TABLE V. Masses of Kr^{85} and Kr^{87}; binding energy of the last neutron.

Masses	Binding energies
$Kr^{85}=84.94090\pm13$	49th neutron: 5.95 ± 0.08 Mev
$Kr^{87}=86.94125\pm12$	50th neutron: 10.88 ± 0.23 Mev
	51st neutron: 5.53 ± 0.08 Mev

the mass of Kr^{86}, we find the mass of Kr^{87} to be 86.94125 ±12. Thulin[7] reports a 3.63-Mev beta in the transition from Kr^{87} to Rb^{87}, which leads to a value of 86.93735 ±27 for the mass of Rb^{87}. Nier[6] reports a value of 86.93709$\pm$17 for Rb^{87}. These values are in satisfactory agreement.

The binding energies of the 49th and 51st neutron calculated from these results in the usual fashion are 5.95$\pm$0.08 Mev and 5.53$\pm$0.08 Mev, respectively.

The binding energy of the 50th neutron may be estimated as 10.88$\pm$0.23 Mev. This variation in the binding energy of the last neutron is expected at the closing of the 50-neutron shell. These results are summarized in Table V.

In conclusion, the authors wish to thank Professor L. C. Biedenharn and Professor E. C. Pollard for many helpful discussions.

[7] S. Thulin, Phys. Rev. 87, 684 (1952).

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The Use of Tensor Operators in Nuclear Collision Problems

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The consequences of the invariance under rotations and reflections of the collision matrix are expressed as a relationship between the expectation values of initial-state and final-state irreducible tensor operators in spin-orbit space. This relationship is used to give a proof of the Eisner-Sachs theorem on the complexity of angular distributions of nuclear reactions and some extensions of this theorem.

IN collision problems and decay problems in nuclear physics, one often encounters calculations in which it is necessary to sum over the spins of certain of the initial or final particles. As a result of such summing it may happen that a number of terms in the result cancel. For example, if two unpolarized particles of spin 1 collide in a p state, it is possible that the total angular momentum is 3 and that the outgoing particles leave in an f state; nevertheless, the resulting angular distribution, after summing over initial and final spins, can contain no power of $\cos\theta$ higher than $\cos^2\theta$. This result follows from a theorem proven by Eisner and Sachs[1] and also by Yang.[2] Intuitively, the argument is that although the initial state has angular momentum 3, it is only polarized to degree 1 and the anisotropy of the angular distribution is a measure of the polarization of the orbital angular momentum. This argument may be made precise by focusing attention on the expectation values of the spin-orbital operators before and after the collision rather than on the initial and scattered wave functions. The consequences of rotational invariance may be expressed in a simple form through the use of irreducible tensor operators.[3] In the first section of this paper the general method is developed, and in the second section it is applied to the proof of the theorem of Eisner and Sachs and some extensions of the theorem. The method has also been applied,[4] using irreducible tensor operators in isotopic spin space, to the problem of the consequences of charge independence for multiple meson production.

[1] E. Eisner and R. G. Sachs, Phys. Rev. 72, 680 (1947); L. Wolfenstein and R. G. Sachs, Phys. Rev. 73, 528 (1948).

[2] C. N. Yang, Phys. Rev. 74, 764 (1948). The proof of Yang is also presented in J. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (J. Wiley and Sons, 1952), p. 535 ff.

[3] G. Racah, Phys. Rev. 62, 438 (1942).

I. GENERAL METHOD

A system which is not completely polarized is in a statistical mixture of pure quantum-mechanical states and is best described in terms of the density matrix[5]

$$\rho=(1/r)\ (\mathrm{Tr}\,\rho)\textstyle\sum_\mu\langle S^\mu\rangle S^\mu, \quad (1)$$

where r is the number of independent pure states to be considered, S^μ is a member of a complete set of base matrices[6] operating on the vectors representing pure states, and $\langle S^\mu\rangle$ is the average value of this operator for the system. The pure states may be considered as sums of products of pure spin states and pure orbital states, and it is assumed that only orbital states with angular momentum L less than or equal to some specified value $L_{\max}$ need be considered.

Instead of considering the transformation properties of the pure states, we shall consider the behavior under

[4] L. Wolfenstein, Phys. Rev. 90, 371 (1953).

