$$

and

$$\langle('A(x_1)'A(x_2)'A(x_3)'A(x_4))_+\rangle = -[D_+(x_1-x_2)D_+(x_3-x_4)+D_+(x_1-x_3)D_+(x_2-x_4) +D_+(x_2-x_3)D_+(x_1-x_4)].$$

To obtain corresponding results for products of the operators A, as in the simple examples (113) and (115), we first consider the even terms of (111). The even function, $\cos x$, can be obtained from an exponential function of x^2 by a suitable operation

$$\cos x = C_t[\exp(-\tfrac{1}{2}tx^2)],$$

where

$$C_t(t^n) = 2^n n!/(2n)!$$

effectively defines the operator C_t, although an explicit integral representation can also be exhibited, as in (81). Hence the even part of (111) is

$$\left\langle\left(\cos\left[\int (dx)J(x)A(x)\right]\right)_+\right\rangle$$
$$= C_t \exp\left[-\tfrac{1}{2}\int (dx)(dx')J(x) \times(-iD_+(x-x')+tA'(x)A'(x'))J(x')\right],$$

which, in view of (112) and (114), yields

$$\begin{aligned}\langle(A(x_1)\cdots A(x_{2n}))_+\rangle &= C_t\,\mathrm{sym}_{(n)}[-iD_+(x_i-x_j)+tA'(x_i)A'(x_j)]\\ &= A'(x_1)\cdots A'(x_{2n})+\cdots\\ &\quad+(-i)^n\,\mathrm{sym}_{(n)}D_+(x_i-x_j). \qquad (116)\end{aligned}$$

As the initial term of the developed version indicates, the effect of the C_t operation is to reinstate the unique counting of each distinct permutation in the expansion of the symmetrant (116).

For the construction of matrices describing odd products of the A, we remark that

$$(\delta/\delta A'(x))\left\langle\left(\exp\left[i\int (dx)JA\right]\right)_+\right\rangle = iJ(x)\left\langle\left(\exp\left[i\int (dx)JA\right]\right)_+\right\rangle.$$

On expanding both sides, we obtain

$$\begin{aligned}(\delta/\delta A'(x))\langle(A(x_1)\cdots A(x_m))_+\rangle &= \delta(x-x_1)\langle(A(x_2)\cdots A(x_m))_+\rangle+\cdots\\ &\quad+\delta(x-x_m)\langle(A(x_1)\cdots A(x_{m-1}))_+\rangle,\end{aligned}$$

which can also be expressed by

$$(\partial/\partial A'(x_m))\langle(A(x_1)\cdots A(x_m))_+\rangle = \langle(A(x_1)\cdots A(x_{m-1}))_+\rangle.$$

Hence the matrices of odd products can be obtained by differentiation from those of even products,

$$\begin{aligned}\langle(A(x_1)\cdots A(x_{2n-1}))_+\rangle &= (\partial/\partial A'(x_{2n}))\langle(A(x_1)\cdots A(x_{2n}))_+\rangle\\ &= (\partial/\partial A'(x_{2n}))C_t\,\mathrm{sym}_{(n)}[-iD_+(x_i-x_j)\\ &\qquad+tA'(x_i)A'(x_j)],\end{aligned}$$

or

$$\langle(A(x_1)\cdots A(x_{2n-1}))_+\rangle = C_t\,\mathrm{sym}_{(n)}[-iD_+(x_i-x_j)+tA'(x_i)A'(x_j),\quad tA'(x_k)],$$

which is intended to indicate a symmetrant that is obtained from (116) by replacing the elements containing the variables x_k, x_{2n} with $tA'(x_k)$.

The Occupation Number Representation

The diagonal matrix elements referring to the vacuum state are obtained by placing all eigenvalues equal to zero,

$$(0|(A(x_1)\cdots A(x_{2n-1}))_+|0)=0,$$
$$(0|(A(x_1)\cdots A(x_{2n}))_+|0) = (-i)^n\,\mathrm{sym}_{(n)}D_+(x_i-x_j), \qquad (117)$$

and, in particular,

$$(0|(A(x)A(x'))_+|0) = -iD_+(x-x'). \qquad (118)$$

We introduce the mode functions

$$\begin{aligned}A_{\lambda k}(x)_\mu &= \left(\frac{(d\mathbf{k})}{(2\pi)^3}\frac{1}{2k_0}\right)^{\frac{1}{2}} e_\mu(\lambda k)e^{ikx},\\ \bar{A}_{\lambda k}(x)_\mu &= \left(\frac{(d\mathbf{k})}{(2\pi)^3}\frac{1}{2k_0}\right)^{\frac{1}{2}} e_\mu(\lambda k)e^{-ikx},\end{aligned} \qquad (119)$$

in terms of which the tensor Green's function (III, 23) appears as

$$\begin{aligned}\delta_{\mu\nu}D_+(x-x') &= i\sum_{\lambda k} A_{\lambda k}(x)_\mu \bar{A}_{\lambda k}(x')_\nu, \quad x_0>x_0'\\ &= i\sum_{\lambda k} \bar{A}_{\lambda k}(x)_\mu A_{\lambda k}(x')_\nu, \quad x_0<x_0'. \qquad (120)\end{aligned}$$

Accordingly,

$$A_\nu'(x') = 2\oint d\sigma_\alpha F_{\alpha\mu}'(x)\delta_{\mu\nu}D_+(x-x')$$

becomes (suppressing the vector indices)

$$A'(x) = \sum_{\lambda k}(A_{\lambda k}(x)e^{-ikx_2}A_{\lambda k}{}^{(+)\prime}+\bar{A}_{\lambda k}(x)e^{ikx_1}A_{\lambda k}{}^{(-)\prime}),$$

where

$$A_{\lambda k}{}^{(-)\prime} = 2i\int_{\sigma_1} d\sigma_\mu F_{\mu\nu}{}^{(-)\prime}(x)A_{\lambda k}(x)_\nu e^{-ikx_1},$$

and

$$A_{\lambda k}{}^{(+)\prime}=-2i\int_{\sigma 2} d\sigma_\mu F_{\mu\nu}{}^{(+)\prime}(x)\bar{A}_{\lambda k}(x)_\nu e^{ikx_2}.$$

The latter quantities are the negatives of $a_{\lambda k}{}^{(\pm)\prime}$ (III, 27, 28). Since these minus signs would be somewhat unfortunate for our present purposes, we notice that the opposite choice of sign in (III, 27, 28) produces relatively trivial changes in the work of III. Thus, in Eq. (III, 34) and its consequences, the signs of the terms contalnlng $J_{\lambda k}$ and $J_{\lambda k}{}^*$ are to be reversed. We now write the eigenfunctions of the isolated electromagnetic field as

$$(F^{(-)\prime}|n)=\prod_{\lambda k}(n!)^{-\frac{1}{2}}(A_{\lambda k}{}^{(-)\prime})^n,$$

$$(n|F^{(+)\prime})=\prod_{\lambda k}(n!)^{-\frac{1}{2}}(A_{\lambda k}{}^{(+)\prime})^n,$$

while the transformation function is

$$(F^{(-)\prime}\sigma_1|F^{(+)\prime}\sigma_2) =\exp[\sum_{\lambda k} A_{\lambda k}{}^{(-)\prime}e^{ikx_1}e^{-ikx_2}A_{\lambda k}{}^{(+)\prime}] =\sum_n(F^{(-)\prime}|n)\exp[iP(n)(x_1-x_2)](n|F^{(+)\prime}).$$

The occupation number matrix of A is derived from (113), written as

$$(F^{(-)\prime}\sigma_1|A(x)|F^{(+)\prime}\sigma_2) =\sum_{n,n'}(F^{(-)\prime}|n)(n\sigma_1|A(x)|n'\sigma_2)(n'|F^{(+)\prime}) =A'(x)(F^{(-)\prime}\sigma_1|F^{(+)\prime}\sigma_2).$$

The substitutions $e^{-ikx_2}A_{\lambda k}{}^{(+)\prime}\rightarrow A_{\lambda k}{}^{(+)\prime}$, $e^{ikx_1}A_{\lambda k}{}^{(-)\prime}\rightarrow A_{\lambda k}{}^{(-)\prime}$ convert this into a generating function for the matrix referring to a standard surface,

$$(n|A(x)|n')=\exp(-iP(n)x_1)\times(n\sigma_1|A(x)|n'\sigma_2)\exp(iP(n')x_2),$$

namely,

$$\sum_{n,n'}(F^{(-)\prime}|n)(n|A(x)|n')(n'|F^{(+)\prime}) =\sum_{\lambda k}(A_{\lambda k}(x)A_{\lambda k}{}^{(+)\prime}+\bar{A}_{\lambda k}(x)A_{\lambda k}{}^{(-)\prime}) \times\sum_n(F^{(-)\prime}|n)(n|F^{(+)\prime}).$$