[5] Use of the density matrix in nuclear physics problems has been made recently by many authors; e.g., H. Tolhoek and S. deGroot, *Physica* 15, 833 (1949); 17, 1 (1951); U. Fano, Phys. Rev. 90, 577 (1953); R. H. Dalitz, Proc. Phys. Soc. (London) A65, 175 (1952); L. Wolfenstein and J. Ashkin, Phys. Rev. 85, 947 (1952). This last paper will be referred to as A.

[6] The notation is the same as in A except that here the operators S^μ operate on the orbital states as well as the spin states; that is, here the orbital angular momentum is treated as another spin. The set called "a complete set of operators" in A is here called "a complete set of base matrices."

rotations and reflections of the operators S^μ in terms of which the density matrix is expressed. The complete set of base matrices S^μ forms a representation of the rotation group which is equivalent to the representation formed by the direct product of two independent representations each equivalent to that of the set of pure states. We denote by $S_{Jp\alpha}{}^m$ an irreducible component of this representation which transforms under rotation like the spherical harmonics $Y_J{}^m(\theta\varphi)$ and under reflection with parity p; the subscript α distinguishes different irreducible components having the same J and p values. From Eq. (4) of A,

$$\rho_f = M\rho_i M^\dagger,$$

where ρ_f and ρ_i are the final and initial density matrices, respectively, and M is the scattering matrix. Using Eq. (1) for ρ_i,

$$\rho_f = \frac{1}{r}(\mathrm{Tr}\,\rho_i) \sum_{J\alpha pm} \langle S^m{}_{Jp\alpha}\rangle_i M S^m{}_{Jp\alpha} M^\dagger. \tag{2}$$

The operators $MS^m{}_{Jp\alpha}M^\dagger$ may be expanded in terms of a complete set of base matrices $S'^{m'}{}_{J'p'\alpha'}$ which operate in the space of the final particles; since M is invariant under rotations and reflections, $MS^m{}_{Jp\alpha}M^\dagger$ has the same transformation properties as $S^m{}_{Jp\alpha}$, so that

$$\begin{aligned} MS^m{}_{Jp\alpha}M^\dagger &= \sum_{\alpha' J' p' m'} b_{\alpha\alpha'}{}^{Jp} \delta_{JJ'}\delta_{mm'}\delta_{pp'} S'^{m'}{}_{J'p'\alpha'} \\ &= \sum_{\alpha'} b_{\alpha\alpha'}{}^{Jp} S'^m{}_{Jp\alpha'}. \end{aligned} \tag{3}$$

It is readily shown by performing a general rotation on both sides of this equation that it is necessary that $b_{\alpha\alpha'}{}^{Jp}$ be independent of m (as has been assumed in the notation), although this result is not actually used in the present paper. Equations (2) and (3) together with Eq. (1) applied now to ρ_f give

$$\langle S'^m{}_{Jp\alpha'}\rangle_f = \textstyle\sum_\alpha c_{\alpha\alpha'}{}^{Jp} \langle S^m{}_{Jp\alpha}\rangle_i, \tag{4}$$

where the coefficients c are equal to the coefficients b times $\mathrm{Tr}\,\rho_i/\mathrm{Tr}\,\rho_f$. This result could have been obtained directly by applying invariance arguments to Eq. (5) of A. Equation (4) is a general statement of the consequences of invariance under rotation and reflection for nuclear collision problems. In particular it states that it is impossible to define a component of a vector (or an irreducible tensor of rank J) in the final state unless that component of a vector (tensor) is defined, that is, has nonzero expectation value, in the initial state.

This discussion may be used as a basis for defining the degree of polarization of a system mentioned in the introduction. *The degree of polarization of a system equals* $\frac{1}{2}J_{\max}$, *where* $J_{\max}$ *is the largest value of* J *for which there exists a nonvanishing* $\langle S^m{}_{Jp\alpha}\rangle$. From Eq. (1) it follows that the degree of polarization describes the complexity of the behavior under rotation of the density matrix. Thus a completely unpolarized system has degree of polarization zero. It is easy to see that a pure state of angular momentum L has a degree of polarization L; in particular, this is true for the initial orbital state in a collision if we consider only one orbital angular momentum L, since then only the pure state with $m=0$ is involved. If several orbital angular momenta are considered in a collision the degree of polarization of the initial orbital motion is equal to the maximum orbital angular momentum $L_{\max}$. A system formed by compounding two other systems with degrees of polarization d_1 and d_2 has a degree of polarization less than or equal to (d_1+d_2) and greater than or equal to d_1-d_2. This can be proved by using for the operators S^μ products of two operators each of which operates in the space of only one of the original systems. In particular, for a collision between two unpolarized nuclei of arbitrary spin with a maximum relative orbital angular momentum $L_{\max}$, the degree of polarization is $L_{\max}$, since this may be considered a combination of the unpolarized spin systems $(d_1=0)$ and the orbital system $(d_2=L_{\max})$.