Now

$$A_{\lambda k}{}^{(-)\prime}(F^{(-)\prime}|n-1_{\lambda k})=(F^{(-)\prime}|n)n_{\lambda k}{}^{\frac{1}{2}}$$

and

$$(n-1_{\lambda k}|F^{(+)\prime})A_{\lambda k}{}^{(+)\prime}=n_{\lambda k}{}^{\frac{1}{2}}(n|F^{(+)\prime}),$$

so that the nonvanishing matrix elements of $A(x)$ are

$$(n-1_{\lambda k}|A(x)|n)=n_{\lambda k}{}^{\frac{1}{2}}A_{\lambda k}(x)$$

and

$$(n|A(x)|n-1_{\lambda k})=n_{\lambda k}{}^{\frac{1}{2}}\bar{A}_{\lambda k}(x).$$

The generating function for the matrices of all ordered products, referred to the standard surface, is

$$\sum_{n,n'}(F^{(-)\prime}|n)\times\left(n\left|\left(\exp\left[i\int(dx)J(x)A(x)\right]\right)_+\right|n'\right)(n'|F^{(+)\prime})$$
$$=(F^{(-)\prime}|F^{(+)\prime})\exp\left[i\tfrac{1}{2}\int(dx)(dx')J(x)\times D_+(x-x')J(x')+i\int(dx)J(x)A'(x)\right]$$
$$=\exp\left[i\tfrac{1}{2}\int(dx)(dx')JD_+J\right]\times\prod_{\lambda k}\exp\left[A_{\lambda k}{}^{(-)\prime}A_{\lambda k}{}^{(+)\prime}+iA_{\lambda k}{}^{(-)\prime}\times\int(dx)J(x)\bar{A}_{\lambda k}(x)+iA_{\lambda k}{}^{(+)\prime}\times\int(dx)J(x)A_{\lambda k}(x)\right].\quad(121)$$

The expansion into eigenfunctions is greatly simplified by exploiting the infinitesimal nature of $(d\mathbf{k})$. Thus the second exponential factor of (121) becomes

$$\prod_{\lambda k}\sum_{n,n'}\frac{(A^{(-)\prime})^n}{(n!)^{\frac{1}{2}}}\left[\delta_{n,n'}\left(1-n\int(dx)JA\int(dx)J\bar{A}\right)+\delta_{n,n'+1}n^{\frac{1}{2}}i\int(dx)J\bar{A}+\delta_{n',n+1}n'^{\frac{1}{2}}i\times\int(dx)JA\right]\frac{(A^{(+)\prime})^n}{(n'!)^{\frac{1}{2}}}.\quad(122)$$

A change in the occupation number of a given mode by two, for example, would lead to a matrix element proportional to $(d\mathbf{k})$, and thus to a transition probability proportional to $(d\mathbf{k})^2$.

We first consider diagonal matrix elements, which are contained in the following terms of (122),

$$\prod_{\lambda k}\sum_n\frac{(A^{(-)\prime})^n}{(n!)^{\frac{1}{2}}}\left[1-n\int(dx)JA\int(dx)J\bar{A}\right]\frac{(A^{(+)\prime})^n}{(n!)^{\frac{1}{2}}}$$
$$=\sum_n(F^{(-)\prime}|n)\exp\left[-\int(dx)(dx')J(x)\times(\sum_{\lambda k}n_{\lambda k}A_{\lambda k}(x)\bar{A}_{\lambda k}(x'))J(x')\right](n|F^{(+)\prime}).$$

Thus

$$\left(n\left|\left(\exp\left[i\int (dx)J(x)A(x)\right]\right)_+\right|n\right)$$
$$=\exp\left[-\tfrac{1}{2}\int (dx)(dx')J(x)\left(-iD_+(x-x')+\sum_{\lambda k} 2n_{\lambda k}A_{\lambda k}(x)\bar{A}_{\lambda k}(x')\right)J(x')\right],$$

which asserts that

$$(n|(A(x_1)\cdots A(x_{2l-1}))_+|n)=0,$$

and that

$$(n|(A(x_1)\cdots A(x_{2l}))_+|n)=\mathrm{sym}_{(l)}[-iD_+(x_i-x_j)+\textstyle\sum_{\lambda k} n_{\lambda k}\times(A_{\lambda k}(x_i)\bar{A}_{\lambda k}(x_j)+\bar{A}_{\lambda k}(x_i)A_{\lambda k}(x_j))]. \quad (123)$$

The elementary example of the latter result is

$$(n|(A(x)A(x'))_+|n)=-iD_+(x-x')+\textstyle\sum_{\lambda k} n_{\lambda k}(A_{\lambda k}(x)\bar{A}_{\lambda k}(x')+\bar{A}_{\lambda k}(x)A_{\lambda k}(x')). \quad (124)$$

We introduce a classification of modes in the general matrix element: class a, those for which $n_{\lambda k}=n_{\lambda k}'-1$; class b, modes with $n_{\lambda k}=n_{\lambda k}'$; class c, those with $n_{\lambda k}=n_{\lambda k}'+1$. A typical term in the expansion of the product (122) can then be written

$$(F^{(-)\prime}|n)\prod_a\left[n'^{\frac{1}{2}}i\int (dx)JA\right]\times\prod_b\left[1-n\int(dx)JA\int(dx)J\bar{A}\right]\times\prod_c\left[n^{\frac{1}{2}}i\int(dx)J\bar{A}\right](n'|F^{(+)\prime}).$$

Accordingly,

$$\left(n\left|\left(\exp\left[i\int (dx)J(x)A(x)\right]\right)_+\right|n'\right)$$
$$=\exp\left[i\tfrac{1}{2}\int(dx)(dx')J(x)D_+^{(n)}(x,x')J(x')\right]\times\prod_a\left[n'^{\frac{1}{2}}i\int(dx)JA_{\lambda k}\right]\prod_c\left[n^{\frac{1}{2}}i\int(dx)J\bar{A}_{\lambda k}\right], \quad (125)$$

in which we have introduced the symbol

$$D_+^{(n)}(x,x')=D_+(x-x')+i\sum_b n_{\lambda k}\times[A_{\lambda k}(x)\bar{A}_{\lambda k}(x')+\bar{A}_{\lambda k}(x)A_{\lambda k}(x')].$$

We shall use $A_\alpha(x)$ to denote collectively the a mode functions, $A_{\lambda k}(x)$, and the c mode functions $\bar{A}_{\lambda k}(x)$ (a complete unification would be achieved by transposing the matrices with respect to the occupation numbers of the c modes, which would introduce a time-reversed description for emission processes). With this notation, (125) appears as

$$\left(n\left|\left(\exp\left[i\int (dx)J(x)A(x)\right]\right)_+\right|n'\right)$$
$$=\prod_a(n'^{\frac{1}{2}})\prod_c(n^{\frac{1}{2}})\prod_\alpha\left[i\int(dx)J(x)A_\alpha(x)\right]\times\exp\left[i\tfrac{1}{2}\int(dx)(dx')JD_+^{(n)}J\right].$$

The expansion of this generating function leads to the following formula for the general nonvanishing matrix element,

$$(n|(A(x_1)\cdots A(x_m))_+|n')=\prod_a(n'^{\frac{1}{2}})\prod_c(n^{\frac{1}{2}})(-i)^r\,\mathrm{sym}_{(N,r)}\times[A_\alpha(x_i);\,D_+^{(n)}(x_j,x_k)], \quad (126)$$

where N is the number of $\alpha=a+c$ modes, and r is a non-negative integer such that

$$m=N+2r.$$

We have introduced an extension of the symmetrant which is constructed from N different functions of a single variable, $A_\alpha(x_i)$, and a symmetrical function of two variables, $D_+^{(n)}(x_j,x_k)$, taken r times,

$$\mathrm{sym}_{(N,r)}[A_\alpha(x_i);\,D_+^{(n)}(x_j,x_k)]=\sum_{\text{perm}}A_{\alpha 1}(x_{i_1})\cdots A_{\alpha N}(x_{i_N})\times D_+^{(n)}(x_{i_{N+1}},x_{i_{N+2}})\cdots D_+^{(n)}(x_{i_{m-1}},x_{i_m}),$$

where the summation is extended over the $(N+2r)!/2^r r!$ distinct permutations of the indexes $i_1\cdots i_m$, which are some rearrangement of the integers $1\cdots m$.

The diagonal matrix elements correspond to the situation where $N=0$, $m=2r$. At the opposite extreme is $r=0$, $N=m$, where the symmetrant reduces to

$$\mathrm{sym}_{(m,0)}[A_\alpha(x_i)]=\sum_{\text{perm}}A_{\alpha 1}(x_{i_1})\cdots A_{\alpha m}(x_{i_m}).$$

This sum of $m!$ permutations is obtained from the corresponding determinant by omitting the alternating sign factor.