Part of the consequences of Eq. (4) can now be restated as the following theorem: *the degree of polarization of the final state cannot be greater than that of the initial.*

II. APPLICATION TO ANGULAR DISTRIBUTIONS

The complexity of angular distributions is limited by the following theorem:[1,2] If in a collision of unpolarized particles of arbitrary spin the maximum orbital angular momentum that need be considered is $L_{\max}$, then the angular distribution of the outgoing particles when expanded in spherical harmonics will contain no spherical harmonics of degree greater than $2L_{\max}$. This theorem follows easily as a corollary of the general theorem stated above.

From the discussion at the end of the first section, the initial state has a degree of polarization equal to $L_{\max}$. We consider the spherical harmonics $Y_j{}^k(\theta'\varphi')$, (times the unity operator in the final spin spaces) as multiplicative operators in the space of the final particles;[7] these are clearly irreducible subsets of the S'^μ with J', p', and m' equal to j, $(-1)^j$, and k, respectively. Since the degree of polarization of the final state cannot be more than $L_{\max}$, the expectation value $\langle Y_j{}^k(\theta'\varphi')\rangle$ must be zero if j is greater than $2L_{\max}$. From the orthogonality of the spherical harmonics it is evident that $\langle Y_j{}^k(\theta'\varphi')\rangle$ is directly proportional to the coefficient of $Y_j{}^k(\theta'\varphi')$ in the expansion of the outgoing intensity (after all outgoing spin states have been summed over) in spherical harmonics. This completes the proof.

Other well-known results can be obtained using Eq. (4). Since, for the initial nonvanishing operators, m must be zero, it follows that m' (or k) must be zero and that the angular distribution is symmetrical about

[7] In order that the S'^μ have a finite dimensionality, it is necessary to impose some limit $L'_{\max}$ on the outgoing orbital angular momenta. The spherical harmonics are allowable operators then only if they are multiplied on the left by the sum of all projection operators for states of orbital angular momentum L' from 0 to $L'_{\max}$. Since this sum is invariant under rotation it does not affect the argument. The interpretation of $Y_j{}^k$ requires only that $L'_{\max}$ be chosen at least as large as the true maximum possible value of L'.

the axis of incidence. If the initial system has a well-defined parity, then p must be $(+1)$ for all nonvanishing operators, since every operator connecting states of the same parity has even parity. It follows that p' must equal $(+1)$ and therefore j must be even; that is, only even spherical harmonics can enter the angular distribution.

It is interesting to note that the spherical harmonics do not constitute a complete set of base matrices even for the orbital states alone. Indeed, for a given $L'_{\max}$ there are $(2L'_{\max}+1)^2$ spherical harmonics, but $(L'_{\max}+1)^4$ operators are needed for the complete set. On the other hand, the spherical harmonics exhaust the operators whose expectation values can be found from the outgoing angular distribution alone. Thus, even if the final particles have zero spin, the angular distribution tells less than the final density matrix in the most general case, so that it would seem possible in principle to make more complete observations on the final state than simply the angular distribution. For example, an equal statistical mixture of the three p states yields the same angular distribution, but does not have the same density matrix, as a pure S state. In order to distinguish between these, one would have to observe the interference between portions of the waves coming out at some angle relative to each other; while such observations might be possible in principle through the use of additional scatterings, they clearly are not practical.

These results may be easily extended to the case where the spins of the initial particles are polarized or to the angular distribution of the polarization produced in a reaction in which the incident particles are unpolarized.[8] Since the results are essentially the same for the two cases, we comment only on the former problem. The polarization[9] of a particle of spin S must be specified by the expectation values of a set of irreducible tensor operators $T^m{}_{jp}$ with j varying from 0 to $2S$ and p even. For a collision of polarized particles of spin S and spin i, the operators, $S^m{}_{Jp}$ acting on the initial states are sums of products of

$$T^{m_1}{}_{j_1+},\quad U^{m_2}{}_{j_2+},\quad L^0{}_{j_3p},$$

where T and U operate on the spin states of spin S and i, respectively, and L operates on the orbital states. Considering the effect of particular tensor operators T and U, we find the maximum value of J is $(j_1+j_2+2L_{\max})$ and for this value of J the parity p is even, since the operator $L_{2L_{\max}}$ has even parity because it connects pure states having the same parity. It follows from the previous argument that $\langle Y_j{}^k\rangle$ can be nonzero only if $j\leq(j_1+j_2+2L_{\max})$. However, the equality can hold only if $(j_1+j_2+2L_{\max})$ is even, in order that $p'(=(-1)^j)$ be even. *Thus, if* j_1+j_2 *is even, the maximum degree spherical harmonic allowable is* $(j_1+j_2+2L_{\max})$; *if* j_1+j_2 *is odd, the maximum is* $(j_1+j_2+2L_{\max}-1)$.

In particular, for polarized particles of spin $\frac{1}{2}$ colliding with an unpolarized nucleus, $j_1=1$ and $j_2=0$ with the result that the maximum is $2L_{\max}$ just as for the unpolarized case. The same will be true for polarized particles of higher spin, e.g., deuterons, provided only the part of the polarization specified by a vector is considered. In the general case of an incident polarized deuteron[10] we have $j_1=2$ and (still assuming $j_2=0$) the maximum degree spherical harmonic allowable is $2L_{\max}+2$. By carrying out these arguments in more detail one can find the most general angular dependence in the term in the angular distribution associated with each particular tensor operator describing the initial polarization; to get this result it is necessary to use the fact that $c^{Jp}{}_{\alpha\alpha'}$ in Eq. (4) is independent of m.

III. DISCUSSION

The main advantage of the method presented here is that it gives the invariance principle directly in terms of observables rather than in terms of wave functions. This is of particular value when the road leading from the wave functions to the observables is fairly involved because of averagings over an initial statistical mixture and summings over final unobserved variables.

The use of tensor operators to specify a system has some disadvantages also. In the case of a pure state it is very redundant; thus for a pure state of spin 1, there are clearly only five independent real parameters necessary to specify the state (including its normalization), but there are nine tensor operators whose expectation values are to be specified. The four relations between these nine are not linear.[10] Tensor operators acting on the spin state alone are nearly always useful in specifying the polarization of the spin,[8,11] but it is important to note that the operators $S^m{}_{Jp\alpha}$ entering Eq. (4) are not these but rather are the irreducible components of the products of these spin operators with orbital operators. These orbital operators (particularly for the initial state) are generally unfamiliar; however, in applications one need only make explicit use of the final state operators which are spherical harmonics. For detailed discussions beyond the general kinds of arguments given here, it is nearly always desirable to introduce explicitly the matrix elements relating initial and final wave functions. This then requires summing over spins, for which purpose the methods of Racah are particularly suitable.[12,13]

[8] L. Wolfenstein, Phys. Rev. **75**, 1664 (1949); R. J. Blin-Stoyle, Proc. Phys. Soc. (London) **A64**, 700 (1951); A. Simon and T. A. Welton, Phys. Rev. **89**, 886 (1953); A. Simon, Phys. Rev. **90**, 326 (1953).

[9] The term polarization is sometimes used in the more restricted sense of the expectation value of the vector $T_1{}^m$; see B. Bleaney, Proc. Phys. Soc. (London) **A64**, 315 (1951).

[10] W. Lakin and L. Wolfenstein, U. S. Office of Naval Research Technical Report (unpublished); Phys. Rev. **90**, 365 (1953).

[11] See H. Tolhoek and J. Cox, Physica **18**, 357, 359 (1952) for an application to the decay of a nucleus polarized by use of low temperatures.

[12] J. M. Blatt and L. C. Biedenharn, Revs. Modern Phys. **24**, 258 (1952).

[13] *Note added in proof:*—F. Coester and J. Jauch, Helv. Phys. Acta **26**, 3 (1953) use methods similar to those of this paper for deriving more detailed results.